Free constructions and quotients of d-frames

Tomáš Jakl [a] (joint work with Achim Jung and Aleš Pultr)

ToLo VI in Tbilisi
2 July 2018

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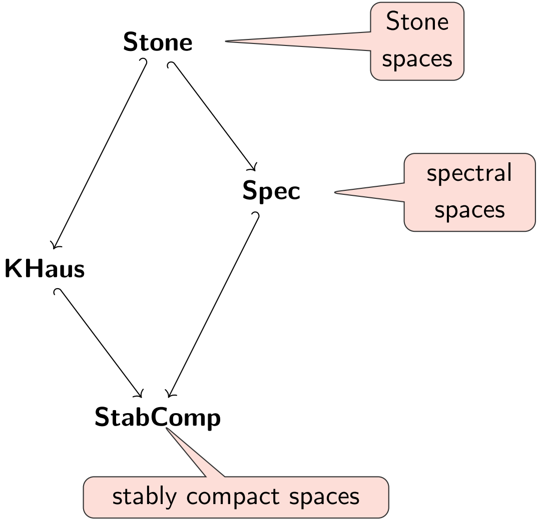
Why think bitopologically? (1/2)



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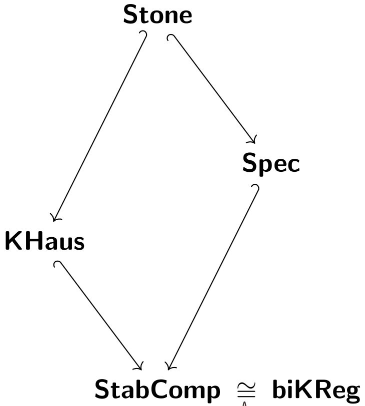

$$(X, \tau) \mapsto (X, \tau, \tau^d)$$

$$(X, \tau_+) \leftarrow\!\shortmid (X, \tau_+, \tau_-)$$

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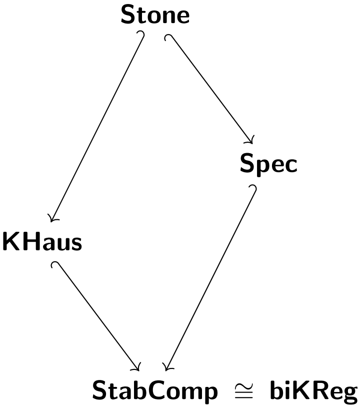

A space (X, τ) is **stably compact** if it is T_0, compact, locally compact, coherent, and well-filtered.

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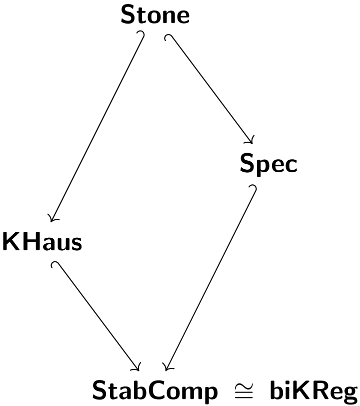

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vs.

A bispace (X, τ_+, τ_-) is **compact regular** if

- $\bigcup \mathcal{U}_+ \cup \bigcup \mathcal{U}_- = X$ implies $\bigcup \mathcal{F}_+ \cup \bigcup \mathcal{F}_- = X$ for some finite $\mathcal{F}_+ \subseteq \mathcal{U}_+$ and $\mathcal{F}_- \subseteq \mathcal{U}_-$
- $x \in U_+$ implies $x \in V_+ \subseteq \overline{V_+}^{\tau_-} \subseteq U_+$ for some $V_+ \in \tau_+$
- $x \in U_-$ symmetrically

Why think bitopologically? (2/2)

We have isomorphisms of categories

$$\textbf{CompactPospaces} \;\cong\; \textbf{biKReg} \;\cong\; \textbf{StabComp}$$

$$(X, \tau, \leq) \mapsto (X, \tau^{\uparrow}, \tau^{\downarrow})$$
$$(X, \tau_+ \vee \tau_-, \leq_{\tau_+}) \mapsfrom (X, \tau_+, \tau_-)$$

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such that

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This talk is about **d-frames** = algebraic duals of bitopological spaces.

$$\mathbf{biTop} \underset{\Sigma_d}{\overset{\Omega_d}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{d\text{-}Frm}$$

$$\mathbf{Top} \underset{\Sigma}{\overset{\Omega}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Frm}$$

Frames $=$ complete lattices $(L, \bigvee, \wedge, 0, 1)$ such that

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\}.$$

E.g. $\Omega(X, \tau) = (\tau, \bigcup, \cap, \emptyset, X)$.

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d-Frames $=$ structures $(L_+, L_-, \mathrm{con}, \mathrm{tot})$ such that

- L_+ and L_- are frames, and
- $\mathrm{con}, \mathrm{tot} \subseteq L_+ \times L_-$
- $+$ some axioms

e.g. $\dfrac{}{(1, 0) \in \mathrm{con}}$ $\qquad$ $\dfrac{(x_+, x_-) \in \mathrm{con},\ x'_+ \leq x_+,\ x'_- \leq x_-}{(x'_+, x'_-) \in con}$

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In particular, $\Omega_d(X, \tau_+, \tau_-) = (\tau_+, \tau_-, \mathrm{con}_X, \mathrm{tot}_X)$ where

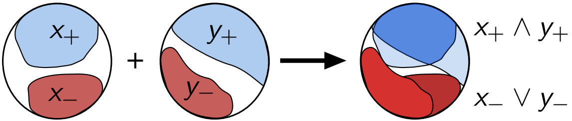
$$(U_+, U_-) \in \mathrm{con}_X \quad \text{iff} \quad U_+ \cap U_- = \emptyset$$

$$(U_+, U_-) \in \mathrm{tot}_X \quad \text{iff} \quad U_+ \cup U_- = X$$

Axioms of con and tot

1. A few **finitary axioms**, e.g.

$$(x_+, x_-),\ (y_+, y_-) \in \text{con} \implies (x_+ \wedge y_+,\ x_- \vee y_-) \in \text{con}$$



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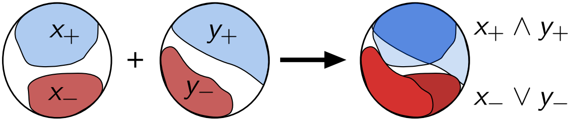

2. **DCPO axiom**:

$\{(x_+^i, x_-^i)\}_i \subseteq^\uparrow \mathsf{con}$

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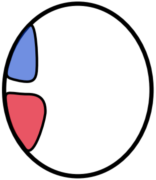

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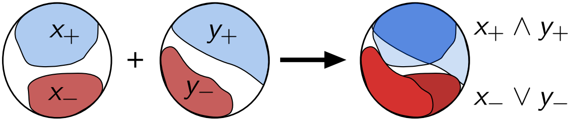

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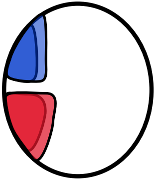

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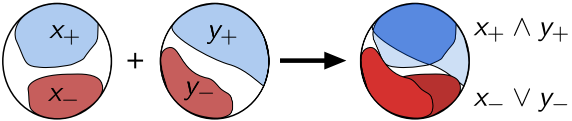

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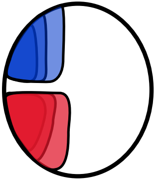

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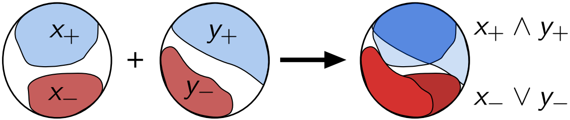

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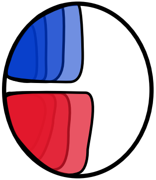

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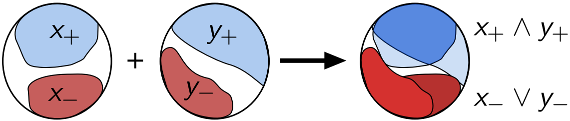

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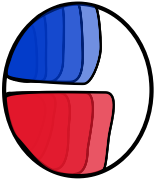

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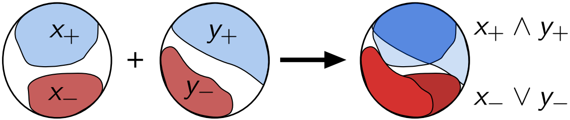

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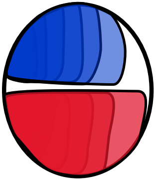

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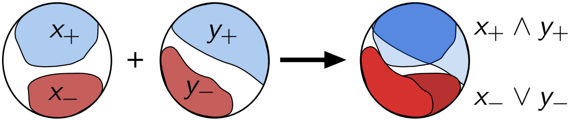

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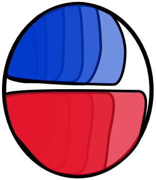
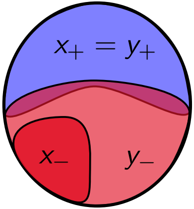


3. **con-tot axioms**:

$(x_+, x_-) \in \mathrm{con}, (y_+, y_-) \in \mathrm{tot}$,

and $x_+ = y_+ \implies x_- \leq y_-$.

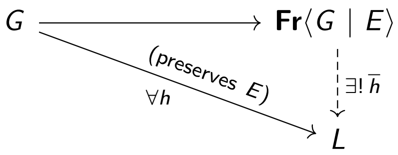


Free constructions (presenting by generators and relations)

A **frame presentation** (G, E) consists of

- a set of generators G
- equations $E \subseteq \mathbf{Fr}\langle G \rangle \times \mathbf{Fr}\langle G \rangle$

$\mathbf{Fr}\langle G \mid E \rangle \cong \mathbf{Fr}\langle G \rangle / E$



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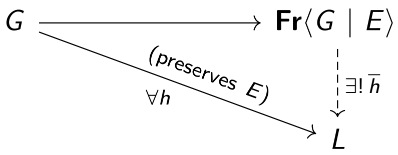


A **d-frame presentation** $(\mathbb{G}, \mathbb{E})$ consists of

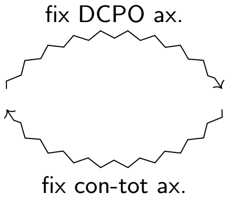
- sets of generators $\mathbb{G} = (G_+, G_-)$
- sets of equations $\mathbb{E} = (E_+, E_-, E_{\text{con}}, E_{\text{tot}})$ where

$$E_+ \subseteq \mathbf{Fr}\langle G_+\rangle \times \mathbf{Fr}\langle G_+\rangle \quad E_- \subseteq \mathbf{Fr}\langle G_-\rangle \times \mathbf{Fr}\langle G_-\rangle$$

$$E_{\text{con}},\ E_{\text{tot}} \subseteq \mathbf{Fr}\langle G_+\rangle \times \mathbf{Fr}\langle G_-\rangle$$

Our strategy to construct $\mathbf{dFr}\langle \mathbb{G} \mid \mathbb{E} \rangle$

We aim to improve $\mathbb{E}$ in steps (possibly transfinitely many).



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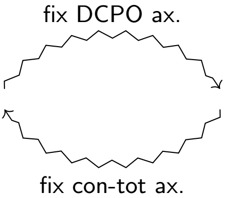

After we finish, compute

$$\Gamma(\mathbb{E}) \;=\; (\mathbf{Fr}\langle G_+ \mid E_+ \rangle\,,\; \mathbf{Fr}\langle G_- \mid E_- \rangle\,,\; q[E_{\mathsf{con}}],\; q[E_{\mathsf{tot}}])$$

where

$$q\colon \mathbf{Fr}\langle G_+ \rangle \times \mathbf{Fr}\langle G_- \rangle \twoheadrightarrow \mathbf{Fr}\langle G_+ \mid E_+ \rangle \times \mathbf{Fr}\langle G_- \mid E_- \rangle\,.$$

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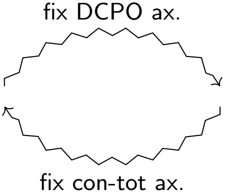

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Instead of quotienting, we work with preorders $\leq^{E_+}$ and $\leq^{E_-}$

Steps towards $\mathbf{dFr}\langle \mathbb{G} \mid \mathbb{E} \rangle$ (finitary axioms)

(1a) If $\mathbb{E} = (E_+, E_-, E_{\mathrm{con}}, E_{\mathrm{tot}})$ is such that the *finitary axioms* hold for

$$(\mathbf{Fr}\langle G_+ \rangle, \mathbf{Fr}\langle G_- \rangle, E_{\mathrm{con}}, E_{\mathrm{tot}})$$

then

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(1b) For a general $\mathbb{E}$ define

$$u_{\text{fin}}(\mathbb{E}) \;=\; (E_+,\, E_-,\, E^{\dagger}_{\text{con}},\, E^{\dagger}_{\text{tot}})$$

Where $E^{\dagger}_{\text{con}}$ and $E^{\dagger}_{\text{tot}}$ are the closures E_{con} and E_{tot} under the finitary axioms.

Then, $\Gamma(u_{\text{fin}}(\mathbb{E}))$ always satisfies the finitary axioms.

Steps towards $\mathbf{dFr}\langle \mathbb{G} \mid \mathbb{E} \rangle$ (DCPO axiom)

(2a) Let $\mathbb{E} = (\emptyset, \emptyset, E_{\text{con}}, E_{\text{tot}})$. Then $\Gamma(\mathbb{E})$ satisfies the DCPO axiom iff

$$(\mathbf{Fr}\langle G_+ \rangle, \mathbf{Fr}\langle G_- \rangle, E_{\text{con}}, E_{\text{tot}})$$

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satisfies it, where

$(x, y) \in E^*_{\mathsf{con}}$ iff $x =^{E_+} x'$ and $y =^{E_-} y'$, for some $(x', y') \in E_{\mathsf{con}}$.

$q[E^*_{\mathsf{con}}] = q[E_{\mathsf{con}}]$

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(2c) Set $u_{\text{DCPO}} = (E_+,\ E_-,\ E^\dagger_{\text{con}},\ E_{\text{tot}})$ where $E^\dagger_{\text{con}}$ is the DCPO-closure of E^*_{con}.

Then, $\Gamma(u_{\text{DCPO}}(\mathbb{E}))$ always satisfies the DCPO axiom.

(3) Let $\mathbb{E} = (E_+, E_-, E_{\text{con}}, E_{\text{tot}})$.

Set $E_+^\dagger$ and $E_-^\dagger$ such that $(y, z) \in E_-^\dagger$ iff

$$\exists x \in \mathbf{Fr}\langle G \rangle \quad (x, y) \in E_{\text{con}} \text{ and } (x, z) \in E_{\text{tot}}$$

(and $E_+^\dagger$ symmetrically).

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Defining

$$u_{\text{ct}} = (E_+ \cup E_+^\dagger,\ E_- \cup E_-^\dagger,\ E_{\text{con}}^*,\ E_{\text{tot}}^*)$$

gives

$$\mathbb{E} \ \subseteq\ u_{\text{ct}}(\mathbb{E}) \ \subseteq\ u_{\text{ct}}^2(\mathbb{E}) \ \subseteq\ u_{\text{ct}}^3(\mathbb{E}) \ \subseteq\ \ldots \ \subseteq\ u_{\text{ct}}^\infty(\mathbb{E}).$$

$$E_+ \subseteq \mathbf{Fr}\langle G_+ \rangle \times \mathbf{Fr}\langle G_+ \rangle \quad E_- \subseteq \mathbf{Fr}\langle G_- \rangle \times \mathbf{Fr}\langle G_- \rangle$$
$$E_{\text{con}},\ E_{\text{tot}} \quad \subseteq \quad \mathbf{Fr}\langle G_+ \rangle \times \mathbf{Fr}\langle G_- \rangle$$

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Then, $\Gamma(u_{\text{ct}}(\mathbb{E}))$ always satisfies con-tot axioms.

Free d-frame $\mathbf{dFr}\langle \mathbb{G} \mid \mathbb{E} \rangle$

No finite combination of u_{fin}, u_{DCPO} and u_{ct}^{∞} yields a closure operator!

Define

$$\mathbf{dFr}\langle \mathbb{G} \mid \mathbb{E} \rangle \;=\; \Gamma(u^{\infty}(u_{\text{fin}}(\mathbb{E})))$$

where

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$$\mathbb{E} \;\subseteq\; u(\mathbb{E}) \;\subseteq\; u^2(\mathbb{E}) \;\subseteq\; u^3(\mathbb{E}) \;\subseteq\; \ldots \;\subseteq\; u^{\infty}(\mathbb{E}).$$

Free d-frame $\mathbf{dFr}\langle \mathbb{G} \mid \mathbb{E} \rangle$

No finite combination of u_{fin}, u_{DCPO} and u_{ct}^{∞} yields a closure operator!

Define

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Theorem

Let $(\mathbb{G}, \mathbb{E})$ *be a d-frame presentation. Then* $\mathbf{dFr}\langle \mathbb{G} \mid \mathbb{E} \rangle$ *is the free d-frame.*

Applications

Application: d-Frame of real numbers $\mathcal{L}(\mathbb{R})$

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(for con and tot)

$$((q', -), (-, q)) \in \mathrm{con} \quad \text{if} \quad q \leq q',$$
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A concrete description of $\mathcal{L}(\mathbb{R})$

1. Every presentation can be rewritten into $\mathbb{E} = (\emptyset, \emptyset, E_{\mathsf{con}}, E_{\mathsf{tot}})$ over $(\mathbf{Fr}\langle G_+ \mid E_+\rangle, \mathbf{Fr}\langle G_+ \mid E_+\rangle)$,
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$$(U_u, U_d) \in \mathsf{tot}_{\mathcal{L}(\mathbb{R})} \quad \text{iff} \quad U_u \cap U_d \neq \emptyset,\ U_u = \mathbb{Q},\ \text{or}\ U_d = \mathbb{Q}$$

Spectra of d-frames

Frames: The points of ΣL are the completely prime filters $P \subseteq L$ and the topology consists of

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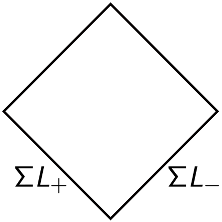

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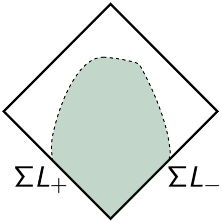

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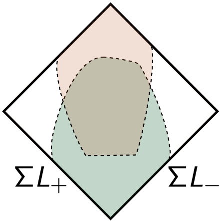

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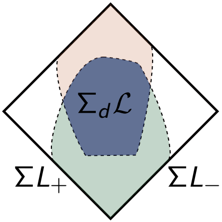
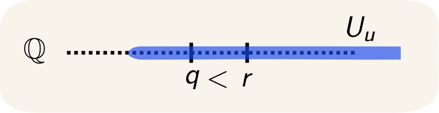


Spectrum of $\mathcal{L}(\mathbb{R})$, i.e. of $(\mathsf{OpenUp}(\mathbb{Q}), \mathsf{OpenDown}(\mathbb{Q}), \mathsf{con}_{\mathcal{L}(\mathbb{R})}, \mathsf{tot}_{\mathcal{L}(\mathbb{R})})$

$\mathbb{R} \cup \{+\infty\} \cong \Sigma(\mathsf{OpenUp}(\mathbb{Q}))$ $\quad r \mapsto \{U_u \mid q < r \text{ for some } q \in U_u\}$



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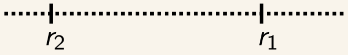

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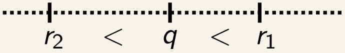

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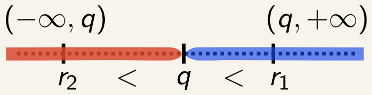

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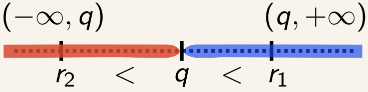

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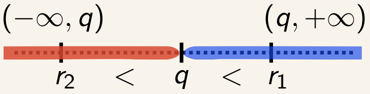

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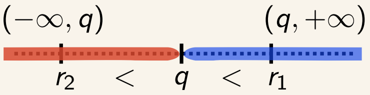

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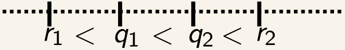

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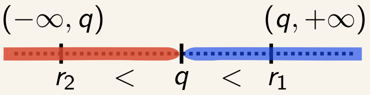

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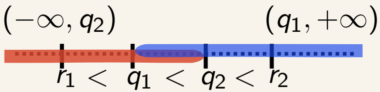

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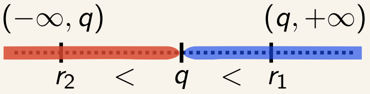

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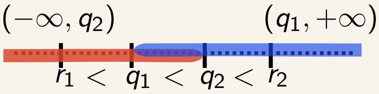

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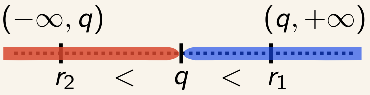
$$\implies \quad r_2 \not< r_1 \quad \text{and} \quad r_1 \not< r_2 \quad \implies r_1 = r_2$$



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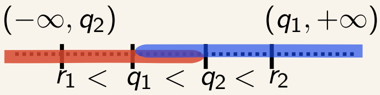

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A note on Dedekind reals

Every $r \in \mathbb{R} \rightsquigarrow$ pair of completely prime filters

$$\{U_u \mid q < r \text{ for some } q \in U_u\} \text{ and } \{U_l \mid r < q \text{ for some } q \in U_l\}$$

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Our d-frame presentation yields a representation of reals as Dedekind cuts!

Quotients

The quotient of a d-frame $(L_+, L_-, \text{con}, \text{tot})$ by a pair of relations $R = (R_+, R_-)$ is obtained as

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Some quotients obtained by adding equations to con or tot. In general, the quotient by $R = (R_+, R_-, R_{\text{con}}, R_{\text{tot}})$ is obtained as

$$\mathbf{dFr}\langle L_+, L_- \mid \leq_+ \cup R_+,\ \leq_- \cup R_-,\ \text{con} \cup R_{\text{con}},\ \text{tot} \cup R_{\text{tot}}\rangle$$

Quotients

The quotient of a d-frame $(L_+, L_-, \text{con}, \text{tot})$ by a pair of relations $R = (R_+, R_-)$ is obtained as

$$\mathbf{dFr}\langle L_+, L_- \mid \leq_+ \cup R_+,\ \leq_- \cup R_-,\ \text{con},\ \text{tot}\rangle$$

Some quotients obtained by adding equations to con or tot. In general, the quotient by $R = (R_+, R_-, R_{\text{con}}, R_{\text{tot}})$ is obtained as

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Observation (Sanity check)

Every d-frame is a quotient of a free d-frame.

Proof $(L_+, L_-, \text{con}, \text{tot})$ is a quotient of $\mathbf{dFr}\langle L_+, L_- \mid \emptyset\rangle$. □

Closing remarks

1. It's a technique for mixed "algebraic+relational" structures (no need to restrict to frames).

2. $\mathbb{E}$'s such that $u(\mathbb{E}) = \mathbb{E}$ form a complete lattice:

$$\mathbf{dFr}\langle \mathbb{G} \mid \mathbb{E} \rangle = \Gamma(\mathbb{E}^\infty)$$

   where $\mathbb{E}^\infty$ is the smallest $\mathbb{E} \subseteq \mathbb{E}^\infty$ such that $u(\mathbb{E}^\infty) = \mathbb{E}^\infty$.

   Open problem: What is the structure of this complete lattice?

3. The iterative procedure gives us (a) sufficient conditions for *nice* presentations, (b) proof of completeness of a paraconsistent geometric logic.

   Other *nice* presentations: coproducts, Vietoris, **d-KReg** from a proximity lattice, ...

Thank you!

References:

1. Achim Jung, M. Andrew Moshier: On the bitopological nature of Stone duality, Technical Report, 2006.
2. Tomáš Jakl: d-Frames as algebraic duals of bitopological spaces, Ph.D. thesis, 2018.
3. Tomáš Jakl, Achim Jung, Aleš Pultr: Quotients of d-frames, submitted.
4. Tomáš Jakl, Achim Jung: Free constructions of d-frames, in preparation.

Bonus slides

Paraconsistent geometric logic of d-frame

Semantically, $(x_+, x_-) \in L_+ \times L_-$ represents an evidence for

affirmation and **refutation**, respectively.

Terms: $\mathcal{V}ar$, $\bot$, $\top$, tt, ff, $\bigsqcup A$, $\alpha \sqcap \beta$

Judgements: $\alpha \Rightarrow \beta$, $\mathfrak{con}(\alpha)$, and $\mathfrak{tot}(\alpha)$

Rules:

1. "$\Rightarrow$" is a preorder, $\bigsqcup$ is a join, $\wedge$ is a meet, $\bot$ the smallest element and $\top$ the largest element, $\bigvee$ distributes over $\wedge$
2. Rules for judgements mimicking d-frame axioms, e.g.

$$\frac{\alpha \sqsubseteq \beta \quad \mathbf{con}(\alpha)}{\mathbf{con}(\beta)}$$

d-Frame quotients for completeness

Theorem (Completeness)
$\Gamma \vDash \varphi$ *implies* $\Gamma \vdash \varphi$.

Proof sketch.

1. $\mathbf{dFr}\langle \mathcal{V}ar \mid \sigma\Gamma \rangle \vDash \Gamma$ ($\sigma\Gamma = \Gamma$ closed under substitutions)
2. $\mathbf{dFr}\langle \mathcal{V}ar \mid \sigma\Gamma \rangle \vDash \varphi$
3. Every application of $u(\cdot)$ yields one step in $\sigma\Gamma \vdash \varphi$. □

Priestley duality revisited

For a distributive lattice D, define $\mathcal{L}(D)$

$$\mathbf{dFr}\Big\langle \langle d_+\rangle, \langle d_-\rangle : d \in D \;\Big|\; \begin{aligned} &\langle d_+\rangle \vee \langle e_+\rangle = \langle d_+ \vee e_+\rangle, \;\; \langle 0_+\rangle = 0, \\ &\langle d_+\rangle \wedge \langle e_+\rangle = \langle d_+ \wedge e_+\rangle, \;\; \langle 1_+\rangle = 1, \\ &\langle d_1\rangle \vee \langle e_+\rangle = \langle d_+ \wedge e_+\rangle, \;\; \langle 0_+\rangle = 1, \\ &\langle d_1\rangle \wedge \langle e_+\rangle = \langle d_+ \vee e_+\rangle, \;\; \langle 1_+\rangle = 0, \\ &(\forall d \in D) \quad (\langle d_+\rangle, \langle d_-\rangle) \in \mathrm{con}, \\ &\qquad\qquad\qquad (\langle d_+\rangle, \langle d_-\rangle) \in \mathrm{tot}, \end{aligned} \Big\rangle.$$

Priestley duality revisited

For a distributive lattice D, define $\mathcal{L}(D)$

$$\mathbf{dFr}\Big\langle \langle d_+\rangle, \langle d_-\rangle : d \in D \;\Big|\; \begin{aligned} &\langle d_+\rangle \vee \langle e_+\rangle = \langle d_+ \vee e_+\rangle, \;\; \langle 0_+\rangle = 0, \\ &\langle d_+\rangle \wedge \langle e_+\rangle = \langle d_+ \wedge e_+\rangle, \;\; \langle 1_+\rangle = 1, \\ &\langle d_1\rangle \vee \langle e_+\rangle = \langle d_+ \wedge e_+\rangle, \;\; \langle 0_+\rangle = 1, \\ &\langle d_1\rangle \wedge \langle e_+\rangle = \langle d_+ \vee e_+\rangle, \;\; \langle 1_+\rangle = 0, \\ &(\forall d \in D) \quad (\langle d_+\rangle, \langle d_-\rangle) \in \mathrm{con}, \\ &\qquad\qquad\quad\; (\langle d_+\rangle, \langle d_-\rangle) \in \mathrm{tot}, \end{aligned} \Big\rangle.$$

Then, $\mathcal{L}(D) \cong (\mathrm{Idl}(D), \mathrm{Filt}(D), \mathrm{con}_D, \mathrm{tot}_D)$ such that

$$(I, F) \in \mathrm{con}_D \quad \text{iff} \quad \forall i \in I, f \in F.\; i \leq f$$

$$(I, F) \in \mathrm{tot}_D \quad \text{iff} \quad I \cap F \neq \emptyset$$

Priestley duality revisited

For a distributive lattice D, define $\mathcal{L}(D)$

$$\mathbf{dFr}\Big\langle \langle d_+\rangle , \langle d_-\rangle : d \in D \;\Big|\; \begin{array}{l} \langle d_+\rangle \vee \langle e_+\rangle = \langle d_+ \vee e_+\rangle, \;\; \langle 0_+\rangle = 0, \\ \langle d_+\rangle \wedge \langle e_+\rangle = \langle d_+ \wedge e_+\rangle, \;\; \langle 1_+\rangle = 1, \\ \langle d_1\rangle \vee \langle e_+\rangle = \langle d_+ \wedge e_+\rangle, \;\; \langle 0_+\rangle = 1, \\ \langle d_1\rangle \wedge \langle e_+\rangle = \langle d_+ \vee e_+\rangle, \;\; \langle 1_+\rangle = 0, \\ (\forall d \in D) \quad (\langle d_+\rangle , \langle d_-\rangle) \in \mathrm{con}, \\ \qquad\qquad\qquad (\langle d_+\rangle , \langle d_-\rangle) \in \mathrm{tot}, \end{array} \Big\rangle.$$

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and $\mathcal{L}(D) \cong \Omega_d(\mathrm{spec}_d(D))$.