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On the variety of $L_P G$ -algebras

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MV-algebras

The infinitely valued propositional calculi *Ł* have been introduced by Łukasiewicz and Tarski in 1930.

The algebraic models, MV-algebras, for this logic was introduced by Chang in 1959.

MV-algebras

An ***MV-algebra*** is an algebra

$$\mathrm{A} = (A, \oplus, \otimes, ^*, 0, 1)$$

where $(A, \oplus, 0)$ is an *abelian monoid*, and for all $x,y \in A$ the following identities hold:

$x \oplus 1 = 1$, $x^{**} = x$,

$(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x$,

$x \otimes y = (x^* \oplus y^*)^*$.

MV -algebras

It is well known that the *MV*-algebra $S = ([0, 1], \oplus, \otimes, *, 0, 1)$, where $x \oplus y = \min(1, x+y)$, $x \otimes y = \max(0, x+y-1)$, $x^* = 1-x$, generates the variety **MV** of all *MV*-algebras.

Let Q denote the set of rational numbers, for $(0 \neq)\ n \in \omega$ we set $S_n = (S_n, \oplus, \otimes, *, 0, 1)$, where $S_n = \{0, 1/n-1, \dots, n-2/n-1, 1\}$ is also *MV*-algebra.

ℓ-groups

Let (G, u) be ℓ-group with strong unite u.

Then $\Gamma(G,u)$ = ([0,u], $\oplus$, *, 0) (Chang 1959, Mundici 1986) is an MV-algebra, where

$[0,u] = \{a \in G : 0 \leq a \leq u\}$,

$a \oplus b = (a + b) \wedge u$,

$a^* = u - a$.

ℓ-groups

A *lattice-ordered abelian group* (ℓ-group) is an algebra $(G, +, -, 0, \vee, \wedge)$ such that $(G, +, -, 0)$ is a abelian group, $(G, \vee, \wedge)$ is a lattice, and + distributes over $\vee$ and $\wedge$.

A *strong unite* of ℓ-group G is an element $u > 0$ of G such that for every $a \in G$, there exists a natural number m with $a \leq mu$.

Examples

$C_0 = \Gamma(Z,1)$,

$C_1 = C = \Gamma(Z \times_{lex} Z, (1, 0))$
with generator $(0, 1) = c_1 (= c)$,

$C_m = \Gamma(Z \times_{lex} \ldots \times_{lex} Z, (1, 0, \ldots , 0))$

with generators

$c_1(= (0, 0, \ldots , 1)), \ldots , c_m(= (0, 1, \ldots , 0))$,

where the number of factors Z is equal to m+1,

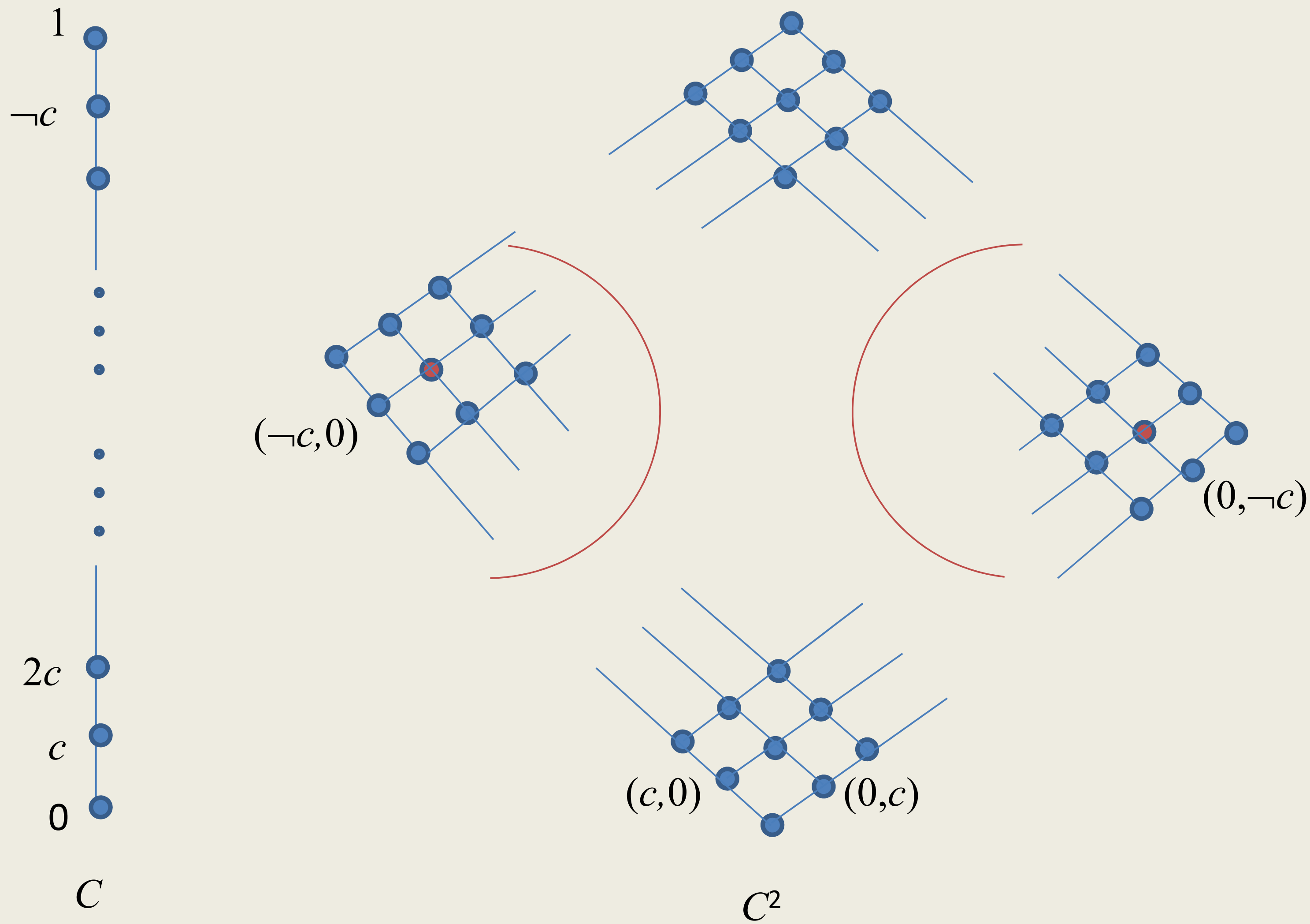
$m > 1$ and $\times_{lex}$ is the lexicographic product.

Perfect MV-algebras

From the variety of *MV*-algebras **MV** select the subvariety **MV(C)** which is defined by the following identity:

$$\text{(Perf)} \quad 2(x^2) = (2x)^2,$$

that is **MV(C)** = **MV**+ (Perf) (Di Nola, Lettieri 1993) .


1
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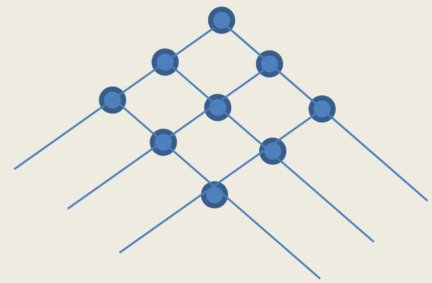
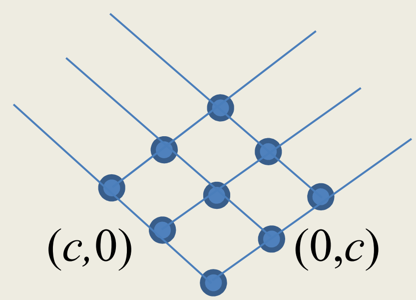
Perfect *MV* -algebras





$\mathrm{Rad}(C^2) \cup \neg\mathrm{Rad}(C^2)$

Logic $Ł_P$

$Ł_P$ is the logic corresponding to the variety generated by perfect *MV*-algebras which coincides with the set of all Łukasiewicz formulas that are valid in all perfect *MV*-chains, or equivalently that are valid in the *MV*-algebra *C*. Actually, $Ł_P$ is the logic obtained by adding to the axioms of Łukasiewicz sentential calculus the following axiom:

$$Ł_P\text{:}\quad (\alpha \veebar \alpha)\&(\alpha \veebar \alpha) \leftrightarrow (\alpha \,\&\, \alpha) \veebar (\alpha \,\&\, \alpha)$$

Heyting algebra

A *Heyting algebra*

$$(H, \vee, \wedge, \rightarrow, 0, 1)$$

is a bounded distributive lattice

$$(H, \vee, \wedge, 0, 1)$$

with an additional binary operation

$\rightarrow : H \times H \rightarrow H$ such that for any $a, b \in H$

$$x \leq a \rightarrow b \text{ iff } a \wedge x \leq b.$$

(Here $x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$.)

Gödel algebra

Gödel algebras are Heyting algebras with the linearity condition:

$$(x \to y) \vee (y \to x) = 1.$$

Let **G** be the variety of Gödel algebras

$L_P G$-algebra

An algebra

$$(A, \otimes, \oplus, {}^*, \wedge, \vee, \rightarrow, 0, 1)$$

is called *$L_P G$-algebra* if

$$(A, \otimes, \oplus, {}^*, 0, 1)$$

is L_P-algebra (i. e. an algebra from the variety generated by perfect *MV*-algebras) and

$$(A, \wedge, \vee, \rightarrow, 0, 1)$$

is a Gödel algebra (i. e. Heyting algebra satisfying the identity

$$(x \rightarrow y) \vee (y \rightarrow x) = 1).$$

$L_P G$-algebra

1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;

2) $x \oplus y = y \oplus x$;

3) $x \oplus 0 = x$;

4) $x \oplus 1 = 1$;

5) $0^* = 1$;

6) $1^* = 0$;

7) $x \otimes y = (x^* \oplus y^*)^*$;

8) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$;

9) $2(x^2) = (2x)^2$

10) $x \vee y = (x \otimes y^*) \oplus y$;

11) $x \wedge y = (x \oplus y^*) \otimes y$;

12) $(x \rightarrow y) \wedge y = y$;

13) $(x \wedge (x \rightarrow y) = x \wedge y$;

14) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$;

15) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$;

16) $(x \rightarrow 0)^* \leq ((x \rightarrow 0) \rightarrow 0)$;

17) $(x \rightarrow y)^* \leq (x^* \oplus y)$.

$L_P G$-algebra

The algebras

$C_m = \Gamma(Z \times_{lex} \ldots \times_{lex} Z, (1, 0, \ldots, 0))$, $m \in \omega$

are $L_P G$-algebras. Denote by the same symbol the $L_P G$-algebra

$$(C_m, \otimes, \oplus, {}^*, \wedge, \vee, \rightarrow, 0, 1).$$

$L_P G$-algebra

Theorem 1. *The variety* $\mathbf{L_P G}$ *is generated by the algebra* $(C, \otimes, \oplus, *, \wedge, \vee, \rightarrow, 0, 1)$.

Heying-Brouwer logic

- Heyting-Brouwer logic (alias symmetric Intuitionistic logic Int^2) was introduced by C. Rauszer (1974) as a Hilbert calculus with an algebraic semantics.
- The variety of Skolem (Heyting-Brouwerian) algebras are algebraic models for symmetric Intuitionistic logic Int^2 (Rauszer 1974, Esakia 1978) .

$L_P G$-algebra

- *Let* $(A, \otimes, \oplus, {}^*, \wedge, \vee, \to, 0, 1)$ *be* $L_P G$*-algebra. Then A is a bi-Heyting (Heyting-Browerian) algebra, where the pseudo-difference*

$$b - a = (a^* \to b^*)^* \quad and \qquad \ulcorner a = (\neg a^*)^* .$$

Let A be an $L_P G$-algebra. A subset $F \subset T$ is said to be a *Skolem filter* [for *Heyting-Browerian* algebras Rauszer 1974, Esakia 1978], if F is a MV-filter (i. e. $1 \in F$, if $x \in F$ and $x \leq y$, then $y \in F$, if $x, y \in$ F, then $x \otimes$ y $\in F$) and if $x \in F$, then $\neg \ulcorner x \in F$.

$L_P G$-algebra

Theorem 2. *Let* $(A, \otimes, \oplus, {}^*, \wedge, \vee, \to, 0, 1)$ *be* $L_P G$*-algebra and* F *Skolem filter. Then the equivalence relation*

$$x \equiv y \Leftrightarrow (x^* \oplus y) \wedge (y^* \oplus x) \in F$$

is a congruence relation for $L_P G$*-algebra* A*.*

A lattice of congruences of an $L_P G$*-algebra* A *is isomorphic to a lattice of Skolem filters of* $L_P G$*-algebra* A.

$L_P G$-algebra

Theorem 3. *The logic Ł$_P$G is recursively axiomatizable and charcharacterized by a recursively enumerable class of recursive algebras*

$L_P G$-algebra

Theorem 4. *The logic Ł$_P$G is decidable.*

Topological spaces

A topological space X is said to be an *MV -space* if there exists an MV -algebra A such that Spec(A) and X are homeomorphic. It is well known that Spec(A) with the specialization order (which coincides with the inclusion between prime filters) forms a root system. Actually any MV-space is a Priestly space which is a root system.

An MV-space is a Priestley space X such that $R(x)$ is a chain for any $x \in X$ and a morphism between MV -spaces is a strongly isotone map, i. e. a continuous map $f : X \to Y$ such that f(R(x)) = R(f(x)) for all $x \in X$.

Belluce's functor

- On each MV-algebra *A* Belluce has defined a binary relation ≡ by the following stipulation: x ≡ y iff supp(x) = supp(y), where supp(x) is defined as the set of all prime ideals of *A* not containing the element x.

Belluce's functor

$\equiv$ is a congruence with respect to $\oplus$ and $\wedge$. The resulting set $\beta(A)(= A/\equiv)$ of equivalence classes is a bounded distributive lattice, called the Belluce lattice of A. For each $x \in A$ let us denote by $\beta(x)$ the equivalence class of x.

Let $f : A \to B$ be an MV-homomorphism. Then $\beta(f)$ is a lattice homomorphism from $\beta(A)$ to $\beta(B)$ which is defined as follows: $\beta(f)(\beta(x)) = \beta(f(x))$.

β defines a covariant functor from the category of MV-algebras to the category of bounded distributive lattices. Moreover MV-space of A and Priestly space of $\beta(A)$ are homeomorphic (Belluce).

Belluce's functor

Dually, on each *MV*-algebra *A* is defined a binary relation ≡* by the following stipulation: x ≡* y iff supp*(*x*) = supp*(*y*), where supp*(*x*) is defined as the set of all prime filters of *A* containing the element *x*.

Belluce's functor

$\equiv^*$ is a congruence with respect to $\otimes$ and $\vee$. The resulting set $\beta^*(A)(= A/\equiv)$ of equivalence classes is a bounded distributive lattice. For each $x \in A$ let us denote by $\beta^*(x)$ the equivalence class of x.

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Belluce's functor

Theorem 5. *Let* $(A, \otimes, \oplus, {}^*, \wedge, \vee, \rightarrow, 0, 1)$ *be* L_PG*-algebra. Then* $\beta^*(A)$ *is a Gödel algebra.*

Belluce's functor

Theorem 5. *Let* $(A, \otimes, \oplus, \ ^*, \wedge, \vee, \rightarrow, 0, 1)$ *be* L_P*G-algebra. Then* $\beta^*(A)$ *is a Gödel algebra.*

Theorem 6. *Let* $(A, \otimes, \oplus, \ ^*, \wedge, \vee, \rightarrow, 0, 1)$ *be* L_P*G-algebra. Then* $\beta^*(A)$ *is a bi-Heyting algebra, i. e. the distributive lattice where there exist Heyting implication and pseudo-difference.*

Belluce's functor

Theorem 7. *Let* $(A, \otimes, \oplus, {}^*, \wedge, \vee, \to, 0, 1)$ *be* $L_P G$*-algebra. Then the topological spaces of* A *and* $\beta^*(A)$ *are homeomorphic.*

The space Spec($\beta^*(A)$*) (= the set of prime filters of Gödel algebra* $\beta^*(A)$*) of* $\beta^*(A)$ *is a cardinal sum of chains.*

$L_P G$-space

- The set of prime Skolem filters of $L_P G$-algebra *A, ordered by inclusion,* is named by *$L_P G$-space*.

  Let $\mathbf{L_P GS}$ be the category of $L_P G$-spaces and strongly isotone symmetric maps

  $f: X \to Y$, i. e.

  $f(R_X(x)) = R_Y(f(x))$ and $f(R^{-1}{}_X(x)) = R^{-1}{}_Y(f(x))$.

Belluce's functor

Theorem 8. *Let* $\{A_i\}_{i\in I}$ *be a family of* L_PG *-algebras. Then*

$$\beta^*(\prod_{i\in I} A_i) \cong \prod_{i\in I} \beta^*(A_i)$$

Theorem 9. *Let* $f : A \to B$ *be a injective* L_PG *-homomorphism between* L_PG -algebras A and B. *Then* $\beta^*(f) : \beta^*(A) \to \beta^*(B)$

is a L_PG *–algebra injective homomorphism.*

Belluce's functor

Theorem 10. $\beta^*(F_{\mathbf{LpG}}(n))$ *is bi-Heyting algebra.*

Priestley space

A *Boolean space* X is zero-dimensional, compact and Hausdorf topological space.

A *Priestley space* is a triple (X,R), where X is a Boolean space and R is an order relation on X such that, for all $x,y \in X$ with xRy, there exists a clopen up-set V with $x \in V$ and $y \notin V$.

A morphism between Priestley spaces is a continuous order-preserving map.

Heyting space

Heyting space (or Esakia space) X is a Priestley space such that $R^{-1}(U)$ is open for every open subset U of X.

A morphism between Heyting spaces, called a strongly isotone map, is a continuous map $f : X \to Y$ such that $f(R(x)) = R(f(x))$ for all $x \in X$.

Heyting space

- *There exists the dual equivalence between the categories of bounded distributive lattices* **D** *and Priestley spaces* **PS**

- *There exists the dual equivalence between the categories* **HA** *of Heyting algebras and Heyting spaces* **HS**.

Gödel algebra

A Heyting algebra A is said to be *Gödel algebra* if it satisfies the linearity condition:

$$(a \to b) \vee (b \to a) = 1.$$

It is well known that the Heyting spaces for Gödel algebras form root systems. Specifically, Heyting algebra is a Gödel algebra if its set of prime lattice filters is a root system (ordered by inclusion). So we can define *Gödel space* X as such kind Heyting space that $R(x)$ is a chain for any $x \in X$.

Gödel algebra

- *There exists the dual equivalence between the categories of Gödel algebra* **G** *and* Gödel *spaces* **GS**
- Let $\mathbf{H}^2$ be the variety of be-Heyting algebras (symmetric Heyting algebras).

Let $\mathbf{G}^2$ be the variety selected from $\mathbf{H}^2$ by the identities:

$$(a \to b)\vee(b \to a) = 1,\ (a - b)\wedge(b - a) = 0.$$

Let $\mathbf{G}^2$S be the category of $\mathbf{G}^2$-spaces.

$L_P G$-algebras

Let $\mathbf{L_P G^Q}$ = LSP$\{C_n : n \in \omega)$ be the class of algebras generated from $\{C_n : n \in \omega)$ by the operators of direct product, subalgebras and direct limit.

This class is a full subcategory of the category of $L_P G$-algebras $\mathbf{L_P G}$.

$L_P G$-algebras

Taking into account that **$\mathbf{G}^2$** is locally finite and any algebra can be represented as direct limit of finitely generated subalgebras, we have that

$$\mathbf{G}^2 = \mathrm{LSP}\{\beta^*(C_n) : n \in \omega\}.$$

Duality

Theorem 12. *Let we have two categories*: $\mathbf{L_P G^Q}$ and $\mathbf{G^2 S}$. *Then there exist contravariant functor*

- $\mathtt{S} : \mathbf{L_P G^Q} \to \mathbf{G^2 S}$ *and contravariant functor*
- $\mathtt{K} : \mathbf{G^2 S} \to \mathbf{L_P G^Q}$ *such that* $\mathtt{K}(\mathtt{S}(A)) \cong A$ *for any object* $A \in \mathbf{L_P G^Q}$ *and* $\mathtt{S}(\mathtt{K}(X)) \cong X$ *for any object* $X \in \mathbf{GS}$, *i. e. the functors* $\mathtt{S}$ *and* $\mathtt{K}$ *are dense*.

Moreover, the functor $\mathtt{S} : \mathbf{L_P G^Q} \to \mathbf{G^2 S}$ *is full, but not faithful and the functor* $\mathtt{K} : \mathbf{G^2 S} \to \mathbf{L_P G^Q}$ *is faithful, but not full.*

Free $L_P G$-algebra

Theorem 13. *The algebra* $B^2 \times C^2$ *is a free 1-generated* $L_P G$*-algebra with free generator* (0,1,*c*, *c**), *where B is two-element Boolean algebra.*

Free $L_P G$-algebra

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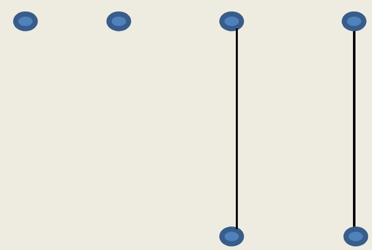

Free $L_P G$-algebra

- *$L_P G$-algebra C_m ($m > 0$) is generated by 2^m different generators and these are minimal number of different generators.*
- *For $1 < n < m$ C_n is generated by infinitely many different m generators.*

Free $L_P G$-algebra

- *m-generated free* $L_P G$*-algebra* $F_{\mathbf{LpG}}(m)$*, where* $1 < m$ *, contains as a homomorphic image the* $L_P G$*-algebras*

$$B^{2^m} \times C_m^{2^m} \text{ and } \prod_{i=1}^{k} C_n^{(i)}$$

for $0 < n < m$*,* $k \in \omega$*.*

$L_P G$*-space* $F_{\mathbf{LpG}}(m)$ *consists of cardinal sum of n-element chains, where* $1 \leq n \leq m$*.*

Projective L_P*G-algebras*

Theorem 14. *An L_PG-algebra A is finitely presented if A = $F_{LpG}(n)/[u]$ for some principal Scolem filter generated by $u \in F_{LpG}(n)$.*

Projective *L_PG-algebras*

Theorem 14. *An L_PG-algebra A is finitely presented if A = $F_{LpG}(n)/[u)$ for some principal filter generated by $u \in F_{LpG}(n)$.*

Theorem 15. *Any finitely presented algebra $A \in \mathbf{L_P G^Q}$ is projective.*

Projective $L_P G$-algebras

Theorem 16. *Let* $A \in \mathbf{L_P G^Q}$. *If* $\mathbb{S}(A)$ *is finite, then* A *is projective in* $\mathbf{L_P G^Q}$.

Corollary 17. *Any finite product of finitely generated totally ordered* $L_P G$*-algebras is projective.*

THANK YOU