Pretoposes and topological representations*

*Joint work with Vincenzo Marra

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Introduction

I will discuss how to construct topological representations for certain categories, i.e. faithful functors $\mathbf{X} \to \mathbf{Top}$.

Purpose: characterise/axiomatise the category **KH** of compact Hausdorff spaces and continuous maps between them.

The characterisation that I will present hinges on the fact that **KH** has both a spatial and an algebraic nature.

The spatial side of KH

The spatial nature of **KH** has proved rich from the duality theoretic viewpoint:

- starting in the 1940s, several dualities for **KH**: Gelfand-Naimark, Kakutani, Krein-Krein, Yosida, Stone. Later, also Banaschewski, Isbell, de Vries;
- Duskin (1969): $\mathbf{KH}^{\mathrm{op}}$ is monadic over **Set**;
- Banaschewski, Rosický (1980s): several (negative) results on the axiomatisability of $\mathbf{KH}^{\mathrm{op}}$;
- Marra, L. R. (2017): finite axiomatisation of a variety of infinitary algebras equivalent to $\mathbf{KH}^{\mathrm{op}}$.

The algebraic side of KH

Surprisingly, **KH** has also an algebraic nature:

- Linton (1966): **KH** is monadic over **Set** (in fact, it is *varietal*);
- Manes (1967): explicit description of compact Hausdorff spaces as the algebras for the ultrafilter monad on **Set**;
- Herrlich-Strecker (1971): exploit this algebraic nature to give a characterisation of **KH** (as the unique non-trivial full epireflective subcategory of Hausdorff spaces which is varietal).

We seek a characterisation of **KH** which is not relative to a particular fixed category. An example of such a characterisation, for the category **Set**, was provided by Lawvere.

Theorem (Lawvere's ETCS, 1964)
If **C** *is a complete category satisfying the eight axioms below, then* **C** *is equivalent to* **Set***.*

- **Ax. 1** **C** *is finitely complete and cocomplete;*
- **Ax. 2** *for any two objects* A, B *in* **C***, there exists* B^A *s.t.. . . ;*
- **Ax. 3** **C** *admits a natural number object;*
- . . .
- **Ax. 8** *there exists an object with more than one element.*

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The topological representation

Coherent categories are

- a categorical generalisation of distributive lattices;
- the categorical semantics for coherent logic $(\bot, \top, \vee, \wedge, \exists)$.

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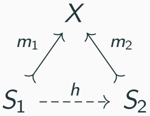
Given two subobjects $m_1 \colon S_1 \rightarrowtail X$ and $m_2 \colon S_2 \rightarrowtail X$, set



$$\boxed{m_1 \leq m_2 \iff \exists h \colon S_1 \to S_2 \text{ with } m_2 \circ h = m_1.}$$

Write $\equiv$ for the equivalence relation induced by the preorder $\leq$. The set of $\equiv$-equivalence classes of subobjects of X, with the partial order $\leq$, is denoted by $\operatorname{Sub} X$.

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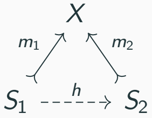

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In the presence of finite limits, $\operatorname{Sub} X$ is a $\wedge$-semilattice and, $\forall f \colon X \to Y$, the associated pullback functor is a $\wedge$-semilattice homomorphism:

$$f^* \colon \operatorname{Sub} Y \to \operatorname{Sub} X.$$

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Definition

A coherent category is a

- regular category, i.e.,
    - finitely complete,
    - with stable image factorisations,
- in which each $\mathrm{Sub}\, X$ is a $\vee$-semilattice and, for every $f\colon X \to Y$, the pullback functor

$$f^*\colon \mathrm{Sub}\, Y \to \mathrm{Sub}\, X$$

  is a $\vee$-semilattice homomorphism (hence a lattice homomorphism).

For every $f\colon X \to Y$ and $S \in \operatorname{Sub} X$, denote by $\exists_f(S)$ the image of S through f. The map

$$\exists_f\colon \operatorname{Sub} X \to \operatorname{Sub} Y,\ S \mapsto \exists_f(S)$$

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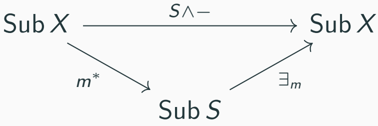
Lemma

For every X, $\operatorname{Sub} X$ is a (bounded) distributive lattice.

Proof.

Let $m\colon S \rightarrowtail X$ be a subobject.



$S \wedge (T \vee U) = (S \wedge T) \vee (S \wedge U)$

□

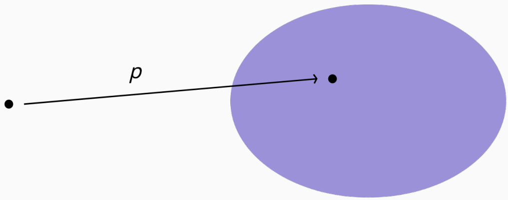
(non-)Examples

- $\mathbf{Set}_f$, **Set**, **BStone** and **KH** are coherent categories;
- every (elementary) topos is a coherent category;
- **Top** is not coherent (regular epis are not stable);
- any Abelian category (more generally, any pointed category) with two non-isomorphic objects is not coherent;
- for every equational theory T in an algebraic signature containing at least one constant symbol, Mod T is not coherent.

Points

Let $\mathbf{X}$ be a category admitting a terminal object $\mathbf{1}$, and X an object of $\mathbf{X}$. A point of X is a morphism

$$p\colon \mathbf{1} \to X.$$



Points

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Define the functor of points

$$\mathrm{pt} = \hom_{\mathbf{X}}(\mathbf{1}, -)\colon \mathbf{X} \to \mathbf{Set}$$

(Throughout, we assume $\mathbf{X}$ is locally small, hence well-powered.)

Idea: give a topological representation of the category $\mathbf{X}$ by lifting $\mathrm{pt}\colon \mathbf{X} \to \mathbf{Set}$ to a functor $\mathbf{X} \to \mathbf{Top}$.

Definition

The category $\mathbf{X}$ is well-pointed if, given any two distinct morphisms $f, g\colon X \rightrightarrows Y$ in $\mathbf{X}$, there is a point $p\colon \mathbf{1} \to X$ such that

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Observe that:

- $\mathbf{X}$ is well-pointed $\Leftrightarrow$ $\mathrm{pt}\colon \mathbf{X} \to \mathbf{Set}$ is faithful;
- if $\mathbf{X}$ is well-pointed and $\sum_{\mathrm{pt}\, X} \mathbf{1}$ exists in $\mathbf{X}$, then the following is an epimorphism:

$$\sum_{\mathrm{pt}\, X} \mathbf{1} \to X.$$

Lemma

Let $\mathbf{X}$ *be a well-pointed category with initial object* $\mathbf{0}$ *and terminal object* $\mathbf{1}$. *Suppose the unique morphism* $\mathbf{0} \to \mathbf{1}$ *is an extremal mono. Then,*

- *every non-initial object has at least one point;*
- *the points of* X *are precisely the atoms of* $\operatorname{Sub} X$.

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Remark:

- A mono m is extremal if $m = f \circ e$, with e epi, implies e iso;
- $\mathbf{0} \to \mathbf{1}$ is an extremal mono iff for every non-initial object X there is an object Y, and two distinct morphisms $f, g\colon X \rightrightarrows Y$.

For every object X and subobject $S \in \mathrm{Sub}\,X$, define

$$\mathbb{V}(S) = \{p\colon \mathbf{1} \to X \mid p \text{ factors through the subobject } S \to X\},$$

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The operator $\mathbb{V} \colon \operatorname{Sub} X \to \wp(\operatorname{pt} X)$ preserves all infima existing in the poset $\operatorname{Sub} X$. Hence, if $\operatorname{Sub} X$ is complete, $\mathbb{V}$ has a lower adjoint $\mathbb{I} \colon \wp(\operatorname{pt} X) \to \operatorname{Sub} X$ given by

$$\mathbb{I}(T) = \bigwedge \{S \in \operatorname{Sub} X \mid \text{ each } p \in T \text{ factors through } S\}.$$

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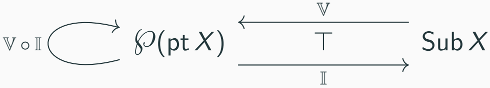

Lemma

Let $\mathbf{X}$ *be a non-trivial, well-pointed, coherent category in which each poset* $\operatorname{Sub} X$ *is complete. If* $\mathbf{0} \to \mathbf{1}$ *is an extremal mono, then the following statements hold.*

- *For each* $X \in \mathbf{X}$*, the closure operator* $\mathbb{V} \circ \mathbb{I}$ *on* $\wp(\operatorname{pt} X)$ *is topological.*
- *For each* $f\colon X \to Y$ *in* $\mathbf{X}$*, the function* $\operatorname{pt} f\colon \operatorname{pt} X \to \operatorname{pt} Y$ *is continuous and closed.*

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- *For each* $f : X \to Y$ *in* $\mathbf{X}$, *the function* $\operatorname{pt} f : \operatorname{pt} X \to \operatorname{pt} Y$ *is continuous and closed.*
- *If* $\operatorname{Sub} X$ *is atomic, then each* $S \in \operatorname{Sub} X$ *is a fixed point of the operator* $\mathbb{V} \circ \mathbb{I}$.

Obs.: $\operatorname{Sub} X$ is atomic for every $X \in \mathbf{X} \Leftrightarrow \operatorname{pt} : \mathbf{X} \to \mathbf{Set}$ is conservative. The two equivalent conditions are satisfied if, e.g., every mono in $\mathbf{X}$ is regular, or every epi is regular.

Under the hypotheses of the previous lemma, the functor of points

$$\mathrm{pt}\colon \mathbf{X} \to \mathbf{Set}$$

can be lifted to a (faithful) functor

$$\mathrm{Spec}\colon \mathbf{X} \to \mathbf{Top},$$

which sends an object X to the set $\mathrm{pt}\, X$ equipped with the topology induced by the operator $\mathbb{V} \circ \mathbb{I}$.

This yields a topological representation of $\mathbf{X}$.

Question: when does the functor Spec land in $\mathbf{KH}$?

Filtrality

VARIETÀ A QUOZIENTI FILTRALI

ROBERTO MAGARI * **

1. Premessa.

In alcuni recenti lavori (R. Magari [7], [8], [9]) [1]) ho studiato la varietà V generata da una data algebra $\boldsymbol{W}$ sotto due distinte ipotesi:

1) che $\boldsymbol{W}$ *sia funzionalmente completa* (finita o infinita) [2]);

2) che $\boldsymbol{W}$ abbia due elementi.

Molti risultati ottenuti in [7], [8] possono essere generalizzati assumendo l'ipotesi che $\boldsymbol{W}$ sia *semplice* e che ogni congruenza di ogni potenza sottodiretta di $\boldsymbol{W}$ sia associata a un filtro dell'insieme degli indici nel modo indicato nel successivo n. 2.

Nella presente nota saranno studiate più in generale le *classi filtrali*, ossia le classi K di algebre simili tali che ogni congruenza di ogni prodotto sottodiretto di elementi di K sia associata a un filtro dell'insieme degli indici.

I risultati principali sono dati dai teorr. 1, 3, 4, 6, 7, 8 e dal cor 1.

Il risultato di semicategoricità nel caso $K=\{\boldsymbol{W}\}$ con W finita si può ricavare dai risultati di A. Astromoff [1] e di A. L. Foster e A. F. Pixley [6] e viene dimostrato direttamente per completezza.

Gli usuali concetti di algebra universale vengono usati senza particolari richiami e sono reperibili in P. M. Cohn [2]. (Per una breve esposizione in lingua italiana ved. anche R. Magari [10]).

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* La presente stesura definitiva con qualche modifica è pervenuta il 24 ottobre 1968.

** Lavoro eseguito nell'ambito dell'attività del Comitato Nazionale per la Matematica del C.N.R. (anno '68-'69, gruppo 37).

1) Rimando ai lavori citati per i concetti usati e per le convenzioni e notazioni.

2) « functionally strictly complete » nel senso di A. L. Foster [4] in cui il concetto è riservato però alle algebre finite.

$\mathcal{L}$ a class of Birkhoff algebras of the same similarity type, and $\{A_i \mid i \in I\} \subseteq \mathcal{L}$. If B is a subalgebra of $\prod_{i \in I} A_i$, and F is a filter of $\wp(I)$, then the equivalence relation ϑ_F given by

$$\forall b, b' \in B,\ (b, b') \in \vartheta_F \iff \{i \in I \mid b_i = b'_i\} \in F$$

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Definition (Magari, 1969)

$\mathcal{L}$ is filtral if, whenever $B \rightarrowtail \prod_{i \in I} A_i$ is a subdirect product of members of $\mathcal{L}$, $F \mapsto \vartheta_F$ is a surjection onto the set of congruences of B.

$\mathcal{L}$ is semifiltral if the previous condition is satisfied whenever B is a direct product of members of $\mathcal{L}$. (Hence $F \mapsto \vartheta_F$ is a bijection).

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- If $\mathcal{L}$ is (semi)filtral, then each of its members is simple;
- $\mathcal{L} = \{A\}$ is filtral if A is the two-element Boolean algebra, or if A has a reduct of finite distributive lattice.

Let $\mathbf{X}$ be a category with a terminal object $\mathbf{1}$ such that arbitrary copowers of $\mathbf{1}$ exist in $\mathbf{X}$, and $\operatorname{Sub} X$ is complete for every $X \in \mathbf{X}$.

Fix $X \in \mathbf{X}$. Every filter F of $\wp(\operatorname{pt} X)$ gives a subobject of $\sum_{\operatorname{pt} X} \mathbf{1}$:

$$F \longmapsto k(F) = \bigwedge \{S \in \operatorname{Sub} \sum_{\operatorname{pt} X} \mathbf{1} \mid \operatorname{pt} S \cap \operatorname{pt} X \in F\}.$$

Definition

$\mathbf{X}$ is filtral if, for each X in $\mathbf{X}$, the following map is bijective:

$$\boxed{k\colon \operatorname{Filt}(\wp(\operatorname{pt} X)) \to \operatorname{Sub} \sum_{\operatorname{pt} X} \mathbf{1}.}$$

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- The definition can be easily adapted to deal with regular subobjects, which dualize congruences in a variety. If we do so, then:
- the condition above dualizes semifiltrality, in the sense of Magari, for $\mathcal{L} = \{A\}$, where A is initial in the variety that it generates.

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- **Set** and $\mathbf{Set}_f$ (relax assumption on the copowers of $\mathbf{1}$), are filtral.
- **KH** and **BStone** are filtral: for every discrete space I, the closed subsets of the Stone-Čech compactification $\beta(I)$ of I are in bijection with the filters of $\wp(I)$.

Theorem (Marra, R.)

Let $\mathbf{X}$ *be a non-trivial, well-pointed, coherent category s.t.* $\mathbf{0} \to \mathbf{1}$ *is an extremal mono. Suppose* $\mathbf{X}$ *admits arbitrary copowers of* $\mathbf{1}$*, and* $\mathrm{Sub}\, X$ *is complete and atomic for every* $X \in \mathbf{X}$*. Consider the conditions:*

1. $\mathbf{X}$ *is filtral;*
2. $\mathrm{Spec}\, X \in \mathbf{KH}$ *for every* $X \in \mathbf{X}$*.*

Then $\boxed{1 \Rightarrow 2}$*. That is, the functor* $\mathrm{Spec}\colon \mathbf{X} \to \mathbf{Top}$ *co-restricts to*

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$\boxed{2 \Rightarrow 1}$ *holds if every finite coproduct existing in* $\mathbf{X}$ *is disjoint.*

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Proof.

Sketch of $1 \Rightarrow 2$. Filtrality of $\mathbf{X}$ implies that $\operatorname{Spec} \sum_{\operatorname{pt} X} \mathbf{1} \cong \beta(\operatorname{pt} X)$. For every $X \in \mathbf{X}$, consider the epimorphism $\varepsilon\colon \sum_{\operatorname{pt} X} \mathbf{1} \to X$. Then $\operatorname{Spec} \varepsilon\colon \operatorname{Spec} \sum_{\operatorname{pt} X} \mathbf{1} \twoheadrightarrow \operatorname{Spec} X$ exhibits $\operatorname{Spec} X$ as the image of a compact Hausdorff space under a continuous closed map. $\square$

A characterisation of KH

Definition

A pretopos is a coherent category which is

- positive, i.e., finite coproducts exist and are disjoint, and
- effective, i.e., every internal equivalence relation coincides with the kernel pair of its coequaliser.

(Equivalently, a pretopos is an extensive and Barr-exact category).

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(non-)Examples

- $\mathbf{Set}_f$, $\mathbf{Set}$, are pretoposes;
- more generally, every elementary topos is a pretopos;
- **KH** is a pretopos (effectiveness follows, e.g., from monadicity);
- **BStone** is coherent and positive, but not effective. Hence it is not a pretopos. Its pretopos completion is **KH**.

Let $\mathbf{X}$ be a non-trivial, well-pointed, pretopos admitting all coproducts.

Theorem (Marra, R.)
Let $\mathbf{X}$ be a non-trivial, well-pointed, coherent category s.t. $\mathbf{0} \to \mathbf{1}$ is an extremal mono. Suppose $\mathbf{X}$ admits arbitrary copowers of $\mathbf{1}$, and $\operatorname{Sub} X$ is complete and atomic for every $X \in \mathbf{X}$. Consider the conditions:

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Let $\mathbf{X}$ be a non-trivial, well-pointed, pretopos admitting all coproducts. Then $\mathbf{X}$ is filtral if, and only if, $\mathrm{Spec}\colon \mathbf{X} \to \mathbf{Top}$ lands in $\mathbf{KH}$.

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1. $\mathbf{X}$ *is filtral;*
2. $\mathrm{Spec}\, X \in \mathbf{KH}$ *for every* $X \in \mathbf{X}$.

Then $\boxed{1 \Rightarrow 2}$. *That is, the functor* $\mathrm{Spec}\colon \mathbf{X} \to \mathbf{Top}$ *co-restricts to*

$$\mathrm{Spec}\colon \mathbf{X} \to \mathbf{KH}.$$

$\boxed{2 \Rightarrow 1}$ *holds if every finite coproduct existing in $\mathbf{X}$ is disjoint.*

Proof.

Sketch of $1 \Rightarrow 2$. Filtrality of $\mathbf{X}$ implies that $\mathrm{Spec} \sum_{\mathrm{pt}\, X} \mathbf{1} \cong \beta(\mathrm{pt}\, X)$. For every $X \in \mathbf{X}$, consider the epimorphism $\varepsilon\colon \sum_{\mathrm{pt}\, X} \mathbf{1} \to X$. Then $\mathrm{Spec}\, \varepsilon\colon \mathrm{Spec} \sum_{\mathrm{pt}\, X} \mathbf{1} \twoheadrightarrow \mathrm{Spec}\, X$ exhibits $\mathrm{Spec}\, X$ as the image of a compact Hausdorff space under a continuous closed map. □

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- because epi+mono=iso in a pretopos

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- this means that, for every $X \in \mathbf{X}$, the lattice homomorphism $\mathrm{Sub}\, X \to \mathrm{Sub}\, \mathrm{Spec}\, X$ is surjective. It follows from the fact that every closed subset of $\mathrm{Spec}\, X$ is of the form $\mathbb{V}(S)$, for some $S \in \mathrm{Sub}\, X$.

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> **Proposition** (Makkai)
> A coherent functor between pretoposes is an equivalence iff it is conservative, full on subobjects, and it covers its codomain.

- that is, for every $Y \in \mathbf{KH}$ there is $X \in \mathbf{X}$ and an epimorphism $\mathrm{Spec}\, X \twoheadrightarrow Y$. Use the fact that, $\forall \tilde{X} \in \mathbf{X}$, $\mathrm{Spec} \sum_{\mathrm{pt}\, \tilde{X}} \mathbf{1} \cong \beta(\mathrm{pt}\, \tilde{X})$, and every compact Hausdorff space is the continuous image of the Stone-Čech compactification of a discrete space.

Comments and questions

For varieties of Birkhoff algebras, filtrality is related to a certain generalization of the inconsistency lemma

$$\Gamma \cup \{\alpha\} \text{ is inconsistent } \Leftrightarrow \Gamma \vdash \neg\alpha.$$

(Raftery, "Inconsistency lemmas in algebraic logic", Math. Log. Quart. **59**)

Is there a logical counterpart to "filtrality for categories"?

Comments and questions

If $\mathbf{X}$ is a well-pointed, positive, coherent category which is filtral (+some properties already discussed), there is a faithful functor $\text{Spec}\colon \mathbf{X} \to \mathbf{KH}$.

Where are finite sets and Boolean spaces, in this picture?

Comments and questions

If $\mathbf{X}$ is a well-pointed, positive, coherent category which is filtral (+some properties already discussed), there is a faithful functor $\mathrm{Spec}\colon \mathbf{X} \to \mathbf{KH}$.

Where are finite sets and Boolean spaces, in this picture?

Definition

An object X is decidable if its diagonal is complemented. That is, if $\delta\colon X \to X \times X$ denotes the diagonal morphism, there is $\epsilon\colon Y \to X \times X$ such that the following is a coproduct diagram:

$$X \xrightarrow{\delta} X \times X \xleftarrow{\epsilon} Y.$$

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- $\mathrm{Spec}\colon \mathbf{X} \to \mathbf{KH}$ restricts to an equivalence $\mathbf{Dec}(\mathbf{X}) \to \mathbf{Set}_f$;
- if $\mathbf{X}$ is complete, then taking inverse limits in $\mathbf{X}$ of decidable objects yields a full subcategory equivalent to **BStone**.

Comments and questions

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Date:  6 August 1996
From:  Peter Freyd
The phrase DISTRIBUTIVE CATEGORY is established as
referring to a category with finite products and...

                  ...[LONG MESSAGE]...

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Date: 7 August 1996
From: P. T. Johnstone
Not much of it, if you mean what is in Johnstone's
published work, rather than in Johnstone's mind.

Thank you!