Some open problems on intermediate logics determined by classes of polyhedra

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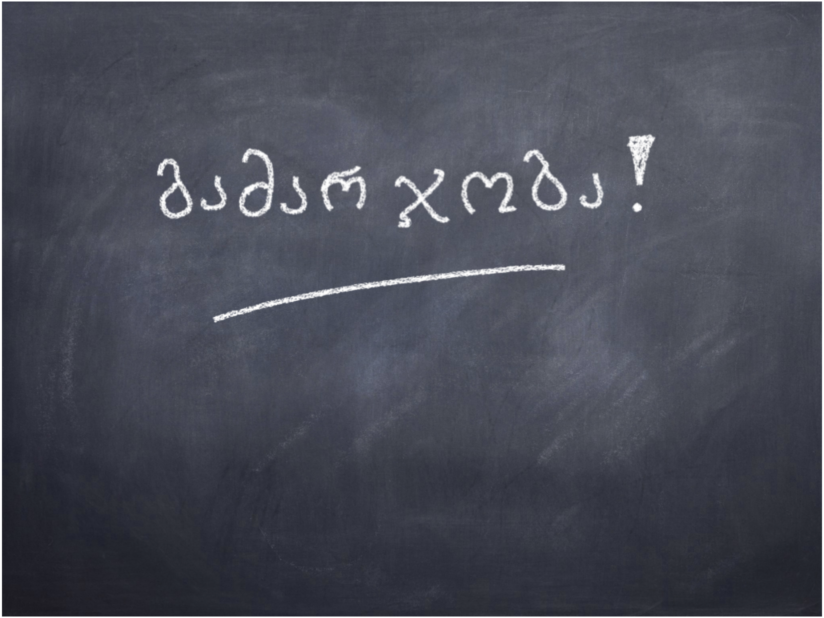

გამარჯობა!

The Heyting and co-Heyting structure of a space P

For P any topological space, the lattice $\mathscr{O}(P)$ of open sets of P is a Heyting algebra.

Heyting structure (open sets)

$$U \to V \coloneqq \mathrm{int}\,(\,P \setminus (U \setminus V)\,)$$
$$\neg U \coloneqq U \to \bot = \mathrm{int}\,(P \setminus U).$$

Co-Heyting structure (closed sets)

$$C \leftarrow D \coloneqq \mathrm{cl}\,(\,C \setminus D\,)$$
$$\llcorner D \coloneqq \top \leftarrow D = \mathrm{cl}\,(P \setminus D).$$

Intermediate logic of a class of spaces C

Int is intuitionistic logic.

For P a space,

$$\mathsf{Log}\, P \subseteq \mathsf{Int}$$

is the logic of formulæ that are valid in the Heyting algebra of opens of P.

For C a class of spaces,

$$\mathsf{Log}\, \mathsf{C} := \bigcap_{P \in \mathsf{C}} \mathsf{Log}\, P.$$

Intuitionistic logic is the logic of spaces

Spatial Completeness Theorem for Int (A.Tarski, 1938)

If C is the class of all topological spaces,

$$\mathrm{Log}\, \mathsf{C} = \mathsf{Int}.$$

Also, writing $\mathbb{R}^n$ for n-dimensional Euclidean space,

$$\mathrm{Log}\, \mathbb{R}^n = \mathsf{Int}$$

for each $n \geqslant 1$. Also, writing $2^{\mathbb{N}}$ for the Cantor space,

$$\mathrm{Log}\, 2^{\mathbb{N}} = \mathsf{Int}.$$

Spaces with additional structure

Suppose space P has additional algebro-geometric structure that determines distinguished subspaces of P. E.g.:

P (topological) linear space	$\Rightarrow$	linear subspaces.
P algebraic variety	$\Rightarrow$	subvarieties.
P differentiable manifold	$\Rightarrow$	differentiable submanifolds.
P semi-algebraic set	$\Rightarrow$	semi-algebraic subsets
P a polyhedron	$\Rightarrow$	subpolyhedra

Sometimes the distinguished open or closed subspaces form a Heyting subalgebra of the open sets, or a co-Heyting subalgebra of the closed sets, of the space P. This happens e.g. in the last two examples. We will look at the polyhedral case, and recall from (N. Bezhanishvili et al., 2018) that we still get completeness.

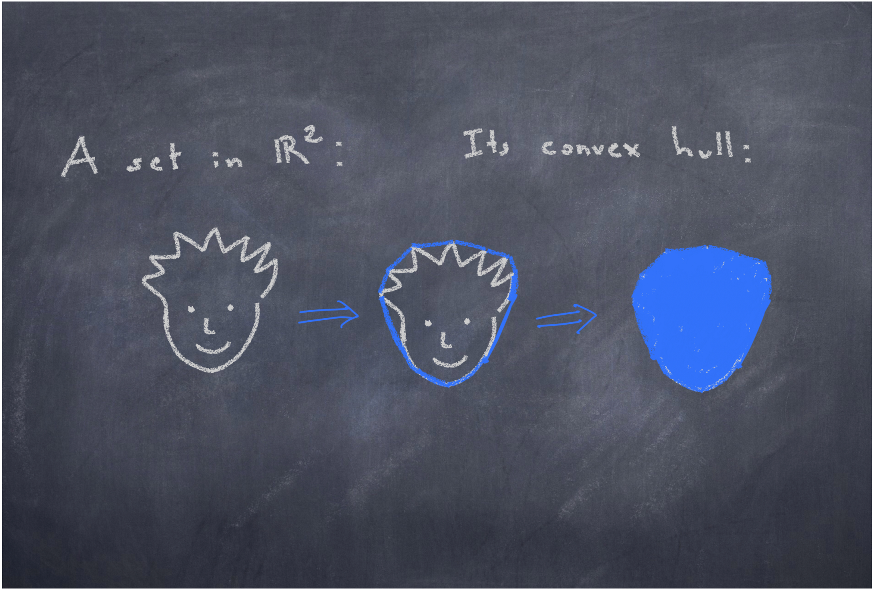
Polytopes

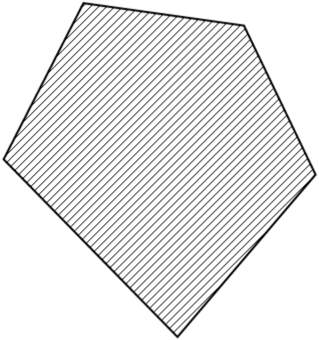
The convex hull of a set $S \subseteq \mathbb{R}^n$ is the collection of all convex combinations of elements of S:

$$\operatorname{conv} S = \left\{ \sum_{i=1}^{m} r_i v_i \mid v_i \in S \text{ and } 0 \leqslant r_i \in \mathbb{R} \text{ with } \sum_{i=1}^{m} r_i = 1 \right\} .$$

The set $P \subseteq \mathbb{R}^n$ is called a polytope if there is a finite $S \subseteq \mathbb{R}^n$ with

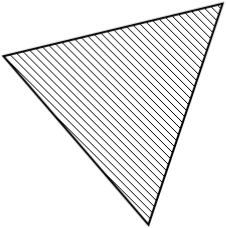
$$P = \operatorname{conv} S.$$


A set in $\mathbb{R}^2$:
Its convex hull:



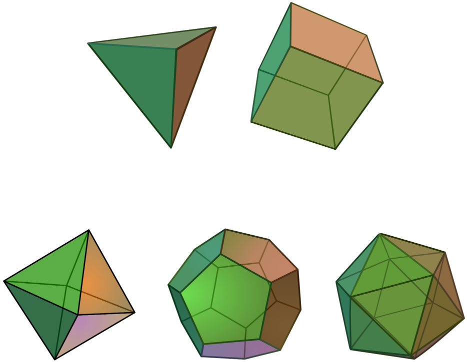
A polytope in $\mathbb{R}^2$.

The vertices of a polytope are those points not contained in the convex hull of any two other points. They are finitely many. Every polytope has a canonical convex-hull representation: it is the convex hull of its vertices.



A simplex in $\mathbb{R}^2$.

A simplex is the convex hull of a finite set of affinely independent points, or equivalently, it is a polytope with affinely independent vertices.



The Five Platonic Solids

(*In Plato's Timaeus, ca. 350 B.C., after his friend mathematician Theaetetus.*)

The set of polytopes in $\mathbb{R}^n$ is closed under finite intersections, but not under finite unions: it is not a sublattice of $2^{\mathbb{R}^n}$.

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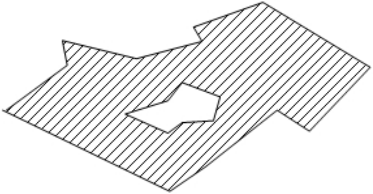
A (compact) polyhedron in $\mathbb{R}^n$ is a union of finitely many polytopes in $\mathbb{R}^n$.



A polyhedron in $\mathbb{R}^2$.

Polyhedra have a dimension. Use intuition. (Short definition: $\dim P \leqslant d$ iff there is simplex $\sigma \subseteq P$ with $d+1$ vertices.) A d-simplex, i.e. a simplex with $d+1$ vertices, has dimension d.

The Heyting structure of polyhedra

Definition (Distributive lattice of subpolyhedra)

For P a polyhedron, $\mathrm{Sub_o}\, P$ is the sublattice of $\mathscr{O}(P)$ consisting of open subpolyhedra, that is, complements $P \setminus Q$ of polyhedra $Q \subseteq P$.

Lemma

For any polyhedron P, $\mathrm{Sub_o}\, P$ is a Heyting subalgebra of $\mathscr{O}(P)$.

One shows (for co-Heyting) that $\mathrm{cl}\,(\, C \setminus D\,)$ is a polyhedron whenever $C, D \subseteq P$ are subpolyhedra of P.

Bounded Depth

The bounded-depth schemata — Krull dimension for Heyting algebras:

$$\mathrm{BD}_d \coloneqq \begin{cases} (\,\alpha_0 \vee \neg\alpha_0\,) & \text{if } d = 0, \text{ and} \\ (\,\alpha_d \vee (\alpha_d \to \mathrm{BD}_{d-1})\,) & \text{if } d \geqslant 1. \end{cases}$$

Standard Lemma

For any non-trivial Heyting algebra H and each $d \in \mathbb{N}$, TFAE.

1. H satisfies the equation $\mathrm{BD}_d = \top$, and fails each equation $\mathrm{BD}_{d'} = \top$ with $1 \leqslant d' < d$.
2. The longest chain of prime filters in H has cardinality $d + 1$.
3. The frame dual to H (=its Esakia-Priestley space) has depth d.

(Note to self: Bounded depth for co-Heyting, and dimension of boundaries $\partial x \coloneqq x \wedge \neg x$.)

Intuitionistic logic is the logic of polyhedra

Write P for the class of all polyhedra, and P_d for the class of all polyhedra of dimension $\leqslant d$.
For a class C of polyhedra, write $\mathsf{Log_p}\,\mathsf{C}$ for the set of formulæ that are valid in each Heyting algebra $\mathrm{Sub_o}\,P$, $P \in \mathsf{C}$.

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Theorem

For each integer $d \geqslant 0$,

$$\mathsf{Log_p}\,\mathsf{P}_d = \mathsf{Int} + \mathrm{BD}_d.$$

Also,

$$\mathsf{Log_p}\,\mathsf{P} = \mathsf{Int}.$$

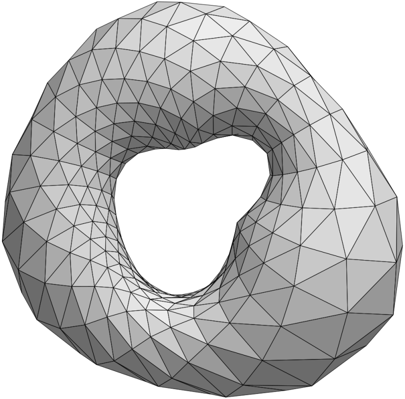
N. Bezhanishvili, V.M., A. Pedrini, D. McNeill, *Tarski's Theorem on ituitionistic logic, for polyhedra*, Annals of Pure and Applied Logic, 2018.

where the inclusion preserves the Heyting structure by Lemma 3.6. Since the equation $\alpha = \top$ fails in Up A, it also fails in the larger algebra $\mathrm{Sub_o}\, P$; equivalently, $\alpha \notin \mathsf{Log}\, P \supseteq \mathsf{Log}\, \mathsf{P}_d$, and the proof of the first statement is complete. The second statement follows easily from the first using Lemma 2.2. □

Remark 6.1. Intuitionistic logic is capable of expressing properties of polyhedra other than their dimension. To show this, let $\mathscr{P}$ consist of the class of all polyhedra that are, as topological spaces, closed (= without boundary) topological manifolds. Then $\mathsf{Log}\, \mathscr{P}$ contains intuitionistic logic properly. Indeed, it is a classical theorem that for any triangulation Σ of any d-dimensional manifold $M \in \mathscr{P}$, each $(d-1)$-simplex $\sigma \in \Sigma$ is a face of exactly two d-simplices of Σ. It follows from our results above that $\mathsf{Log}\, \mathscr{P}$ contains (all instances of) the well-known *bounded top-width axiom schema* of index 2, cf. [11, p. 112], which is refuted by intuitionistic logic. The problem of determining which intermediate logics are complete for classes of polyhedra is open; e.g., what is the logic of the class $\mathscr{P}$ of all closed triangulable manifolds? □

Question. What is the logic of all triangulable manifolds?

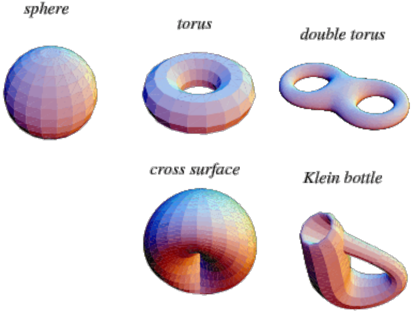
PL and combinatorial manifolds



A triangulated torus

Topological manifolds

A topological d-manifold is a second-countable Hausdorff space that is locally Euclidean of dimension d, i.e., each point has a neighbourhood homeomorphic to $\mathbb{R}^d$. (The dimension d is uniquely determined because of Brouwer's Invariance of Domain Theorem.) We only consider compact manifolds.

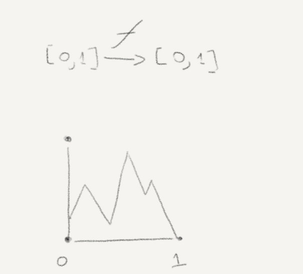


It is tempting to define a polyhedral manifold as a polyhedron that is also, as a space, a d-manifold—but that is not standard usage. Need more work.

The PL category

Any polyhedron $P \subseteq \mathbb{R}^n$ is a topological compact second-countable Hausdorff space, as a subspace of $\mathbb{R}^n$.

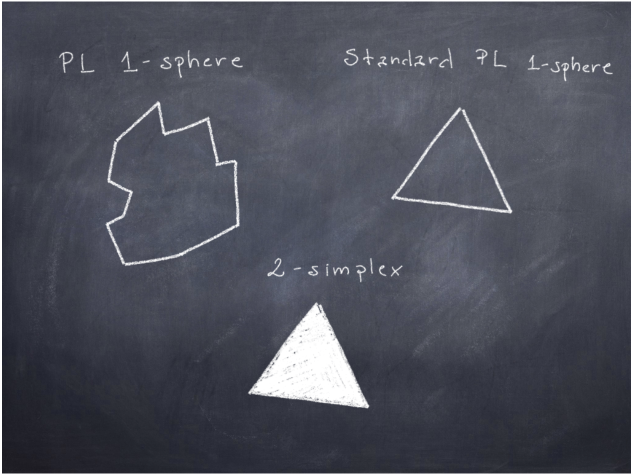
The natural morphisms between polyhedra, however, are those continuous maps that are (affine) linear in patches: the piecewise linear (PL) maps. Short definition: $f\colon P \to Q$ is PL if its graph $\Gamma(f) \subseteq P \times Q$ is a polyhedron.



This gives rise to the category PL of compact polyhedra and their PL maps. (Note to self: PL vs. TOP; your motivation.)

PL manifolds

A PL d-sphere is a polyhedron that is PL-homeomorphic to the boundary polyhedron of a $(d+1)$-simplex.



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Definition

A PL d-manifold is a polyhedron P such that each point $p \in P$ has a neighbourhood homeomorphic to $\mathbb{R}^d$ and such that its closure a PL d-ball. Write

$$\mathsf{M} \subseteq \mathsf{P}$$

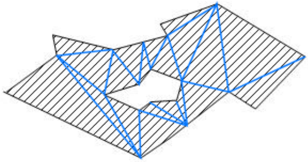
for the class of all PL manifolds, and

$$\mathsf{M_d} \subseteq \mathsf{M}$$

for the subclass of all PL d-manifolds.

(Note to Self: Possibly hint at (i) R. Edwards-J. Cannon solution to Double Suspension Conjecture, and why naive definition fails miserably, and (ii) the Hauptvermutung.)

Triangulations



A triangulated polyhedron in $\mathbb{R}^2$*.*

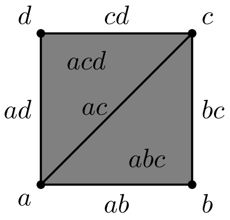
A triangulation is a finite set Σ of simplices in $\mathbb{R}^n$ such that

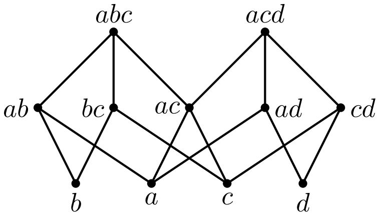
1. If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.
2. If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is either empty, or a common face of σ and τ.

The underlying polyhedron of the triangulation Σ is

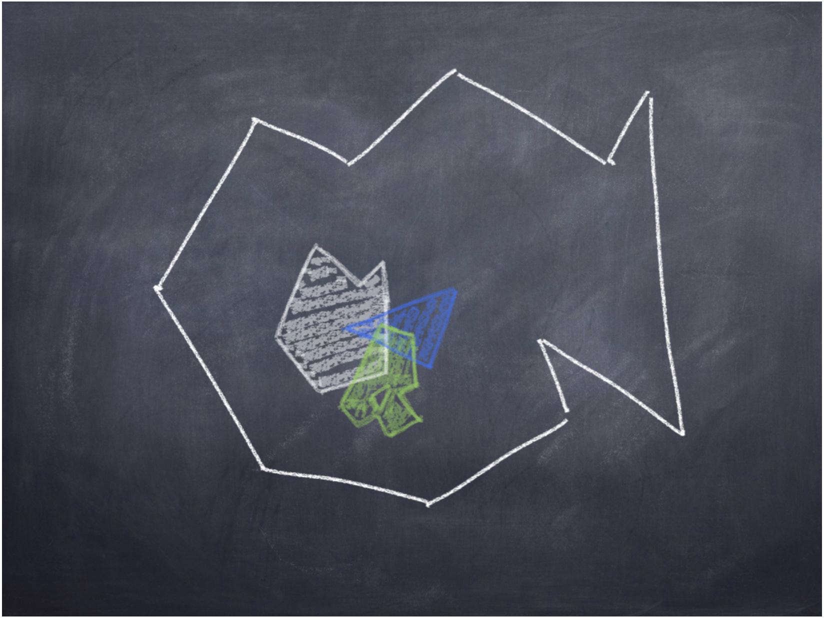
$$|\Sigma| := \bigcup \Sigma \subseteq \mathbb{R}^n.$$

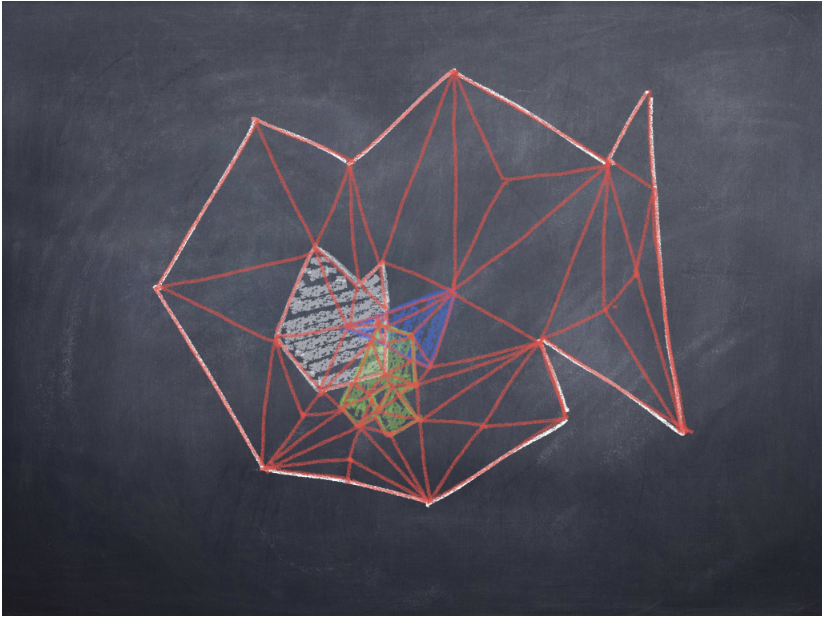
A triangulation of a polyhedron P is a triangulation Σ with $P = |\Sigma|$.





Triangulations determine finite frames: elements are simplices, order is inclusion order. Dually, triangulations of P determine finite subalgebras of $\mathrm{Sub_o}\, P$.





The Triangulation Lemma

Given finitely many polyhedra $P, P_1, \ldots, P_m$ in $\mathbb{R}^n$ with $P_i \subseteq P$ for each $i \in \{1, \ldots, m\}$, there exists a triangulation Σ of P such that, for each $i \in \{1, \ldots, m\}$, the collection

$$\Sigma_i \coloneqq \{\sigma \in \Sigma \mid \sigma \subseteq P_i\}$$

is a triangulation of P_i, i.e. $|\Sigma_i| = P_i$.

Corollary (Local Finiteness)

For any polyhedron P,

$$\mathrm{Sub_o}\, P$$

is a locally finite Heyting subalgebra of $\mathscr{O}(P)$.

Combinatorial manifolds: links and stars

Given triangulation Σ. We regard it as a poset under inclusion. Given simplex $\sigma \in \Sigma$.

The (open) star of σ is

$$\mathrm{st}\,\sigma := \uparrow\sigma.$$

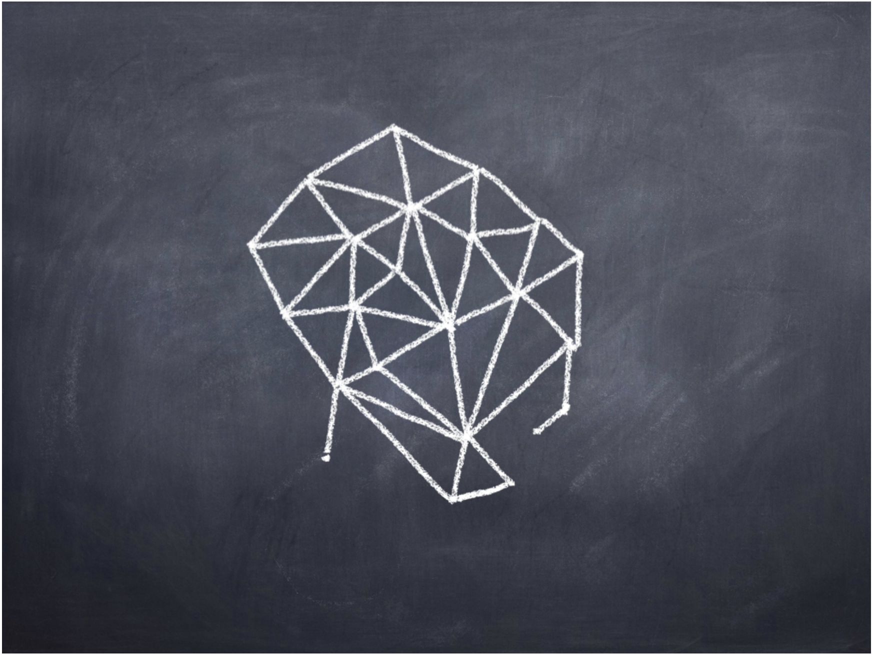
The closed star of σ is

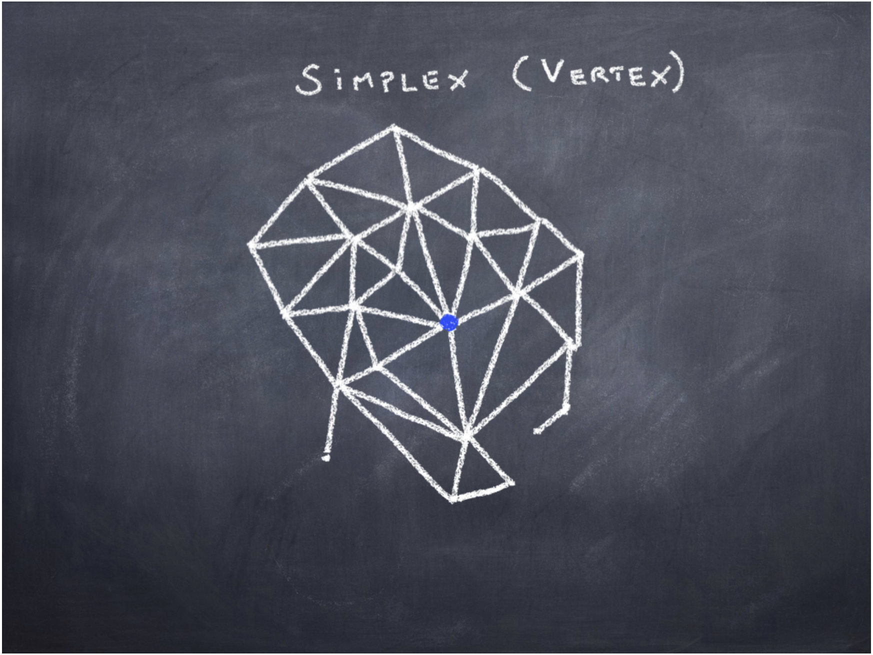
$$\overline{\mathrm{st}\,\sigma} := \downarrow\mathrm{st}\,\sigma = \downarrow\uparrow\sigma.$$

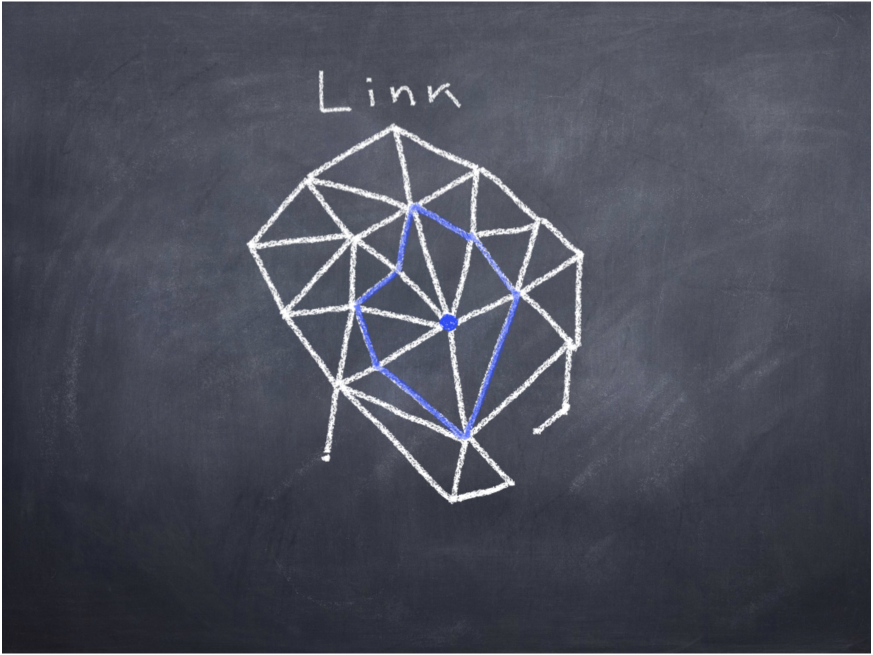
The link of σ is

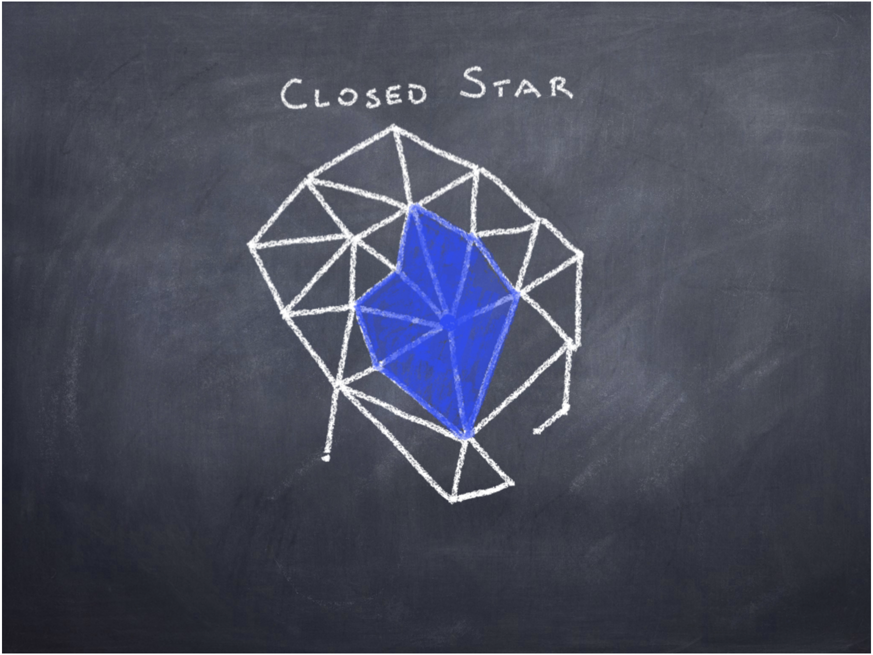
$$\mathrm{lk}\,\sigma := \overline{\mathrm{st}\,\sigma} \setminus \mathrm{st}\,\sigma.$$

Closed stars and links are (sub)triangulations. Stars are not.




SIMPLEX (VERTEX)


Link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Combinatorial manifolds

Definition

A combinatorial d-manifold is a triangulation Σ with the property that the link of each vertex $\sigma \in \Sigma$ is a PL-sphere of dimension $d-1$.

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Theorem (see e.g. J.F.P. Hudson, Piecewise-linear topology, 1969)

(*Underlying polyhedra of*) *combinatorial manifolds are exactly PL manifolds.*

Corollary

Fix any triangulation Σ of a PL manifold. (i) *The link of any simplex of Σ is a PL-sphere.* (ii) *Each simplex in Σ of codimension 1 is a face of exactly two simplices.*

At long last...

...we can prove at least one (very modest) fact about M, the logic of manifolds.

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By the Triangulation Lemma, M is the logic of the following class of finite frames:

All triangulations of each possible PL manifold.

By (ii) in previous corollary, M contains the bounded top-width formulæ of index 2: every triangulation of any PL-manifold, regarded as a poset, has the property that any co-atomic element has precisely 2 successors.

Hence: M $\subsetneq$ Int.

Note. Item (i) in the previous corollary (links in combinatorial manifolds are spheres) will be useful later.

The Alexander-Newman Stellar Theory

By the Triangulation Lemma, M is the logic of the following set of finite frames:

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This is an uncountable set. Can we find a countable set of defining finite posets for M? If yes, can we recursively enumerate it?

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We only care to distinguish triangulations up to the equivalence relation:

Σ and Δ are combinatorially equivalent, written $\Sigma \cong \Delta$, if they are isomorphic as posets.

Easy:

$$\Sigma \cong \Delta \quad \Rightarrow \quad |\Sigma| \text{ and } |\Delta| \text{ are PL-homeomorphic.}$$

The converse is clearly false.

The Alexander-Newman Stellar Theory

Theorem (J.W. Alexander; M. Newman, *circa* 1930)

Given polyhedra P and Q with triangulations Σ and Δ, respectively. Then P and Q are PL-homeomorphic if, and only if, $\Sigma \cong_\star \Delta$.

The (equivalence) relation $\cong_\star$ is defined on the class of all triangulations and extends the relation $\cong$. It is defined in terms of two elementary transformations called stellar subdivisions and stellar welds.

(At the whiteboard.)

Locality

Key Fact. Int can only tells us about local property of manifolds. The global topology does not matter.

This is because if formula α fails (=has countermodel) in a poset F, then it fails in $\uparrow x$ for some $x \in F$.

But we know from (i) in corollary that $\uparrow\sigma$ for $\sigma \in \Sigma$ is always a star whose link is a combinatorial sphere, if Σ is a combinatorial manifold.

In other words, $\uparrow\sigma \setminus \sigma$ is a triangulation of a sphere (as poset).

Proposition

The logic $\mathrm{M_d}$ *is determined by the class of rooted frames obtained from triangulations of spheres of dimension* $\leqslant d$ *by adding a fresh bottom point.*

Standard triangulation of the d-sphere

Take a $(d+1)$-simplex σ and consider the set Σ of its proper faces. This is a triangulation of the boundary $\partial\sigma$ of σ and it is called the standard triangulation of the d-sphere.

Now add to Σ (as poset) a bottom point and call the result $\Sigma_\perp$. This is not a triangulation!

By Alexander-Newman together with (ii) in the corollary, applying stellar subdivisions and welds to Σ in all possible ways, from $\Sigma_\perp$ we obtain all possible posets of the form $\uparrow\sigma$ for $\sigma \in \Sigma$ and Σ a combinatorial manifold. Call

$$\mathscr{F}_d$$

this set of frames.

Recursive enumerability of non-validities

Proposition

For each d, M_d is the logic of the set of finite frames $\mathscr{F}_d$. Moreover, $\mathscr{F}_d$ is recursively enumerable. Hence, $\mathsf{Int} \setminus \mathsf{M_d}$ is recursively enumerable. Analogous results apply to M by dove-tailing.

Remark

Medvedev's logic of finite problems is contained in M—consider the frames $\Sigma_\perp$ with Σ standard triangulation of a d-simplex; these are in $\mathscr{F} := \bigcup \mathscr{F}_d$, and define Medvedev logic. So M is not locally tabular.

(Draw at whiteboard.)

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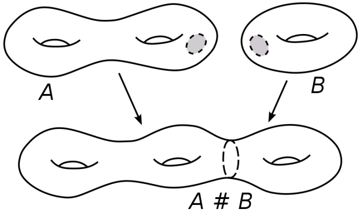
(Draw at whiteboard.)

What about decidability?

Markov's Theorem on Unrecognisability of Manifolds

1. A.A. Markov, *On unsolvability of certain problems in topology* (Russian), 1958. Also: *Unsolvability of the homeomorphy problem*, Proc. ICM, 1960.
2. W.W. Boone, W. Haken, and V. Poénaru, *On the recursively unsolvable problems in topology and their classification*, Contributions to Mathematical Logic (Logic Colloquium), 1968.
3. S. P. Novikov, Unrecognisability of the 4-sphere (Russian), in: I.A. Volodin, V.E. Kuznetsov, A.T. Fomenko, *On the recognition problem of the standard 3-sphere*, 1977.
4. M.A. Shtan'ko, *A theorem of A. A. Markov and algorithmically unrecognisable combinatorial manifolds*, Izv. Math., 2006.
5. A.V. Chernavski and V.P. Leskine, *Unrecognisability of manifolds*, APAL 2006.

Markov-Shtan'ko: For each $k \geqslant 4$, the connected sum of 14 copies of $\mathbb{S}^2 \times \mathbb{S}^{k-2}$ is algorithmically unrecognizable amongst combinatorial manifolds.



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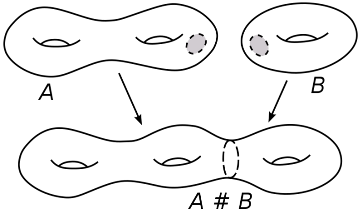

Markov-Novikov: For each $d \geqslant 5$, the d-sphere is algorithmically unrecognizable amongst polyhedra. (Depends on Smale's generalised Poincaré conjecture, Ann. Math., 1961.)

The set $\mathscr{F}_d$ is not recursive for $d \geqslant 6$.

DECISION PROBLEM: Recognising frames for M_d.
INPUT: A finite poset F with bottom $\bot$.
OUTPUT: YES if $F \in \mathscr{F}_d$, NO otherwise.

By our definition of $\mathscr{F}_d$,

$F \setminus \{\bot\}$ is a triangulation of the $(d-1)$-sphere if, and only if, $F \in \mathscr{F}$.

Thus, if we could algorithmically solve this decision problem, then for $d = 6$ we would be able to decide whether any given triangulation is a triangulation of the 5-sphere. This contradicts Markov-Novikov. The problem is undecidable.

Question. Does this tell us anything useful about the logic M of manifolds?

Thank you for your attention!