Completions of Heyting Algebras and Bi-Relational Semantics for IPC

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March 2 2018

Plan

The semantic hierarchy

Kripke frames $\prec$ Topological spaces $\prec$ Locales $\prec$ Heyting algebras

- Kuznetsov's problem (1975): Is every intermediate logic complete with respect to some class of topological spaces?
- Variant: Is every intermediate logic complete with respect to some class of cHA's (locales)?

The semantic hierarchy

- Thanks to Shehtman(1980), we know that some intermediate logics are Kripke incomplete.
- But topological spaces and cHA's are notoriously more abstract than Kripke frames, and the corresponding semantics remain much more obscure.

The semantic hierarchy

- Thanks to Shehtman(1980), we know that some intermediate logics are Kripke incomplete.
- But topological spaces and cHA's are notoriously more abstract than Kripke frames, and the corresponding semantics remain much more obscure.
- However, it has recently been realized (G. Bezhanishvili & Holliday, 2016) that a semantics for IPC in terms of *bi-relational structures* was as general as locale semantics, yet more concrete. This semantics had already been introduced by Fairtlough and Mendler in 1997, although for a different purpose.
- My goal today is to present some results about this semantics, and give you an intuitive grasp of how it compares to Kripke semantics.

Outline

Plan

Stone representation theorem

Theorem (Stone, 1936)

Every Boolean algebra B embeds into the powerset of its dual Stone space.

Definition

Let B be a BA.

- The filter space of B is the topological space (S_B, τ), where S_B is the collection of all filters over B and τ is the upset topology induced by the inclusion ordering on S_B.
- the principal space of B is the topological space (P_B, τ), where P_B is the collection of all principal filters over B and τ is defined similarly.

Theorem

Every Boolean algebra B embeds into the regular open sets of its filter space and into the regular open sets of its principal space.

Topological representations of completions

Lemma

For any Boolean algebra B with dual Stone space X_B:

- $\mathscr{P}(X_B)$ *is isomorphic to the canonical extension of* B*.*
- $\mathrm{RO}(X_B)$ *is isomorphic to the canonical extension of* B*.*

Lemma (Holliday 2015)

For any Boolean algebra B with filter space S_B and principal space P_B:

- $\mathrm{RO}(S_B)$ *is isomorphic to the canonical extension of* B*.*
- $\mathrm{RO}(P_B)$ *is isomorphic to the MacNeille completion of* B*.*

Representation of HAs

Theorem (Esakia)

Any HA L embeds into the upsets of its dual Esakia space.

Lemma

For any HA L with dual space X_L, the canonical extension of L is isomorphic to $Up(X_L)$.

Theorem (G. Bezhanishvili & J. Harding, 2004)

For any HA L with dual Esakia space X_L, the MacNeille completion of L is isomorphic to $\{S \in OpUp(X_L) \,;\, JC(S) = S\}$, where J is the interior operator of the spectral topolgy on X_L.

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- What about constructive versions of those results?

Regular open sets and the double negation nucleus

Lemma (Tarski)

The regular open sets of any topological space form a cBA.

- A point-free argument: In any lattice of open sets $\mathscr{O}$, $\neg U = -C(U)$ for any $U \in \mathscr{O}$.

- Therefore $IC(U) = -C - C(U) = \neg\neg(U)$ for any $U \in \mathscr{O}$.

Regular open sets and the double negation nucleus

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- Therefore $IC(U) = -C - C(U) = \neg\neg(U)$ for any $U \in \mathscr{O}$.
- Key idea: Modify the notion of regular open sets so that the corresponding interior-closure operator is still a nucleus on a lattice of open sets, but not necessarily the double negation nucleus.

Nuclei on subframes

- Let A, B be frames such that A is a subframe of B. Define $\nu : B \to A$ such that for all $b \in B$, $\nu(b) = \bigvee\{a \in A\, ;\ a \leq_B b\}$.

- ν is right-adjoint to the inclusion map $\iota : A \to B$ (hence preserve finite meets in A)

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- Define a map $j : A \to A$ such that $j(a) = \nu\neg_B\neg_B\iota(a)$ for all $a \in A$.

- j is the composition of monotone, multiplicative maps and is increasing on A since $a \leq_B \neg_B\neg_B\iota(a)$. For idempotence: $\nu\neg_B\neg_B\iota\nu\neg_B\neg_B\iota(a) \leq_A \nu\neg_B\neg_B\neg_B\neg_B\iota(a) \leq_A \nu\neg_B\neg_B\iota(a)$

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- so j is a nucleus on A!

Refined bi-topological spaces

Definition

A *refined bi-topological space* is a bi-topological space (X, τ_1, τ_2) such that $\tau_1 \subseteq \tau_2$.

A *bi-relational structure* (bRS) is a refined bi-topological space (X, τ_1, τ_2) such that both τ_1 and τ_2 are Alexandroff topologies.

Lemma

Let (X, τ_1, τ_2) be a refined bi-topological space. Then the operator $I_1 C_2$ (Interior in τ_1, Closure in τ_2) is a nucleus on the frame of opens in τ_1. Therefore $\mathrm{RO}_{12}(X)$ is a cHA.

Proof.

This follows from the previous slide and the fact that $I_1 C_2(U) = I_1 I_2 C_2(U)$ for all $U \subseteq X$. □

Constructive representation theorem for HA

Definition

Let L be a lattice. A *right pseudo-prime pair* over L is a pair (F, I) such that:

- F is a filter, I is an ideal, and $F \cap I = \emptyset$ (compatible pair);
- For any $a \in F$, $b \in I$ and $c \in L$, if $a \wedge c \leq b$, then $c \in I$ (Right Meet Property);

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Lemma ("Constructive PFT")

Let L be a lattice. Then L is distributive iff for any compatible pair (F, I) over L, there exists a right pseudo-prime pair (F^*, I^*) such that $F \subseteq F^*$ and $I \subseteq I^*$.

Constructive representation theorem for HA

Definition

Let L be a Heyting algebra. The *canonical filter-ideal space* is the refined bitopological space (S_L, τ_1, τ_2), where S_L is the set of all pseudo-prime pairs over L, and τ_1 and τ_2 are the upset topologies induced by the filter inclusion ordering and the filter-ideal inclusion ordering respectively.

Theorem

Let L be a Heyting algebra, and (S_L, τ_1, τ_2) its canonical filter-ideal space. Then the Stone map: $|\cdot| : L \to \mathscr{P}(S_L)$ defined by $|a| = \{(F, I) \in S_L \ ; \ a \in F\}$ is a HA-embedding of L into $\mathrm{RO}_{12}(S_L)$.

A note on completions

- For any HA L with canonical filter-ideal space (S_L, τ_1, τ_2), $\mathrm{RO}_{12}(S_L)$ is isomorphic (under PFT) to the upsets of the dual Kripke frame of L (i.e. to the canonical extension of L).

- But one can also slightly modify the definition of (S_L, τ_1, τ_2) in order to represent other completions of L as $\mathrm{RO}_{12}(S_L)$.

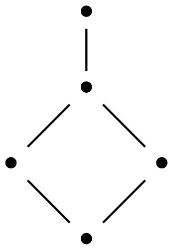
- For example, letting $P_L = \{(\uparrow a, \downarrow a \to b) \; ; \; a, b \in L, a \nleq b\}$, we have that $\mathrm{RO}_{12}(P_L)$ is the *MacNeille completion* of L.

Semantic hierarchy

- As a direct consequence, every cHA can be represented as the refined regular opens of some bi-relational structure.
- This implies that a semantics for *IPC* based on bi-relational structures is as general as Dragalin or locale semantics (in fact, this is precisely FM-semantics).
- On the other hand, bRS are very concrete objects to work with.
- So how close are bi-relational and Kripke semantics to one another?

An example

- Consider the following Heyting algebra:



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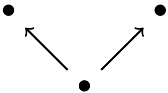

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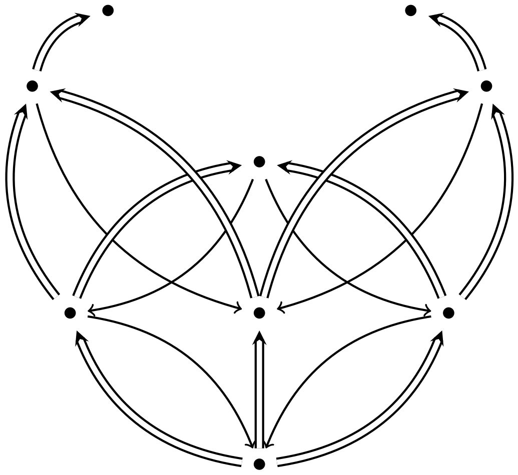


Plan

Bi-relational semantics for IPC

Definition

A bRS model is a tuple $(X, \leq_1, \leq_2, V)$ such that $(X, \leq_1, \leq_2)$ is a bRS an $V : Prop \to RO_{12}(X)$ is a valuation function that assigns refined regular opens to all propositional variables of IPC.

For any bRS model $(X, \leq_1, \leq_2, V)$, satisfaction is defined recursively as follows:

- $x \Vdash \phi$ iff $x \in V(\phi)$ for $\phi \in Prop$;
- $x \Vdash \bot$ never, $x \Vdash \top$ always;
- $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$;
- $x \Vdash \phi \vee \psi$ iff for all $y \geq_1 x$ there is $z \geq_2 y$ such that $z \Vdash \phi$ or $z \Vdash \psi$;
- $x \Vdash \phi \to \psi$ iff for all $y \geq_1 x$, $y \Vdash \phi$ implies $y \Vdash \psi$.

Validity is defined as usual.

Bi-relational semantics for IPC

- Intuitive picture of the semantics: Points in a bRS are partial descriptions of information states.

- Two levels of informativeness:
  - *states* can be more or less informative about *the world*;
  - *descriptions* can be more or less informative about *the states*.

Bi-relational semantics for IPC

- Intuitive picture of the semantics: Points in a bRS are partial descriptions of information states.
- Two levels of informativeness:
  - *states* can be more or less informative about *the world*;
  - *descriptions* can be more or less informative about *the states*.
- For any two points x, y, $x \leq_1 y$ iff every state partially described by y is more informative about the world than some state partially described by x (the states described by y are more informative about the world than the states described by x).
- On the other hand, $x \leq_2 y$ iff every state partially described by y is also partially described by x (y is a more informative description than x).

Bi-relational semantics for IPC

- Kripke frames are precisely those bRS $(X, \leq_1, \leq_2)$ where $\leq_2 = \Delta_X$.
- Possibility frames, on the other hand, are those bRS $(X, \leq_1, \leq_2)$ where $\leq_1 = \leq_2$.
- In more intuitive terms: Kripke frames are those frames where the *second* informativeness level is trivial (every point completely describes a state). Possibility frames are those frames where the *first* informativeness level is trivial (every state is maximally informative about the world).

A glimpse into intermediate logics

(Joint work with Nick Bezhanishvili and Somayeh Chopoghloo)

- $LC = IPC + (p \to q) \vee (q \to p)$ is the logic of right-linear Kripke frames.
- $KC = IPC + \neg p \vee \neg\neg p$ is the logic of directed Kripke frames.
- Can we characterize KC and LC bi-relational structures in a similar manner?

A glimpse into intermediate logics

Definition

1. Let $(X, \leq_1, \leq_2, V)$ be a bRS model. A point $x \in X$ *refutes* a formula ϕ (noted $x \Vdash^- \phi$) if $y \nVdash \phi$ for all $y \geq_2 x$.
2. A point x is *independent* from a point y (noted $x \perp y$) if $\uparrow_2 x \cap \uparrow_1 y = \emptyset$.

Lemma

1. *Let $(X, \leq_1, \leq_2, V)$ be a bRS model. For any $x \in X$, and any formulas ϕ, ψ, $x \Vdash^- \phi \vee \psi$ iff $x \Vdash^- \phi$ and $x \Vdash^- \psi$. Moreover, for any formula ϕ, $x \nVdash \phi$ iff there is $y \geq_1 x$ such that $y \Vdash^- \phi$.*
2. *For any $x, y \in X$, $x \perp y$ implies that $x \notin I_1 C_2(\uparrow_1 y)$.*

A glimpse into intermediate logics

Theorem

1. *LC is valid on a bRS* $(X, \preccurlyeq_1, \preccurlyeq_2)$ *iff for all* $x \in X$ *there are no* $f_1, f_2 : \uparrow_2 x \to X$ *such that:*
   - *for all* $z \succcurlyeq_2 x$, $f_1(z), f_2(z) \succcurlyeq_1 z$, *and*
   - *for all* $z, z' \succcurlyeq_2 x$, $f_1(z) \perp f_2(z')$ *and* $f_2(z) \perp f_1(z')$.
2. *KC is valid on a bRS* $(X, \preccurlyeq_1, \preccurlyeq_2)$ *iff for all* $x \in X$ *there are no* $f_1, f_2 : \uparrow_2 x \to X$ *such that:*
   - *for all* $z \succcurlyeq_2 x$, $(f_1(z), f_2(z) \succcurlyeq_1 z$, *and*
   - *for all* $z' \succcurlyeq_2 x$, $\uparrow_1 f_1(z) \cap \uparrow_1 f_2(z') = \emptyset$.

Surprising examples of LC and KC bRS

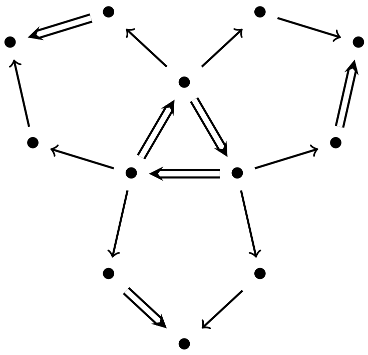


Figure 1: A non-linear LC-bRS

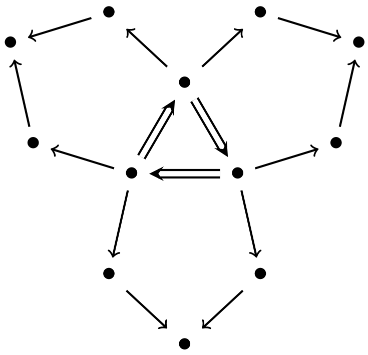


Figure 2: A non-directed KC-bRS

Global Subframes

- By contrast with Kripke frames, LC and KC bi-relational structures do not have a first-order characterization.
- There is however another characterization of LC and KC Kripke frames in terms of subframes:
  1. LC is valid on a Kripke frame $(X, \leq)$ iff the 2-fork is not a subframe of $(X, \leq)$.
  2. KC is valid on a Kripke frame $(X, \leq)$ iff the 2-fork is not a cofinal subframe of $(X, \leq)$.
- A similar characterization can be given in the bi-relational setting, but it requires defining a more abstract notion of subframe.

Global Subframes

Definition

Let $(X, \preccurlyeq_1, \preccurlyeq_2)$ be a bi-relational structure. A *global subframe* of X is a pair $(\mathfrak{S}, (f_i)_{i \in I})$ such that:

1. $\mathfrak{S} := (S, \leq_1, \leq_2)$ is a bi-relational structure and $(f_i)_{i \in I}$ is a collection of maps from $S \to X$ such that:
2. for each $i \in I$, f_i is injective;
3. for each $i \in I$, $x, y \in S$, $k \in \{1, 2\}$: $f_i(x) \preccurlyeq_k f_i(y)$ iff $x \leq_k y$;
4. for each $i \in I$, $x \in S$ and $y \in X$, if $f_i(x) \preccurlyeq_2 y$, then there is $j \in I$ such that $f_j(x) = y$;
5. for each $i, j \in I$, $x, y \in S$, $x \perp y$ implies $f_i(x) \perp f_j(y)$.

Global Subframes

Definition

A global subframe $(\mathfrak{S}, (f_i)_{i \in I}$ is a *cofinal global subframe* if condition 5 above is strengthened as follows:

5'. for each $i \in I$, and $x \in X$, if there is $y \in S$ such that $f_i(y) \preccurlyeq_1 x$, then there is $z \in S$ such that $x \preccurlyeq_1 f_i(z)$.

Theorem

Let $(X, \preccurlyeq_1, \preccurlyeq_2)$ be a bi-relational structure. Then:

1. *LC is valid on $(X, \preccurlyeq_1, \preccurlyeq_2)$ iff the fork is not a global subframe of $(X, \preccurlyeq_1, \preccurlyeq_2)$.*

2. *KC is valid on $(X, \preccurlyeq_1, \preccurlyeq_2)$ iff the fork is not a cofinal global subframe of $(X, \preccurlyeq_1, \preccurlyeq_2)$.*

Plan

A simple observation

- For any bi-topological space (X, τ_1, τ_2), $C_2 : \mathscr{O}_1 \to \mathscr{C}_2$ and $I_1 : \mathscr{C}_1 \to \mathscr{O}_2$ form a monotone Galois connection. Therefore $\mathrm{RO}_{12}(X)$ and $\mathrm{RC}_{21}(X)$ are order-isomorphic.

- In particular, in a bRS $(X, \leq_1, \leq_2)$, $RC_{21}(X)$ is a cHA.

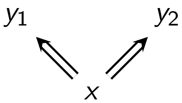
- Refined regular closed sets are always closed under arbitrary unions, but not necessarily under finite intersections.

- In the case of *bRS*, the latter is equivalent to the two relations satisfying the following weak directedness condition:

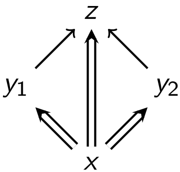
$$\forall x, y, z(x \leq_2 y \wedge x \leq_2 z \to \exists w(y \leq_1 w \wedge z \leq_1 w \wedge x \leq_2 w)$$

Weak Directedness Condition

Equivalently, every diagram of the form:



can be completed as follows:



A Characterization of Spatial Locales

Lemma

For every weakly directed bRS $(X, \leq_1, \leq_2)$, $RC_{21}(X)$ *is a topology on* X.

Theorem

Every spatial locale is isomorphic to the refined regular closed sets of some weakly-directed bRS.

A Characterization of Spatial Locales

Lemma

A locale L is spatial iff for any $a \nleq b \in L$, there is a meet-prime element $i \in L$ such that $a \nleq i$ and $b \leq i$.

Definition

Let L be spatial locale. The *weakly-directed representation* of L is the bRS $(M_L, \leq_1, \leq_2)$ such that:

- $M_L = \{(f, i)\ ;\ f \nleq i \in L, i \text{ meet-prime}\}$;
- $(f, i) \leq_1 (f', i')$ iff $f \geq f'$;
- $(f, i) \leq_2 (f', i')$ iff $f \geq f'$ and $i \leq i'$.

A Characterization of Spatial Locales

- Note that $(M_L, \leq_1, \leq_2)$ is weakly directed: if $(f, i) \leq_2 (f_1, i_1)$, (f_2, i_2), then $f_1, f_2 \not\leq i$, which implies that $(f_1 \wedge f_2 \not\leq i)$ since i is meet-prime.
- Moreover, the map $|\cdot| : L \to RC_{21}(M_L)$ defined by $|a| = \{(f, i)\ ;\ a \not\leq i\}$ is an order-embedding.
- Moreover, for every $U \in \mathrm{RC}_{21}(M_L)$, $U = |\bigvee B|$, where $B = \{b \in L\ ;\ |b| \subseteq B\}$. This means that $|\cdot|$ is an isomorphism.

Corollary

A locale is spatial iff it is isomorphic to the refined regular closed sets of a weakly directed bRS.

EV semantics

- The previous result yields an alternative semantics for IPC which is bi-relational, yet as general as topological semantics.

Definition

An *Effective Verifiability model* (EV model) is a tuple $(X, \leq_1, \leq_2, V)$ such that $(X, \leq_1, \leq_2)$ is a weakly directed bRS, and $V : Prop \to RC_{21}(X)$ is a valuation function that sends the propositional variables of IPC to refined regular closed sets in X.

EV-semantics

Let $(X, \leq_1, \leq_2, V)$ be an EV-model. Satisfaction is recursively defined as follows:

- $x \Vdash \phi$ iff $x \in V(\phi)$ for $\phi \in Prop$;
- $x \Vdash \bot$ never, $x \Vdash \top$ always;
- $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$;
- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$;
- $x \Vdash \phi \rightarrow \psi$ iff there exists $y \geq_2 x$ such that for all $z \geq_1 y$, $z \Vdash \phi$ implies $z \Vdash \psi$.

Validity is defined as usual.

EV-semantics

- Formulas in an EV-model $(X, \leq_1, \leq_2, V)$ are always evaluated as refined regular closed sets.
- Intuitive picture of the semantics: Points are states of information. The first ordering corresponds to an increase in information: $x \leq_1 y$ iff y is more informative than x. On the other hand, $x \leq_2 y$ iff y is a more informative state that can *effectively* been reached from x.
- A formula ϕ is *verified* at a state x iff $x \in I_1(V(\phi))$. On the other hand, ϕ is *assertible* at x iff ϕ can effectively be verified at x, i.e. $x \in C_2 I_1(V(\phi))$.
- One can then justify the weak directedness condition as follows: a conjunction is effectively verifiable if each conjunct is effectively verifiable.

Plan

Open problems

- Can we adapt standard techniques from Kripke semantics to bi-relational semantics?
- In particular, can we define topologically incomplete intermediate logics?
- Can we characterize graph-theoretically some other algebraic or topological properties? Example: join-prime generated algebras, bi-Heyting algebras, Beth frames, separation axioms,...

Thank You!