Neighbourhood and topological completeness for quantified pretransitive modal logic

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Predicate modal language

Let Var be a countably infinite set of (individual) variables and $PL^n = \{P_i^n \mid i \geq 0\}$ be a fixed set of n-ary predicate letters $(n \geq 0)$.
Elements of PL^0 are also called proposition letters.
An modal predicate formulas are defined inductively:

- $\bot$ is a formula;
- P_i^0 is a formula;
- if $x_1, \ldots, x_k \in Var$ then $P_i^k(t_1, \ldots, t_k)$ is a formula;
- if A and B are formulas then $(A \to B)$ is a formula;
- if A is a formula then $\Box A$ is a formula;
- if A is a formula and $x \in Var$ then $\forall x A$ is a formula.

All other connectives $\wedge, \vee, \neg, \exists, \Diamond$ are expressed as usual.

Quantified modal logic

If L is a modal logic then QL is the minimal set of predicate modal formulas such that

- QL includes all formulas from L where propositional variables replaced by corresponding propositional letters (0-ary predicate letters).
- for a fixed 1-ary predicate letter P and a propositional letter q QL includes formulas
  - $\forall x P(x) \to P(y)$,
  - $\forall x(q \to P(x)) \to (q \to \forall x P(x))$;
  - $\forall x(P(x) \to q) \to (\exists x P(x) \to q)$.
- QL is closed under Modus Ponens, necessitation, universal generalization rules, and under $\mathcal{MF}_N$ substitutions

Barcan formula: $BF = \forall x \Box P(x) \to \Box \forall x P(x)$.
Converse Barcan formula: $CBF = \Box \forall x P(x) \to \forall x \Box P(x)$.

Fact 1

QK $\vdash CBF$.

Fact 2

QK $\nvdash BF$.

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Predicate Kripke frames with expanding domains

A valuation ξ on a predicate frame $\mathbb{F}$ is a function sending every predicate letter P_k^m to a family of m-ary predicates on the domains:

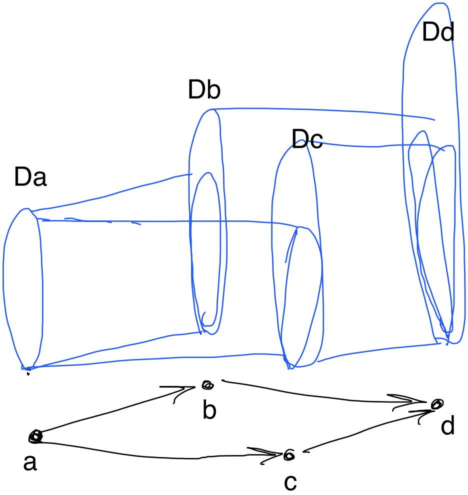
$$\xi(P_k^m) = (\xi_u(P_k^m))_{u\in W}, \text{ where } \xi_u(P_k^m) \subseteq D_{uk}^m.$$

A (predicate) model is $\mathbb{M} = (\mathbb{F}, \xi)$.

$$\begin{array}{lcl}
\mathbb{M}, u \not\models \bot; & & \\
\mathbb{M}, u \models P_i^0 & \iff & \xi_u(P_i^0) \text{ is true;} \\
\mathbb{M}, u \models P_i^m(a_1, \dots, a_m) & \iff & (a_1, \dots, a_m) \in \xi_u(P_i^m); \\
\mathbb{M}, u \models A \to B & \iff & \mathbb{M}, u \not\models A \text{ or } \mathbb{M}, u \models B; \\
\mathbb{M}, u \models \Box A & \iff & \forall v \ (uRv \Rightarrow \mathbb{M}, v \models A); \\
\mathbb{M}, u \models \forall x A(x) & \iff & \forall a \in D_u \ (\mathbb{M}, u \models A(a)).
\end{array}$$

This definition is correct given that the language is enriched with constants from set $\bigcup_{u\in W} D_u$

Predicate Kripke frames with expanding domains



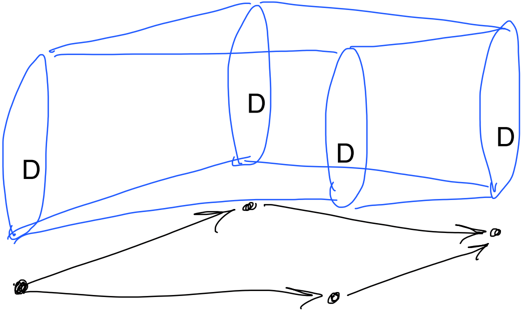
Constant domain

A predicate Kripke frame is with constant domain If all D_u are equal.

Lemma

If $\mathbb{F}$ is a rooted frame then it has a constant domain iff $\mathbb{F} \models BF$.

So QK is not compete w.r.t. Kripke frames with constant domain.



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Neighborhood frames (for normal logics)

Definition

Let X be a nonempty set, then $\mathcal{F} \subseteq 2^X$ is a filter on X if

1. $X \in \mathcal{F}$;
2. if $U_1,\ U_2 \in \mathcal{F}$, then $U_1 \cap U_2 \in \mathcal{F}$;
3. if $U_1 \in F$ and $U_1 \subseteq U_2$, then $U_2 \in \mathcal{F}$.

It is usually required that $\varnothing \notin \mathcal{F}$ ($\mathcal{F}$ is a proper filter), but we will not require this in our paper.

Definition

A (normal) neighborhood frame is $\mathfrak{X} = (X, \tau)$, where $X \neq \varnothing$ and $\tau : X \to 2^{2^X}$, s.t. $\tau(x)$ is a filter on X.
τ — the neighborhood function of $\mathfrak{X}$.
$\tau(x)$ — the family of the neighborhoods of x.

Predicate neighbourhood frames

Definition

A predicate neighborhood frame with constant domain is a couple $\mathbb{X} = (\mathfrak{X}, D^*)$, such that $\mathfrak{X}$ is a neighborhood frame and D^* is a nonempty set.
A valuation ξ on $\mathbb{X}$ is a function sending every predicate letter P_k^m to a family of m-ary predicates on D^*:

$$\xi(P_k^m) = (\xi_u(P_k^m))_{u \in W}, \text{ where } \xi_u(P_k^m) \subseteq (D^*)^m.$$

A neighborhood model on $\mathbb{X}$ is a pair $\mathbb{M} = (\mathbb{X}, \xi)$.
The truth of a closed formula in a model $\mathbb{M}$ at a point $x \in X$ is defined similar to Kripke models, by induction on the length of the formula, and similarly we enrich our language with constants from the set D^*.

$$\begin{array}{lcl}
\mathbb{M}, x \not\models \bot; & & \\
\mathbb{M}, x \models P_i^0 & \iff & \xi_x(P_i^0) \text{ is true}; \\
\mathbb{M}, x \models P_i^m(a_1, \ldots, a_m) & \iff & (a_1, \ldots, a_m) \in \xi_x(P_i^m); \\
\mathbb{M}, x \models A \to B & \iff & \mathbb{M}, x \not\models A \text{ or } \mathbb{M}, x \models B; \\
\mathbb{M}, x \models \Box_i A & \iff & \exists U \in \tau_i(x) \forall y \in U \, (\mathbb{M}, y \models A); \\
\mathbb{M}, x \models \forall x A(x) & \iff & \forall a \in D^* \, (\mathbb{M}, x \models A(a)).
\end{array}$$

Connection with products and expanding products

Kripke product with S5	predicate Kripke frames with constant domain
Gabbay and Shehtman, 1998	
$\models \Box_2\Box_1 p \to \Box_1\Box_2 p$	$\models \forall x \Box P(x) \to \Box \forall x P(x) \qquad (BF)$
$\models \Box_1\Box_2 p \to \Box_2\Box_1 p$	$\models \Box \forall x P(x) \to \forall x \Box P(x) \qquad (CBF)$

expanding Kripke product with S5	predicate Kripke frames with expanding domains
Kurucz and Zakharyaschev, 2003	*Hughes and Cresswell, 1996*
$\not\models \Box_2\Box_1 p \to \Box_1\Box_2 p$	$\not\models \forall x \Box P(x) \to \Box \forall x P(x) \qquad (BF)$
$\models \Box_1\Box_2 p \to \Box_2\Box_1 p$	$\models \Box \forall x P(x) \to \forall x \Box P(x) \qquad (CBF)$

topological product S4 $\times$ S5	predicate topological semantics with constant domain
Kremer, 2011	*Rasiowa and Sikorski, 1963*
$\not\models \Box_2\Box_1 p \to \Box_1\Box_2 p$	$\not\models \forall x \Box P(x) \to \Box \forall x P(x) \qquad (BF)$
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neighborhood product with S5	predicate neighborhood frames with constant domain
K,2014	
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Known completeness results

$$com_{12} = \Box_1\Box_2 p \to \Box_1\Box_2 p$$
$$com_{21} = \Box_2\Box_1 p \to \Box_2\Box_1 p$$
$$chr = \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$$

products of Kripke frames

If L_1 and L_2 are Horn modal logics then $\mathsf{L}_1 \times \mathsf{L}_2 = \mathsf{L}_1 * \mathsf{L}_2 + com_{12} + com_{21} + chr$ (Gabbay, Shehtman'1998)

expanding products of Kripke frames

If L is a one-way Horn modal logic and S5 is a Horn modal logic then $[\mathsf{L}, \mathsf{S5}]^{EX} = \mathsf{L} * \mathsf{S5} + com_{12} + chr$.
(Kurucz and Zakharyaschev'2003)

products of n-frames

If L_1 is a one-way Horn modal logic and L_2 is a Horn modal logic then $\mathsf{L}_1 \times_n \mathsf{S5} = \mathsf{L}_1 * \mathsf{L}_2 + com_{12} + chr$.
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Predicate modal logics

Kripke frames with expanding domains

If L is a one-way Horn modal logic then QL is complete w.r.t. Kripke frames with expanding domains.

n-frames with constant domain

If L is a one-way Horn modal logic then QL is complete w.r.t. n-frames with constant domain.
(K'2018)

Theorem (Arló Costa and Pacuit'2006)

Logic QK *is strongly complete w.r.t. n-frames with constant domain.*

The proof is using the canonical neighborhood model.

Theorem

If L *is a one-way Horn modal logic then* QL *is complete w.r.t. n-frames with constant domain.*

The is using completeness w.r.t. Kripke frames with expanding domains.

Plan.

- Given a Kripke frame $\mathbb{F}$ we construct a "well organized" Kripke frame $\mathbb{F}' = (F', D')$, such that $\mathbb{F}' \twoheadrightarrow \mathbb{F}$ F' is a continuum-branching tree-like frame and $D' = F' \Taurus \mathbb{R}^*$ (set of words in the alphabet $\mathbb{R}$).
- We construct an n-frame $\mathcal{N}_\omega(F')$. As the constant domain we take $\mathbb{R}^*$. And construct a p-morphism $\mathbb{X} \twoheadrightarrow \mathbb{F}'$, where $\mathbb{X} = (\mathcal{N}_\omega(F'), \mathbb{R}^*)$.

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p-morphism

Definition

Let $\mathfrak{X} = (X, \tau, D^*)$ be a neighbourhood frame with constant domain and $F = (W, R, D)$ be a Kripke frame with expanding domain, $D = \bigcup_{w \in W} D_w$. A p-morphism from $\mathbb{X}$ to $\mathbb{F}$ is a couple of functions (φ_0, φ_1), such that:

1. $\varphi_0 : (X, \tau) \twoheadrightarrow \mathcal{N}(W, R)$;
2. $\varphi_1 = (\varphi_{1x})_{x \in X}$ is a family of surjective functions indexed by points in X: $\varphi_{1x} : D^* \to D_{\varphi_0(x)}$;
3. $\forall d \in D^* \, \forall x \in X \, \exists U \in \tau(x) \, \forall y_1, y_2 \in U (\varphi_{1y_1}(d) = \varphi_{1y_2}(d))$.

Notation: $(\varphi_0, \varphi_1) : \mathbb{X} \twoheadrightarrow \mathbb{F}$. We write $\mathbb{X} \twoheadrightarrow \mathbb{F}$ if there exists a p-morphism from $\mathbb{X}$ to $\mathbb{F}$.

Corollaries for the topological semantics with constant domain

Theorem (Rasiowa and Sikorski'1963)

Logic QS4 *is complete with respect to topological spaces.*

Theorem (Kremer'2014)

Logic QS4 *is complete with respect to the set of rational numbers* $\mathbb{Q}$*.*

Theorem

Logics QK4 *and* QD4 *are complete with respect to* T_d *and dence-in-itself* T_d *topological spaces respectively.*

Theorem

Logic QD4 *is complete with respect to the set of rational numbers* $\mathbb{Q}$ *(with derivational modality).*

THANK YOU!!!

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