# one modal logic to rule them all?
# one binder to scan all worlds!

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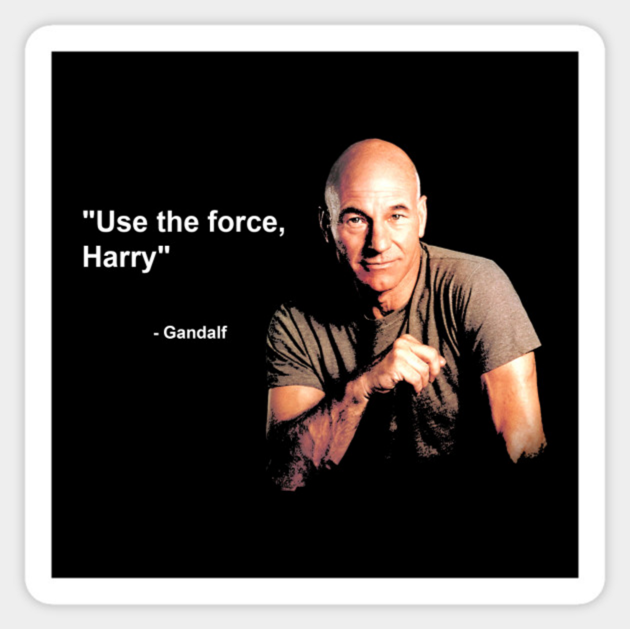
Apologies to participants of forthcoming AiML 2018

Informatik 8
FAU Erlangen-Nürnberg


"Use the force,
Harry"
- Gandalf

# Two problems to solve

- The proliferation of modal "logics"
- The riddle of propositional quantification

# The modal proliferation crisis

- Consider ordinary Kripke semantics
- Each condition on frames—a different "logic"?
  - K: the minimal normal logic
  - D ($\Diamond\top$): non-termination
  - T ($\Box p \to p$): reflexivity
  - K4 ($\Box p \to \Box\Box p$): transitivity
  - S4 (K4 + T): quasiorders
  - S5 (S4 + $p \to \Box\Diamond p$): equivalence relations ...
- By contrast, just one first-order logic (FOL) allowing varying theories!
- Modal logicians are less happy about it than you may think

*[T]hese systems are not "different modal logics", but different special theories of particular kinds of accessibility relation. We do not speak of "different first-order logics" when we vary the underlying model class. There is no good reason for that here, either.*

J. van Benthem, *Modal Logics for Open Minds*

*Another suggestion is that the great proliferation of modal logics is an epidemy from which modal logic ought to be cured.*

R. A. Bull and K. Segerberg, *Basic Modal Logic, HPL*

(in the context of Gentzen systems: some have suggested to keep only those modal logics which allow a natural Natural Deduction calculus . . . )

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  - ⁕ Over state-based semantics or lattice-complete algebras, computationally as bad as ordinary SO logic
  - ⁕ Which axioms? Which rules? Can be nonconservative over a syntactically defined “logic”!

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# The riddle of propositional quantification

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  - ⋇ Over state-based semantics or lattice-complete algebras, computationally as bad as ordinary SO logic
  - ⋇ Which axioms? Which rules? Can be nonconservative over a syntactically defined "logic"!

    Unless you have a semantic result: completeness wrt, e.g., lattice-complete BAEs!
  - ⋇ Even for standard "logics", can yield undesirable principles. See Kaplan's paradox in our paper.

And yet, modal logic twinned with propositional quantification since birth ...

> *[I]t is only through such principles [such as $\exists p(\Diamond p \wedge \Diamond\neg p)$] that the outlines of a logical system can be positively delineated.*
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...okay, at least a trickle : Kripke, Bull, Fine, Kaplan ...

*But very little attention has been paid to second-order modal logic. I predict that it will play an increasingly central role as the framework for many debates in metaphysics and other areas of philosophy, and that this aspect of the 1947 paper will turn out to have been more than sixty years ahead of its time.*

T. Williamson, *Laudatio for R. Barcan Marcus*

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- Other definable global quantificational modalities (GQMs):

$$\langle \exists p \rangle \varphi := \neg [\forall p] \neg \varphi \ \text{“=”} \ \ \exists p \mathsf{E} \varphi$$

$$[\exists p] \varphi := \langle \exists p \rangle \mathsf{A} \varphi \ \text{“=”} \ \ \exists p \mathsf{A} \varphi$$

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$$\langle\forall p\rangle\varphi := \neg[\exists p]\neg\varphi \ \text{“=”} \quad \forall p\mathsf{E}\varphi.$$

- Note this is the most compact syntax:

$$\mathcal{L}_{\mathrm{GQM}} \qquad \varphi ::= p \mid (\varphi \to \varphi) \mid \Box\varphi \mid [\forall p]\varphi,$$

  as $\bot$ can be defined as $[\forall p]p$.

Definition

A Boolean algebra expansion (BAE) is a tuple $\mathfrak{A} = \langle A, \neg, \wedge, \bot, \top, \Box \rangle$ where $\langle A, \neg, \wedge, \bot, \top \rangle$ is a Boolean algebra and $\Box : A \to A$.

Definition

1. A $\mathcal{C}$-BAE (resp. $\mathcal{A}$-BAE) is a BAE whose Boolean reduct is lattice-complete (resp. atomic).
2. A BAO (Boolean Algebra with a (dual) Operator) is a BAE with a normal $\Box$, i.e., $\Box$ distributes over all finite meets.
3. A $\mathcal{V}$-BAO is a BAO where $\Box$ distributes over all existing meets.

   Recall our surprising discovery this property is actually FO-definable. Some use made in this paper too.

## Definition (Algebraic Semantics of GQM)

A valuation $\theta : \mathsf{Prop} \to \mathfrak{A}$ extends to a function $\tilde{\theta} : \mathcal{L}_{\mathrm{GQM}} \to \mathfrak{A}$ as follows:

$$\tilde{\theta}(p) := \theta(p) \qquad \tilde{\theta}(\neg\varphi) := \neg\tilde{\theta}(\varphi)$$

$$\tilde{\theta}(\varphi \wedge \psi) := \tilde{\theta}(\varphi) \wedge \tilde{\theta}(\psi) \qquad \tilde{\theta}(\Box\varphi) := \Box\tilde{\theta}(\varphi)$$

$$\tilde{\theta}([\forall p]\varphi) := \begin{cases} \top & \text{if } \tilde{\gamma}(\varphi) = \top \text{ for all valuations } \gamma \sim_p \theta \\ \bot & \text{otherwise} \end{cases}$$

where $\gamma \sim_p \theta$ denotes that $\gamma$ and $\theta$ disagree at most at $p$.

A formula $\varphi$ is valid in $\mathfrak{A}$ iff for every valuation $\theta$ on $\mathfrak{A}$, $\tilde{\theta}(\varphi) = \top$. Let $\vDash_{\mathsf{GQM}} \varphi$ iff $\varphi$ is valid in all BAEs, in which case $\varphi$ is simply valid.

Lemma (Semantics of Derived Connectives)
*For any valuation $\theta$ on a* BAE $\mathfrak{A}$*:*

$$\tilde{\theta}(\mathsf{A}\varphi) = \begin{cases} \top & \textit{if } \tilde{\theta}(\varphi) = \top \\ \bot & \textit{otherwise} \end{cases} \qquad \tilde{\theta}(\mathsf{E}\varphi) = \begin{cases} \top & \textit{if } \tilde{\theta}(\varphi) \neq \bot \\ \bot & \textit{otherwise} \end{cases}$$

$$\tilde{\theta}(\langle\exists p\rangle\varphi) = \begin{cases} \top & \textit{if } \exists\gamma \sim_p \theta.\tilde{\gamma}(\varphi) \neq \bot \\ \bot & \textit{otherwise} \end{cases} \qquad \tilde{\theta}([\exists p]\varphi) = \begin{cases} \top & \textit{if } \exists\gamma \sim_p \theta.\tilde{\gamma}(\varphi) = \top \\ \bot & \textit{otherwise} \end{cases}$$

$$\tilde{\theta}(\langle\forall p\rangle\varphi) = \begin{cases} \top & \textit{if } \forall\gamma \sim_p \theta.\tilde{\gamma}(\varphi) \neq \bot \\ \bot & \textit{otherwise} \end{cases}.$$

Several definitions of semantic consequence are available, but we go for an algebraic analogue of global model consequence:

Definition

Given $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathrm{GQM}}$, let $\Gamma \vDash^{\mathsf{A}}_{\mathrm{GQM}} \varphi$ iff for any BAE $\mathfrak{A}$ and $\theta : \mathsf{Prop} \to \mathfrak{A}$, if $\tilde{\theta}(\gamma) = \top$ for each $\gamma \in \Gamma$, then $\tilde{\theta}(\varphi) = \top$.

We need now a proof system complete with respect to $\vDash^{\mathsf{A}}_{\mathrm{GQM}}$.

Theorem (Semantic Deduction)

*For any formulas* $\varphi_1, \ldots, \varphi_n, \psi \in \mathcal{L}_{\mathsf{GQM}}$ $\{\varphi_1, \ldots, \varphi_n\} \vDash^{\mathsf{A}}_{\mathrm{GQM}} \psi$ *iff* $\vDash_{\mathsf{GQM}} \mathsf{A}(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \mathsf{A}\psi$.

Definition (Notions of Equivalence)

For any $\varphi, \psi \in \mathcal{L}_{\mathrm{GQM}}$ and class $\mathcal{K}$ of BAEs:

1. $\varphi$ and $\psi$ are equivalent over $\mathcal{K}$ iff for every $\mathfrak{A} \in \mathcal{K}$ and valuation $\theta$ on $\mathfrak{A}$, $\tilde{\theta}(\varphi) = \tilde{\theta}(\psi)$ (or equivalently, $\varphi \leftrightarrow \psi$ is valid in $\mathfrak{A}$);
2. $\varphi$ and $\psi$ are globally equivalent over $\mathcal{K}$ iff for every $\mathfrak{A} \in \mathcal{K}$ and valuation $\theta$ on $\mathfrak{A}$, $\tilde{\theta}(\varphi) = \top$ iff $\tilde{\theta}(\psi) = \top$ (or equivalently, $\mathsf{A}\varphi \leftrightarrow \mathsf{A}\psi$ is valid in $\mathfrak{A}$).
3. $\varphi$ and $\psi$ are equivalent (resp. globally equivalent) iff they are equivalent (resp. globally equivalent) over the class of all BAEs.

Since $\mathcal{L}_{\mathrm{GQM}}$ can be interpreted in arbitrary BAEs, it can be interpreted in any frames that give rise to BAEs, e.g.:

- Kripke frames (corresponding to $\mathcal{CAV}$-BAOs);
- relational possibility frames (corresponding to $\mathcal{CV}$-BAOs);
- neighborhood frames (corresponding to $\mathcal{CA}$-BAEs);
- neighborhood possibility frames (corresponding to $\mathcal{C}$-BAEs);
- discrete general frames (corresponding to $\mathcal{AV}$-BAOs);
- discrete general neighborhood frames (corresponding to $\mathcal{A}$-BAEs);
- general neighborhood frames (corresponding to BAEs).

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The logic GQM is the smallest set of formulas containing the axioms from groups 1, 2 and 3 + closed under the rules from group 4 below.

**1. propositional axioms**

- all classical propositional tautologies.

**2. axioms for $[\forall p]$**

- distribution: $[\forall p](\varphi \to \psi) \to ([\forall p]\varphi \to [\forall p]\psi)$;
- instantiation: $[\forall p]\varphi \to \varphi^p_\psi$ where $\psi$ is substitutable for $p$ in $\varphi$;
- global instantiation: $[\forall p]\varphi \to [\forall r]\varphi^p_\psi$ where $\psi$ is substitutable for $p$ in $\varphi$ and $r$ is not free in $\varphi^p_\psi$;
- quantificational 5 axiom: $\neg[\forall p]\varphi \to [\forall r]\neg[\forall p]\varphi$ where $r$ is not free in $[\forall p]\varphi$.

## 3. axioms binding $[\forall p]$ and $\Box$

- $\Box$-congruence: $[\forall p](\varphi \leftrightarrow \psi) \to (\Box\varphi \leftrightarrow \Box\psi)$.

## 4. rules

- modus ponens: if $\vdash_{\mathsf{GQM}} \varphi$ and $\vdash_{\mathsf{GQM}} \varphi \to \psi$, then $\vdash \psi$;
- $[\forall p]$-necessitation: if $\vdash_{\mathsf{GQM}} \varphi$, then $\vdash_{\mathsf{GQM}} [\forall p]\varphi$;
- universal generalization: if $\vdash_{\mathsf{GQM}} \alpha \to [\forall p]\varphi$ and $q$ is not free in $\alpha$, then $\vdash_{\mathsf{GQM}} \ \alpha \to \ [\forall q][\forall p]\varphi$.

Here '$\vdash_{\mathsf{GQM}} \varphi$' means $\varphi \in \mathsf{GQM}$. We write '$\vdash \varphi$' when no confusion will arise.

## Lemma (Provable Formulas)

1. $\vdash \mathsf{A}(\varphi \to \psi) \to (\mathsf{A}\varphi \to \mathsf{A}\psi)$;
2. $\vdash \mathsf{G}_*(\varphi * \psi) \leftrightarrow (\mathsf{G}_*\varphi * \mathsf{G}_*\psi)$;
3. *if* $\vdash \varphi \to \psi$, *then* $\vdash \mathsf{G}\varphi \to \mathsf{G}\psi$;
4. $\vdash \mathsf{A}\varphi \to \varphi$;
5. $\vdash \varphi \to \mathsf{E}\varphi$;
6. $\vdash \mathsf{E}\varphi \leftrightarrow \mathsf{AE}\varphi$;
7. $\vdash \mathsf{EA}\varphi \leftrightarrow \mathsf{A}\varphi$;
8. $\vdash \mathsf{GG}\varphi \leftrightarrow \mathsf{G}\varphi$;
9. $\vdash \llbracket Qp \rrbracket \mathsf{A}\psi \leftrightarrow [Qp]\psi$;
10. $\vdash \llbracket Qp \rrbracket \mathsf{E}\psi \leftrightarrow \langle Qp \rangle\psi$;
11. $\vdash \llbracket Qp \rrbracket \psi \leftrightarrow \mathsf{A}\llbracket Qp \rrbracket\psi$;
12. $\vdash \llbracket Qp \rrbracket \psi \leftrightarrow \mathsf{E}\llbracket Qp \rrbracket\psi$.

In this statement: for $* \in \{\wedge, \vee\}$, let $\mathsf{G}_*$ be $\mathsf{A}$ if $* = \wedge$ and $\mathsf{E}$ otherwise.

Definition (Global Syntactic Consequence)

Given $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathrm{GQM}}$, let $\Gamma \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi$

iff

$\varphi$ belongs to the smallest set $\Lambda$ of GQM formulas that includes $\Gamma \cup \mathsf{GQM}$ and is closed under modus ponens and $\mathsf{A}$-necessitation: if $\psi \in \Lambda$, then $\mathsf{A}\psi \in \Lambda$.

Theorem (Syntactic Deduction)

*For any formulas* $\varphi_1, \dots, \varphi_n, \psi \in \mathcal{L}_{\mathsf{GQM}}$*:* $\{\varphi_1, \dots, \varphi_n\} \vdash^{\mathsf{A}}_{\mathsf{GQM}} \psi$ *iff* $\vdash_{\mathsf{GQM}} \mathsf{A}(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \mathsf{A}\psi$*.*

Theorem (Soundness)
*For* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathrm{GQM}}$, $\Gamma \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi$ *implies* $\Gamma \vDash^{\mathsf{A}}_{\mathrm{GQM}} \varphi$.

Proof.
Straightforward induction. □

Completeness seems a natural next step. But first, let us cross out an earlier item from our list.

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Let $\mathcal{L}_{\Box}$ ($\mathcal{L}_{\Box\mathsf{A}}$) be the set of GQM formulas in which no GQMs other than $\bot$ (no GQMs other than $\mathsf{A}$, $\mathsf{E}$ and $\bot$) appear.

A congruential modal logic is a set $\mathsf{L} \subseteq \mathcal{L}_{\Box}$

- containing all propositional tautologies and
- closed under uniform substitution, modus ponens, and
- the rule that if $\varphi \leftrightarrow \psi \in \mathsf{L}$, then $\Box\varphi \leftrightarrow \Box\psi \in \mathsf{L}$.

Let $\mathsf{GQM}\text{-}\mathsf{L}$ be the smallest set of formulas that includes $\mathsf{GQM} \cup \mathsf{L}$ and is closed under all three rules of $\mathsf{GQM}$.

Theorem (Conservativity)

*For any* $\varphi \in \mathcal{L}_\Box$, $\varphi \in \mathsf{GQM\text{-}L}$ *iff* $\varphi \in \mathsf{L}$.

Proof.
The Lindenbaum-Tarski algebra for $\mathsf{L}$ is a BAE in which every $\varphi \in \mathsf{GQM\text{-}L}$ is valid and in which any $\mathcal{L}_\Box$ formula not in $\mathsf{L}$ can be refuted. $\Box$

A set $\Sigma \subseteq \mathcal{L}_\Box$ axiomatizes a congruential modal logic $\mathsf{L}$ iff $\mathsf{L}$ is the smallest congruential modal logic such that $\Sigma \subseteq \mathsf{L}$.

Theorem (Modal Monism)
*If $\Sigma$ axiomatizes $\mathsf{L}$, then we have the following equivalence:*
*$\varphi \in \mathsf{L}$ iff there are $\psi_1, \ldots, \psi_n \in \Sigma$ such that $\vdash_{\mathsf{GQM}} [\forall \vec{p}](\psi_1 \wedge \cdots \wedge \psi_n) \rightarrow \varphi$, where $\vec{p}$ is the tuple of variables occurring in $\psi_1, \ldots, \psi_n$.*

We can easily rephrase this Theorem in the language of "theories."

Definition

A $\vdash_{\mathsf{GQM}}$-theory is a set of GQM formulas that includes $\mathsf{GQM}$ and is closed under modus ponens.

Corollary (Logics as Theories)

*If $\Sigma \subseteq \mathcal{L}_{\Box}$ axiomatizes a congruential modal logic $\mathsf{L}$, then we have the following equivalence: $\varphi \in \mathsf{L}$ iff $\varphi$ belongs to the smallest $\vdash_{\mathsf{GQM}}$-theory that includes $[\forall]\Sigma = \{[\forall\vec{p}]\varphi \mid \varphi \in \Sigma$ and $\vec{p}$ are the variables in $\varphi\}$.*

Given this reduction of modal logics to $\vdash_{\mathsf{GQM}}$-theories, we have the following.

Corollary

GQM *theoremhood is undecidable.*

Proof.
In the light of the above Theorem, a decision procedure for GQM would yield a decision procedure for every finitely axiomatizable modal logic. But there are undecidable logics with finite axiomatizations. $\square$

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  More broadly: "internalization" of modal metatheory $\Longleftarrow$ more to do . . .
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  first-order theory of BAEs with a unary discriminator, i.e., the global modality $\mathsf{A}$
- Cannibalize FO completeness!

- A formula is in pure weak prenex form (PWP) iff it is of the form $\{\vec{Q}p\}\mathsf{G}\varphi$ where
  - ⁕ $\{\vec{Q}p\}$ is a sequence of $[\forall p_i]$ and $\langle \exists p_i \rangle$ GQMs only,
  - ⁕ $\mathsf{G}$ is either $\mathsf{A}$ or $\mathsf{E}$,
  - ⁕ and $\varphi$ is a $\mathcal{L}_{\Box\mathsf{A}}$-formula.
- As stated, every formula of the form $\mathsf{A}\phi$ is equivalent to one in PWP
- We have a normal form working for arbitrary GQM formulas like CNFWP, but it is too much for this talk

  conjunction of normal clauses involving as disjuncts nontrivial weak prenex form (NWP), literals or boxed/diamonded modal formulas

Definition

A Boolean algebra expansion with a discriminator ($\mathrm{BAE}_{\mathsf{A}}$) is a tuple $\mathfrak{A} = \langle A, \neg, \wedge, \bot, \top, \Box, \mathsf{A}\rangle$ where $\langle A, \neg, \wedge, \bot, \top, \Box\rangle$ is a BAE and $\mathsf{A}$ is the dual form of the unary discriminator term (Jipsen 1993), i.e., an algebraic counterpart of the global modality:

$$\mathsf{A}a = \top \text{ if } a = \top, \text{ and } \mathsf{A}a = \bot \text{ otherwise.}$$

$\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$ (resp. $\mathrm{FO}_{\mathrm{BAE}}$) is the set of first-order formulas in the $\mathrm{BAE}_{\mathsf{A}}$ (resp. BAE) signature

Recycling $\mathsf{Prop}$ for our set of first-order variables

- The class of all $\mathrm{BAE}_{\mathsf{A}}$s is elementary, although not exactly a variety (an equationally definable class)

  rather, it is the class of all *simple* members of the corresponding variety

  (Jipsen 1993)

  also, we need to focus on nontrivial ones, i.e., those where $\top \neq \bot$
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- BAEs and $\mathrm{BAE}_{\mathsf{A}}$s are in 1-1 correspondence:
  - $\mathrm{BAE}_{\mathsf{A}}$s have BAEs as reducts;
  - every BAE $\mathfrak{A}$ can be trivially extended to a $\mathrm{BAE}_{\mathsf{A}}$ $\mathfrak{A}_{\mathsf{A}}$;

  and both operations are mutual inverses.

# Enderton-style axioms for $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$

- all substitution instances of propositional tautologies;
- $\forall p\varphi \to \varphi^p_t$ where the term $t$ is substitutable for $p$ in $\varphi$;
- $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$;
- $\varphi \to \forall p\varphi$ where $p$ does not occur free in $\varphi$;
- $p \approx p$, and $p \approx q \to (\varphi \to \varphi')$ where $\varphi$ is atomic (i.e., equality) and $\varphi'$ is obtained from $\varphi$ by replacing $p$ in zero or more places by $q$;
- first-order axioms of Boolean algebras;
- $\top \not\approx \bot$;
- $\forall p((p \approx \top \,\&\, \mathsf{A}p \approx \top) \,\mathrm{OR}\, (p \not\approx \top \,\&\, \mathsf{A}p \approx \bot))$.

Every formula of $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$ equivalent to a PWP formula:

where $\sim$ and & are the negation and conjunction connectives in the first-order language, whereas $\neg$ and $\wedge$ in the first-order language are function symbols for the Boolean algebraic operations

$$
\begin{aligned}
(\varphi \approx \psi)_* &:= \mathsf{A}(\varphi \leftrightarrow \psi) \qquad & (\sim\alpha)_* &:= \neg(\alpha)_* \\
(\alpha \mathbin{\&} \beta)_* &:= ((\alpha)_* \wedge (\beta)_*) \qquad & (\forall p \alpha)_* &:= [\forall p](\alpha)_*.
\end{aligned}
$$

Note that the *terms* in the $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$ formula become formulas of $\mathcal{L}_{\mathrm{GQM}}$, with the Boolean function symbols becoming propositional connectives.

In the reverse direction, define for each PWP formula:

$$(\mathsf{A}\varphi)^* := \varphi \approx \top \qquad (\mathsf{E}\varphi)^* := \varphi \not\approx \bot$$
$$([\forall p]\varphi)^* := \forall p(\varphi)^* \qquad (\langle \exists p \rangle \varphi)^* := \exists p(\varphi)^*.$$

Any $\mathsf{A}$ or $\mathsf{E}$ GQMs inside $\varphi$ become function symbols in the $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$ translation.

Lemma (Faithfulness of Translation of $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$)

*For any nontrivial* BAE $\mathfrak{A}$, $\theta : \mathsf{Prop} \to \mathfrak{A}$, *and* $\alpha \in \mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$*:*

$$\mathfrak{A}, \theta \vDash \alpha \textit{ iff } \tilde{\theta}((\alpha)_*) = \top \qquad \textit{and} \qquad \mathfrak{A}, \theta \nvDash \alpha \textit{ iff } \tilde{\theta}((\alpha)_*) = \bot.$$

## Theorem (PWP Equivalence of Consequences)

1. *For any PWP formula* $\varphi \in \mathcal{L}_{\mathrm{GQM}}$, $\varphi \dashv\vdash^{\mathsf{A}}_{\mathsf{GQM}} ((\varphi)^*)_*$.
2. *For any* $\Delta \cup \{\alpha\} \subseteq \mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$, $\Delta \vdash_{\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}} \alpha$ *iff* $(\Delta)_* \vdash^{\mathsf{A}}_{\mathsf{GQM}} (\alpha)_*$.

## Corollary (Cannibalizing $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$-Completeness)

1. *For any* $\Delta \cup \{\alpha\} \subseteq \mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$, $\Delta \vDash_{\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}} \alpha$ *iff* $(\Delta)_* \vdash^{\mathsf{A}}_{\mathsf{GQM}} (\alpha)_*$.
2. *For any set of PWP fomulas* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathrm{GQM}}$, $\Gamma \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi$ *iff* $(\Gamma)^* \vDash_{\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}} (\varphi)^*$.

Similar results used in AAL to show equivalences of closure operators

Theorem (Completeness of GQM)

*For any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathrm{GQM}}$,

$$\Gamma \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi \textit{ iff } \Gamma \vDash^{\mathsf{A}}_{\mathrm{GQM}} \varphi.$$

- Note that the transformation to PWP involves a blowup
- Hence, GQM is more succinct that $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$

  We still need a formal proof of succintness though
- And it's only the global consequence anyway, local consequence more GQM-specific

# That precioussssss thing we're after

- A quantifier ...? modality ...? ... binder? ✓
- Should be interpretable over any semantics, including any algebra ✓
- Consequently, should reduce all modal "logics" to *theories* over some reasonable minimal system ✓

  More broadly: "internalization" of modal metatheory $\Longleftarrow$ more to do ...
- Should have viable proof theory/theoremhood problem ✓

  Decidability can be too much to ask, but at least r.e. should obtain
- Should yield some insight on *ordinary* propositional quantification

# Second-order propositional modal logic

$$\mathcal{L}_{\mathrm{SOPML}} \qquad \varphi ::= p \mid (\varphi \to \varphi) \mid \Box\varphi \mid \forall p \varphi,$$
$$\mathcal{L}_{\mathrm{SOPML}_\mathsf{A}} \qquad \varphi ::= p \mid (\varphi \to \varphi) \mid \Box\varphi \mid \mathsf{A}\varphi \mid \forall p \varphi,$$

- The second language, as stated at the beginning, encodes GQM.
- We have seen that over arbitrary BAEs, $\mathsf{GQM}$ is globally equivalent to $\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}$
- Now we'll add another equivalence: over lattice-complete BAEs, there is a global equivalence between $\mathsf{GQM}$ and $\mathcal{L}_{\mathrm{SOPML}_\mathsf{A}}$

Definition (Algebraic Semantics of SOPML)

We extend a valuation $\theta$ on a $\mathcal{C}$-BAE $\mathfrak{A}$ to a valuation $\tilde{\theta} : \mathcal{L}_{\mathrm{SOPML}_{\mathsf{A}}} \to \mathfrak{A}$ using the standard clauses for $\neg$, $\wedge$, and $\Box$ plus:

$$\tilde{\theta}(\forall p \varphi) = \bigwedge \{\tilde{\gamma}(\varphi) \mid \gamma \sim_p \theta\} \qquad \tilde{\theta}(\mathsf{A}\varphi) = \begin{cases} \top & \text{if } \tilde{\theta}(\varphi) = \top \\ \bot & \text{otherwise.} \end{cases}$$

Dually, $\exists p \varphi$ is interpreted using the join. The definitions of local and global equivalence transfer in the obvious way to $\mathcal{L}_{\mathrm{SOPML}}$ and $\mathcal{L}_{\mathrm{SOPML}_{\mathsf{A}}}$.

Balder ten Cate has shown that over $\mathcal{CAV}$-BAOs, every $\mathcal{L}_{\text{SOPML}}$ formula is equivalent to a prenex one, i.e., a formula of the form $Q_1 p_1 \dots Q_n p_n \varphi$ where $Q_i \in \{\forall, \exists\}$ and $\varphi$ is quantifier-free. In fact, the following more general result holds.

Theorem (Prenex Normal Form for SOPML)

1. *Over* $\mathcal{CV}$-BAO*s, every* SOPML *formula is equivalent to a prenex* SOPML *formula.*
2. *Over* $\mathcal{C}$-BAEs, *every* $\text{SOPML}_{\mathsf{A}}$ *formula is equivalent to a prenex* $\text{SOPML}_{\mathsf{A}}$ *formula.*

Theorem (SOPML to GQM)

- *If $\alpha$ is a prenex* $\mathrm{SOPML}_{\mathsf{A}}$ *formula, then* $\mathsf{A}\alpha$ *is equivalent over* $\mathcal{C}$-BAEs *to a GQM formula.*
- *Every* $SOPML_{\mathsf{A}}$ *formula is globally equivalent over* $\mathcal{C}$-BAEs *to a GQM formula.*

Corollary ($\mathcal{C}$-r.e. Disaster)

*The set of GQM formulas valid over any class of* $\mathcal{C}$-BAEs *containing the class of* $\mathcal{CAV}$-BAO*s satisfying* $\mathsf{S4.2}$ *is not recursively enumerable.*

Proof.

Using an old result by Fine. □

See the paper for an analysis of “Kaplan’s paradox” of propositional quantification in our setting

# Bonus Track: Coq Formalization

- Developed by my student Michael Sammler
- Code available at
  `https://gitlab.cs.fau.de/lo22tobe/GQM-Coq`
- more than 4200 lines of Coq code in 19 files for 15 pages of proofs
- formalization uncovered the following:
  - 10 typos and easy to fix mistakes (e.g. "proof for (vii) is by (vii)")
  - 5 incorrect steps in proofs
  - 1 definition, which needed to be adjusted
- Covers four of the five points on our precioussss list

# Todos

- Succinctness analysis
- More on “internalization” of modal metatheory
- Gentzen-style systems

  recall the origin of the suggestion Bull&Segerberg’s
- Non-classical propositional bases

  Excluded middle can be written as an axiom
- Bisimulation/uniform interpolation GQMs

Proofs for the SOPML part

For the first part, pulling the quantifier out of $\Diamond\forall p\varphi$ is the interesting bit.

For any normal $\Box$, we have the following equivalence:

$$\Diamond\psi \Leftrightarrow \exists q(\Diamond q \wedge \Box(q \to \psi))$$

for a fresh $q$. Thus, we have the following equivalences where $q\#p, \varphi$:

$$\begin{aligned}
\Diamond\forall p\varphi &\Leftrightarrow \exists q(\Diamond q \wedge \Box(q \to \forall p\varphi)) \text{ setting } \psi := \forall p\varphi \\
&\Leftrightarrow \exists q(\Diamond q \wedge \Box\forall p(q \to \varphi)) \text{ because } q \neq p \\
&\Leftrightarrow \exists q(\Diamond q \wedge \forall p\Box(q \to \varphi)) \text{ by } \mathcal{V} \text{ for } \Box \\
&\Leftrightarrow \exists q\forall p(\Diamond q \wedge \Box(q \to \varphi)) \text{ because } q \neq p.
\end{aligned}$$

For the second part, we have $(\star)$: $\mathsf{A}\exists p\psi$ is equivalent to $\forall q\exists p(\mathsf{E}q \to \mathsf{E}(q \wedge \psi))$.

In any $\mathcal{C}$-BAE, we also have the following equivalence:

$$\Diamond\psi \Leftrightarrow \exists q(\Diamond q \wedge \mathsf{A}(q \leftrightarrow \psi))$$

for $q\#\psi$. Thus, we have the following equivalences where $q\#p,\varphi$:

$$\begin{aligned}
\Diamond\forall p\varphi &\Leftrightarrow \exists q(\Diamond q \wedge \mathsf{A}(q \leftrightarrow \forall p\varphi)) \text{ setting } \psi := \forall p\varphi \\
&\Leftrightarrow \exists q(\Diamond q \wedge \mathsf{A}(\forall p(q \to \varphi) \wedge \exists p(\varphi \to q))) \text{ because } q \neq p \\
&\Leftrightarrow \exists q(\Diamond q \wedge \mathsf{A}(\forall p(q \to \varphi) \wedge \exists r(\varphi^p_r \to q))) \text{ for a fresh } r \\
&\Leftrightarrow \exists q(\Diamond q \wedge \mathsf{A}\forall p\exists r\underbrace{((q \to \varphi) \wedge (\varphi^p_r \to q))}_{\alpha}) \text{ because } r \neq p \\
&\Leftrightarrow \exists q(\Diamond q \wedge \forall p\mathsf{A}\exists r\alpha) \text{ by } \mathcal{V} \text{ for } \mathsf{A} \\
&\Leftrightarrow \exists q(\Diamond q \wedge \forall p\forall q'\exists r(\mathsf{E}q' \to \mathsf{E}(q' \wedge \alpha))) \text{ by } (\star) \text{ where } q' \text{ is fresh} \\
&\Leftrightarrow \exists q\forall p\forall q'\exists r(\Diamond q \wedge (\mathsf{E}q' \to \mathsf{E}(q' \wedge \alpha))) \text{ because } q \neq p, q \neq q', q \neq r.
\end{aligned}$$

Still more lemmas ...

Lemma

*The following are valid in all* $\mathcal{C}$-BAEs*:*

1. $\mathsf{A}\forall p\psi \leftrightarrow \forall p\mathsf{A}\psi$;
2. $\mathsf{A}\exists p\psi \leftrightarrow \forall q\mathsf{A}(\mathsf{E}q \rightarrow \exists r\mathsf{A}(\mathsf{E}r \wedge (r \rightarrow q) \wedge \exists p\mathsf{A}(r \rightarrow \psi)))$ *where* $q$ *and* $r$ *do not occur in* $\psi$.

...and with this, we have the main result ...