A Semantic Hierarchy for Intuitionistic Logic

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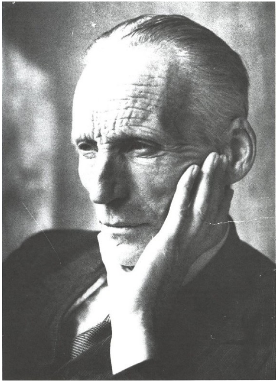
An advertisement for our paper, "A Semantic Hierarchy for Intuitionistic Logic," written for a special issue of *Indagationes Mathematicae* on **L.E.J. Brouwer: Fifty Years Later**.



Luitzen Egbertus Jan Brouwer (1881–1966)

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The Dragalin place in the hierarchy can be expanded as:

Locales $\equiv$ Nuclear $\equiv$ Dragalin $\equiv$ Cover $\equiv$ FM.

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- Kripke $<$ Topological, as Kripke frames produce only those locales that are *completely join-prime generated*, i.e., every element is a join of completely join-prime elements. Many spatial locales are not so generated.
- Topological $<$ Locales, because not all locales are spatial.
- Locales $<$ Algebraic, because not all HAs are complete.

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Contrast this with our knowledge of modal incompleteness with respect to different kinds of algebras—as summarized in, e.g., "Complete Additivity and Modal Incompleteness" by H. & Litak.

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Natural variant: replace 'topological spaces' by 'locales' above.

Beth semantics

Prior to Kripke semantics, Beth proposed a semantics for intuitionistic logic.

Evert Willem Beth (1908–1964)

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- $x \models_v p$ iff every maximal chain[1] through x intersects $v(p)$;
- $x \models_v \varphi \vee \psi$ iff every maximal chain through x intersects $\{y \in X \mid y \models_v \varphi \text{ or } y \models_v \psi\}$.

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If "inevitably" one of the disjuncts of a disjunction will be satisfied, then the disjunction is already satisfied.

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Instead of evaluating formulas in the locale $\mathsf{Up}(X)$ of all upsets, evaluate in the algebra of "fixed" upsets: upsets U such that if every maximal chain through x intersects U, then $x \in U$.

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The join in the algebra is no longer union, but rather:

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Later we will see why the algebra of fixed upsets is a locale, which yields soundness of IPC w.r.t. Beth semantics.

Beth semantics

One of Dummett's (2000) ways of understanding Beth:

> On this approach, we are distinguishing between the *verification* of an atomic statement in a given state of information, and its being *assertible*; the latter notion is represented by truth at a node, and is defined, for all statements, in terms of the verification of atomic statements. The knowledge that a given atomic statement will be verified within a finite time does not itself constitute a verification of it, but is sufficient ground to entitle us to assert it. (p. 139)

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While in Kripke semantics, $x \models_v p$ iff $x \in v(p)$, Dummett suggests that in Beth semantics we can make a distinction:

- $x \in v(p)$ means that p is verified in x;
- $x \models_v p$ means that in x, it is *known* that p *will be verified*.

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  In Beth semantics, $x \models p \vee q$ if it is known that however the future unfolds, one of the disjuncts will be verified.

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Given Shehtman's result that there are Kripke-incomplete but topologically-complete SI-logics, either there are Kripke-incomplete but Beth-complete SI-logics or there are Beth-incomplete but topologically-complete SI-logics.

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Question: Which is it? Both?

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Recall: the locales produced by Kripke frames are the completely join-prime generated locales, and the locales produced by topological spaces are the spatial locales.

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Problem: characterize the locales produced by Beth frames.

The essence of Beth semantics

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In the algebra of fixed upsets mentioned before, the join is:

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A ***nucleus*** on an HA H is a function $j : H \to H$ satisfying:

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A ***nuclear algebra*** is a pair (H, j) of an HA H and nucleus j on H.

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Beyond Beth to nuclear semantics

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But we can generalize:

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A valuation on a nuclear frame assigns to proposition letters elements of $\mathsf{Up}(X)$ as usual, and the definition of $\models$ simply replaces the Beth nucleus j_b with the given nucleus j:

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In short: evaluate formulas in the locale $\mathsf{Up}(X)_j$.

Soundness of IPC is then immediate, since H_j is an HA whenever HA is. **Completeness** follows from Kripke completeness (j is identity) or Beth completeness ($j = j_b$).

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It also reasonable that a conjunction is assertible iff each conjunct is assertible, so j should be **multiplicative**.

The generality of nuclear semantics

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Can we achieve this kind of generality with a semantics that replaces the algebraic j with some more concrete data?

Dragalin semantics

Albert Grigor'evich Dragalin (1941-1998)

Dragalin semantics

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Maybe they aren't maximal; maybe they aren't chains; maybe they are only directed; maybe they are not even directed, etc.

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(1°) $\varnothing \notin D(s)$.
Intuitively: the empty set is not a development of anything.

(2°) if $t \in S \in D(s)$, then $\exists x \in S : s \leq x$ and $t \leq x$.
Intuitively: every stage t in a development of s is *compatible* with s, in that s and t have a common extension x.

(3°) if $s \leq t$, then $\forall T \in D(t)\ \exists S \in D(s) : S \subseteq {\downarrow}T$.
Intuitively: if at some "future" stage t a development T will become available, then it is already possible to follow a development bounded by T.

(4°) if $t \in S \in D(s)$, then $\exists T \in D(t) : T \subseteq {\downarrow}S$.
Intuitively: we "can always stay inside" a development, in the sense that for every stage t in S, we can follow a development T from t that is bounded by S.

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Intuitively: the stages in a development starting from s are extensions of s.

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Intuitively: we "can always stay inside" a development in the sense that for every state t in S, we can follow a development T from t that is included in S.

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Proposition (Dragalin)

For any Dragalin frame (X, D), the function j_D on $\mathsf{Up}(X)$ defined by

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Dragalin semantics: given a Dragalin frame (X, D), apply the earlier nuclear semantics to (X, j_D).

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For every nuclear frame (X, j)*, there is a Dragalin frame* (X, D) *such that* $j_D = j$*.*

Super-sketch. As is well known, the nuclei on $\mathsf{Up}(X)$ form a locale in which each j can be written as a meet of special nuclei w_{j_a}. We show that each of these special nuclei can be captured by a D function, and the meet of nuclei can be captured by an operation on D functions.

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Question: can every SI-logic be characterized by some class of locales? Could Dragalin frames help us?

Relation of Dragalin to Cover Semantics

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He calls this cover semantics.

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Dragalin $\equiv$ Cover.

In our manuscript, "Development Frames", we systematically relate the Beth-Dragalin style path or development semantics to Scott-Montague style neighborhood or cover semantics.

FM-semantics

A (normal) FM-frame is a triple $(Y, \leq_1, \leq_2)$ where Y is a set, $\leq_1$ and $\leq_2$ are preorders on X, and $\leq_2$ is a subrelation of $\leq_1$.

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For any FM-frame $(Y, \leq_1, \leq_2)$, the operation $\Box_1 \Diamond_2$ is a nucleus on the Heyting algebra $\mathsf{Up}(Y, \leq_1)$.

Thus, we can apply nuclear semantics and work with the locale

$$\mathsf{Up}(Y, \leq_1)_{\Box_1 \Diamond_2}.$$

From Dragalin to FM

Surprisingly, FM is as general as Dragalin semantics:

Theorem (Bezhanishvili and Holliday 2016)

For any (normal) Dragalin frame (X, D)*, there is a (normal) FM-frame* $(Y, \leq_1, \leq_2)$ *such that the nuclear algebras* $(\mathsf{Up}(X), j_D)$ *and* $(\mathsf{Up}(Y, \leq_1), \Box_1 \Diamond_2)$ *are isomorphic.*

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Super-sketch. Any Dragalin frame can be made "convex", and any convex (normal) Dragalin frame $(X, \leq, D)$ can be turned into a (normal) FM-frame $(Y, \leq_1, \leq_2)$ as follows:

- $Y = \{(x, S) \mid x \in X, S \in D(x)\}$;
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Corollary

Every locale is isomorphic to one arising from an FM-frame.

Direct from Locales to FM-frames

The FM-frame obtained by following our constructions for Locale $\Rightarrow$ Dragalin $\Rightarrow$ FM is a substructure of the following.

Definition

The *canonical FM-frame* of a locale L is the normal FM-frame $(X_L, \leq_1, \leq_2)$ defined as follows, where $\leq$ is the order in L:

1. $X_L = \{(a,b) \in L^2 \mid a \not\leq b\}$;
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This is essentially the approach of Massas (2016), except he constructs a smaller substructure of the canonical FM-frame.

Relation of FM to Urquhart and Allwein

Generalizing Urquhart, a *doubly preordered structure* is a triple $(X, \leq_1, \leq_2)$ where X is a set and $\leq_1$ and $\leq_2$ are preorders on X.

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Let the *canonical structure* of a complete lattice L be $(X, \leq_1, \leq_2)$:

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If L is a complete lattice, then L is isomorphic to the lattice of $\Box_1\Diamond_2$-fixpoints of the canonical structure of L.

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If L is a locale, we can cut down $\leq_2$ to be a subrelation of $\leq_1$. That's FM-semantics!

Conclusion

We have sketched the semantic hierarchy:

$$\text{Kripke} < \text{Beth} < \text{Topological} < \text{Dragalin} < \text{Algebraic}.$$

$$\text{Locales} \equiv \text{Nuclear} \equiv \text{Dragalin} \equiv \text{Cover} \equiv \text{FM}.$$

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Can the more concrete representations of locales help answer the question of locale (in)completeness of SI-logics?

Kripke $<$ Beth $<$ Topological $<$ Dragalin $<$ Algebraic.

Locales $\equiv$ Nuclear $\equiv$ Dragalin $\equiv$ Cover $\equiv$ FM.

Thank you!

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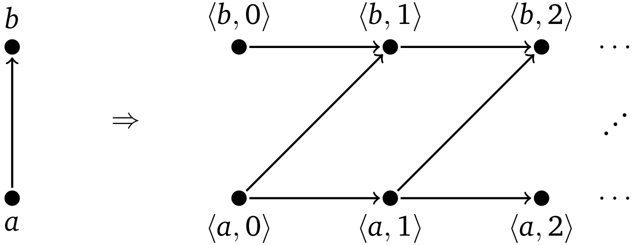
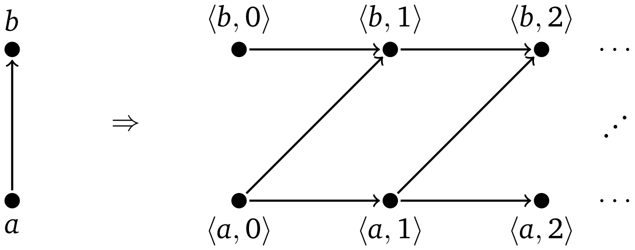
A state in the Bethification records the current time and one's current location in the Kripke frame.



Bethification (right) of a Kripke frame (left).



Bethification (right) of a Kripke frame (left).

Bethification Theorem: Let $\mathfrak{F}$ be a poset and $\mathfrak{F}_b$ its Bethification. Then $\mathsf{Up}(\mathfrak{F})$ is isomorphic to the locale of fixpoints of the Beth nucleus on $\mathsf{Up}(\mathfrak{F}_b)$.

From Beth to Topological

Given a poset $\mathfrak{F} = (X, \leqslant)$, let Y be the set of all maximal chains in X, and for $U \subseteq X$, let $[U] = \{\alpha \in Y \mid \alpha \cap U \neq \varnothing\}$.

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Then the pair (Y, Ω) with

$$\Omega = \{[U] \mid U \text{ is a fixpoint of the Beth nucleus on } \mathsf{Up}(\mathfrak{F})\}$$

is a topological space,

From Beth to Topological

Given a poset $\mathfrak{F} = (X, \leqslant)$, let Y be the set of all maximal chains in X, and for $U \subseteq X$, let $[U] = \{\alpha \in Y \mid \alpha \cap U \neq \varnothing\}$.

Then the pair (Y, Ω) with

$$\Omega = \{[U] \mid U \text{ is a fixpoint of the Beth nucleus on } \mathsf{Up}(\mathfrak{F})\}$$

is a topological space, and the locale of fixpoints of the Beth nucleus on $\mathsf{Up}(\mathfrak{F})$ is isomorphic to the locale of open sets of the topological space (Y, Ω).

From Topological to Dragalin

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Then $(\Omega, \leq, D)$ is a Dragalin frame, and $\Omega(X)$ is isomorphic to the locale of fixpoints of the Dragalin nucleus j_D on $\mathsf{Up}(\Omega, \leq)$.