The topological space of pre-orders on abelian groups

Almudena Colacito
Joint work with Vincenzo Marra

Mathematisches Institut
Universität Bern

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Introduction

In the literature, several ways of representing abelian ℓ-groups by means of topological spaces can be found.

In 1971, K. Keimel showed that every abelian ℓ-group can be represented as a group of sections of a sheaf of 'local' ℓ-groups. The base space for Keimel's sheaf representation is the so-called spectrum of the ℓ-group.

Other classical representations use, respectively, the minimal spectrum and the Stone space of the Boolean algebra of polars.

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The topological space of (right-)orders on a group G appeared for the first time in 2004, in a work by A. Sikora.

In his original paper, some of the results about this topological space were applied to obtain consequences in algebraic topology and commutative algebra.

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Abelian Lattice-Ordered Groups

An *abelian ℓ-group* is an algebra $(H, \wedge, \vee, +, -, 0)$, where $(H, +, -, 0)$ is an abelian group and $(H, \wedge, \vee)$ a lattice whose corresponding lattice-order is translation-invariant.

The class of abelian ℓ-groups is a *representable variety* of ℓ-groups, meaning that *subdirectly irreducible* members are *chains*.

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Prime Ideals

The variety of abelian ℓ-groups is *ideal-determined*: *congruences* are in one-to-one correspondence with *convex ℓ-subgroups* (*ideals*).

Congruences whose quotient is *totally ordered* correspond to *prime ideals* (we consider prime ideals to be *proper*).

The *spectrum* $\operatorname{Spec} H$ of an abelian ℓ-group H is the collection of its *prime ideals*. The set of prime ideals forms a *root system* under inclusion.

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Orders on Abelian Groups

An abelian group G is *orderable* if its elements can be *(totally) ordered* via a *translation-invariant* relation. We call such relation an *order on G*.

An abelian group is *orderable* if, and only if, it is *torsion-free*.

Given an order $\leq$ on G, the set of its *non-negative elements* $C \subseteq G$ is a submonoid of G with the properties $C \cup -C = G$ and $C \cap -C = \{0\}$, and we call such submonoid a *(total) cone* for G.

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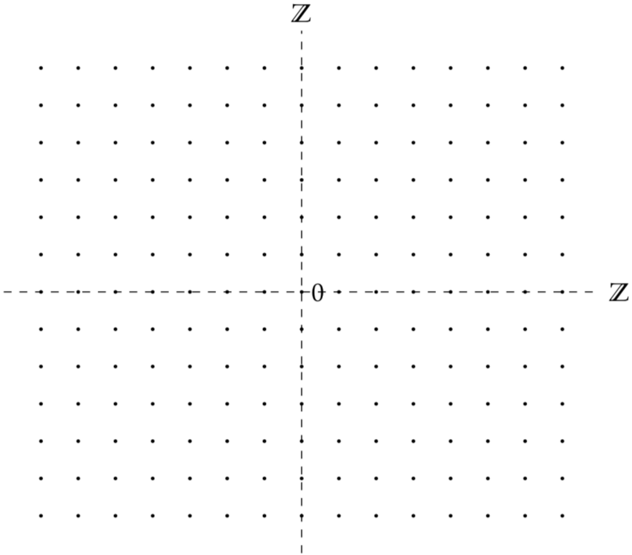

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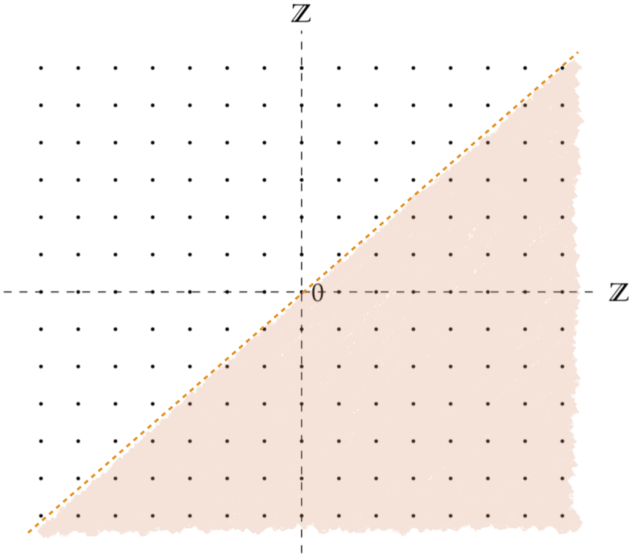

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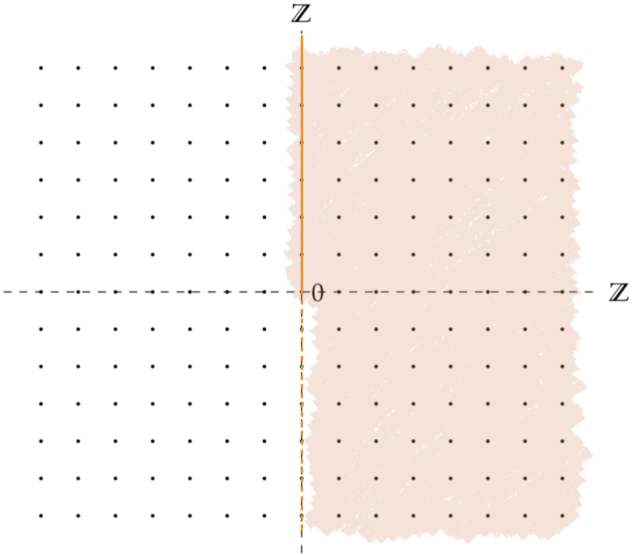

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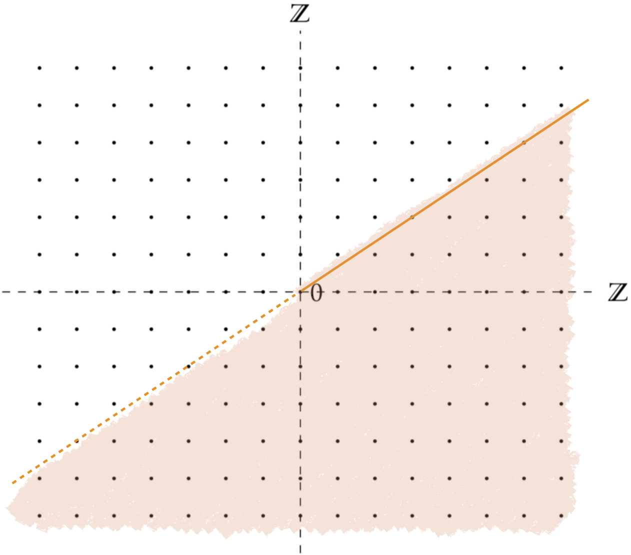

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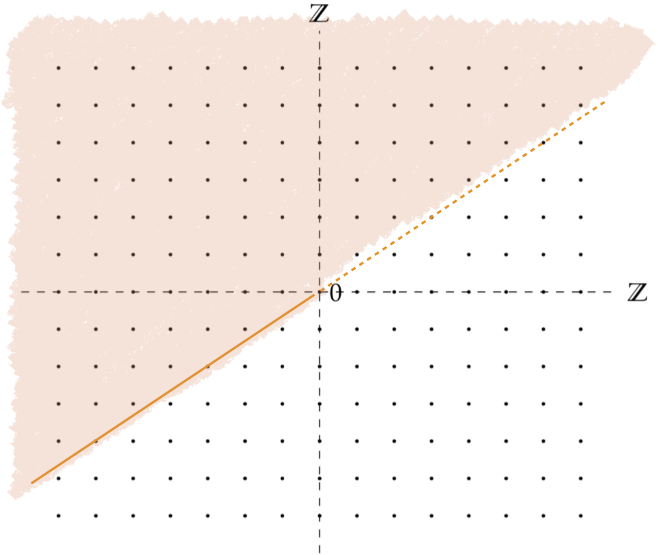

Free Abelian ℓ-Groups

Problem. Given a torsion-free abelian group G, we would like to establish a relation between $\mathscr{O}(G)$ and $\operatorname{Spec} H$, for some abelian ℓ-group H.

Where should we look for H?

For a torsion-free abelian group G, there exists an abelian ℓ-group $F(G)$ and a group homomorphism $\eta_G \colon G \to F(G)$ characterised by the following universal property: *For each group homomorphism $p \colon G \to H$, with H an abelian ℓ-group, there is exactly one ℓ-homomorphism $h \colon F(G) \to H$ such that the following diagram*

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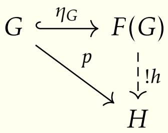

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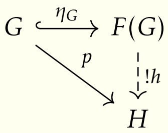

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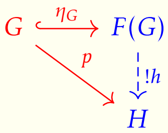

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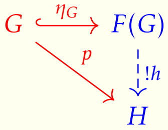

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Theorem (Weinberg, 1963)

Given a torsion-free abelian group G, the free abelian ℓ-group $F(G)$ over G is isomorphic to the ℓ-subgroup of the direct product ℓ-group

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The Minimal Layer

Problem. Given a torsion-free abelian group G, we would like to establish a relation between $\mathscr{O}(G)$ and $\operatorname{Min} F(G)$.

Given a cone C for G, consider the following diagram:

$$\begin{array}{ccc} G & \hookrightarrow & F(G) \\ & \searrow^{1_G} & \downarrow h_C \\ & & G_C \end{array}$$

If we denote by h_C the unique ℓ-homomorphism from $F(G)$ onto the totally ordered group G_C, the kernel $\ker h_C$ is a prime ideal. The idea is to make h_C as injective as possible on G, and hence, look for the 'smallest possible' prime ideals.

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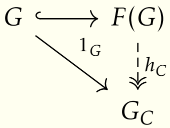

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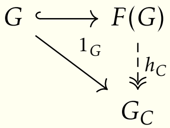

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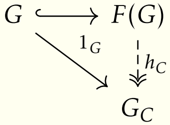

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Zooming Out

If the set of orders of a torsion-free abelian group G corresponds to the minimal spectrum of the free abelian ℓ-group $F(G)$ over G, is there a topological space arising from G that corresponds to $\operatorname{Spec} F(G)$?

Pre-Orders on Abelian Groups

A *(total) pre-order* on a torsion-free abelian group G is a *translation-invariant pre-order on G*.

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The Minimal Layer

The set $\mathscr{P}(G)$ of pre-cones for a torsion-free abelian group G forms a root system ordered by inclusion.

It is immediate that *cones are minimal* in $\mathscr{P}(G)$. Moreover:

Lemma

Every pre-cone for a torsion-free abelian group G extends a cone for G.

Main Idea.
The subgroup $C \cap -C$ is itself torsion-free abelian and hence, orderable. □

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The set $\mathscr{P}(G)$ of pre-cones for a torsion-free abelian group G forms a root system ordered by inclusion.

It is immediate that *cones are minimal* in $\mathscr{P}(G)$. Moreover:

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Given a pre-cone C for G, we consider the following diagram:

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\downarrow{\scriptstyle 1_G} & & \\
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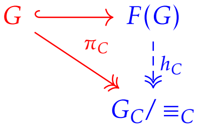

But then, if h_C is the unique ℓ-homomorphism from $F(G)$ onto $G_C/\equiv_C$:

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The map $\mathscr{P}(G) \xrightarrow{K_G} \operatorname{Spec} F(G)$ *defined by*

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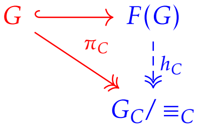

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Main Idea. The crucial steps of the proof rely on the following facts:

- The behaviour of an element $a \in G$ in a given pre-order C can be 'recognized' in $F(G)$ by the behaviour of the elements $h_C(a \wedge 0)$ and $h_C(a \vee 0)$ in G_C.

  E.g., to prove injectivity: observe that if two pre-cones $C, D \in \mathscr{P}(G)$ are different, there is an element $a \in C$ that is certainly *strictly negative* in the pre-order induced by D. Therefore, $\ker h_C \neq \ker h_D$, since $(a \wedge 0)$ will be contained in $\ker h_C$ but not in $\ker h_D$.

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Weinberg's Construction

Corollary (Weinberg, 1963)

Given a torsion-free abelian group G, the free abelian ℓ-group $F(G)$ over G is isomorphic to the ℓ-subgroup of the direct product ℓ-group

$$\prod_{C \in \mathscr{O}(G)} G_C$$

generated by the set $\{(a, a, \ldots, a, \ldots) \mid a \in G\}$.

The Spectral Space

For an abelian ℓ-group H, we consider $\operatorname{Spec} H$ endowed with the *'hull-kernel'* (or *Zariski*) *topology*, with basic open sets:

$$\mathbb{S}(x) = \{\mathfrak{p} \in \operatorname{Spec} H \mid x \notin \mathfrak{p}\},$$

for $x \in H$. The restriction of these sets to $\operatorname{Min} H$ form a basis for the subspace topology on $\operatorname{Min} H$:

$$\mathbb{S}_m(x) = \{\mathfrak{m} \in \operatorname{Min} H \mid x \notin \mathfrak{m}\}.$$

The resulting space $\operatorname{Min} H$ is a *Hausdorff zero-dimensional* space, not necessarily compact. The closed basis given by the sets

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We consider the topology on $\mathscr{P}(G)$ with subbasis given by the following sets, for $a \in G$:

$$\mathbb{P}_a = \{C \in \mathscr{P}(G) \mid a \in C \text{ and } a \notin -C\}.$$

The subset $\mathscr{O}(G)$ with the subspace topology amounts to Sikora's topological space of orderings (2004), and is therefore *compact* and *totally disconnected*. The following subsets

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Given a torsion-free abelian group G:

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The space $\mathscr{P}(G)$ is homeomorphic to $\operatorname{Spec} F(G)$ via a map that restricts to a homeomorphism between $\mathscr{O}(G)$ and $\operatorname{Min} F(G)$.

Main Idea. An element $a \in G$ is strictly positive in C if, and only if, $h_C(a \vee 0)$ is strictly positive in G_C.

Now, since $a \vee 0 \geq 0$ in $F(G)$ and h_C is order-preserving, $h_C(a \vee 0)$ is strictly positive exactly when $a \vee 0 \notin \ker h_C$. Therefore,

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Some Consequences

Given a torsion-free abelian group G:

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$\operatorname{Min} F(G)$ *is compact.*

Theorem (Ball, Marra, McNeill, and Pedrini, 2018)

For an abelian ℓ-group H, the following are equivalent:

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$(\{\mathbb{V}_m(x)\}_{x\in F(G)}, \cap, \cup)$ *is a Boolean algebra, and the space $\mathscr{O}(G)$ is its dual Stone space.*

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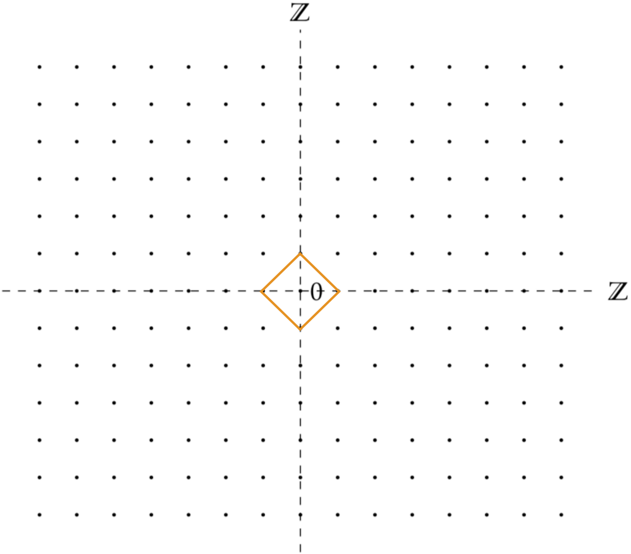

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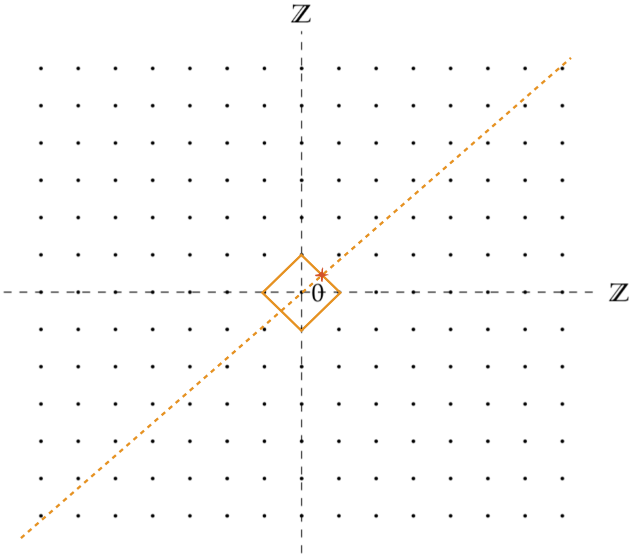

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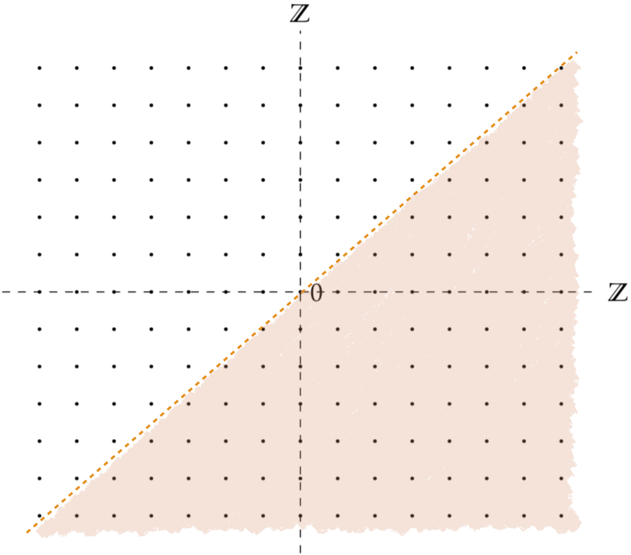

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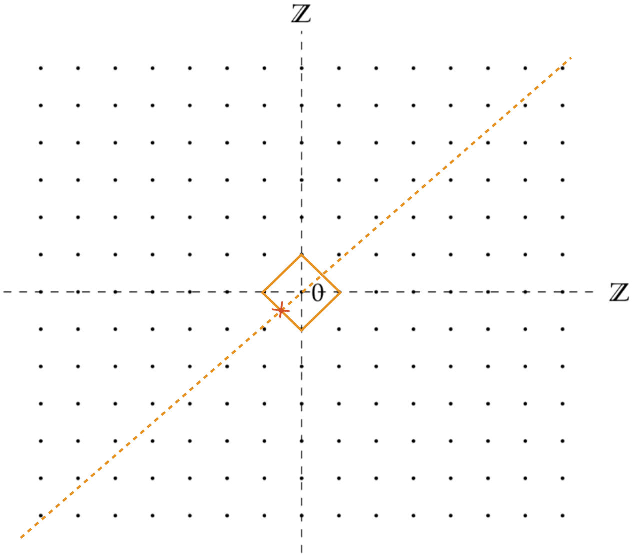

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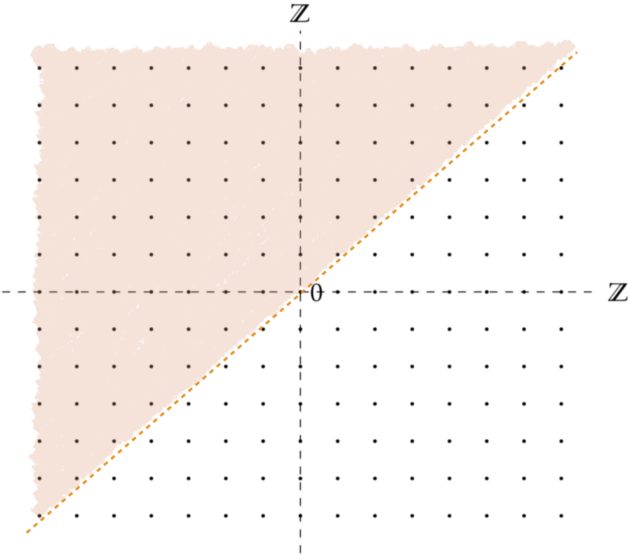

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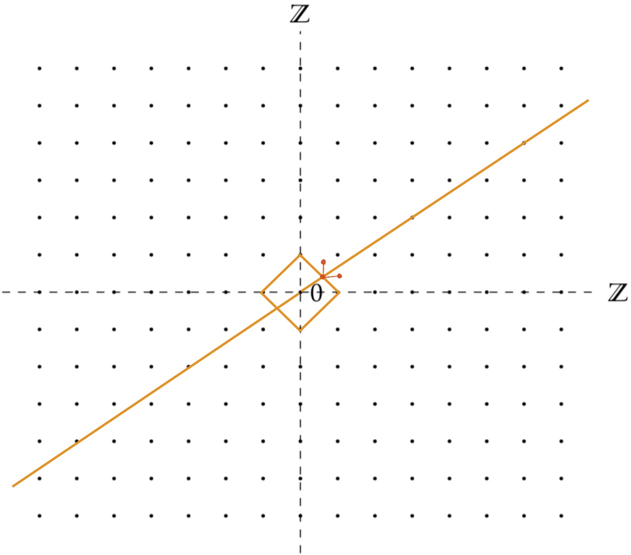

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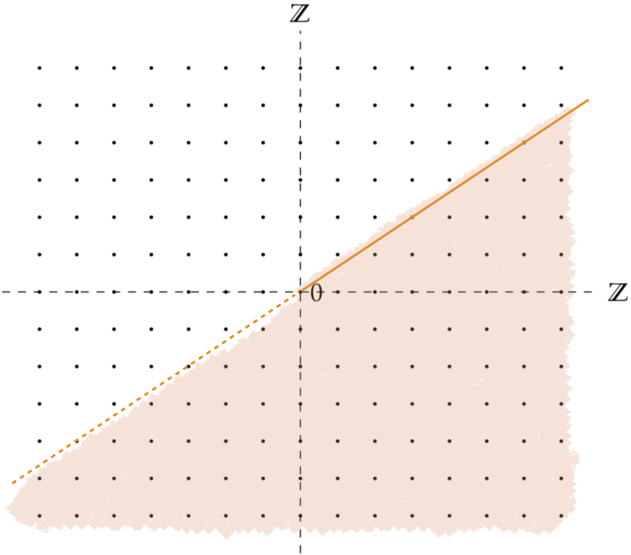

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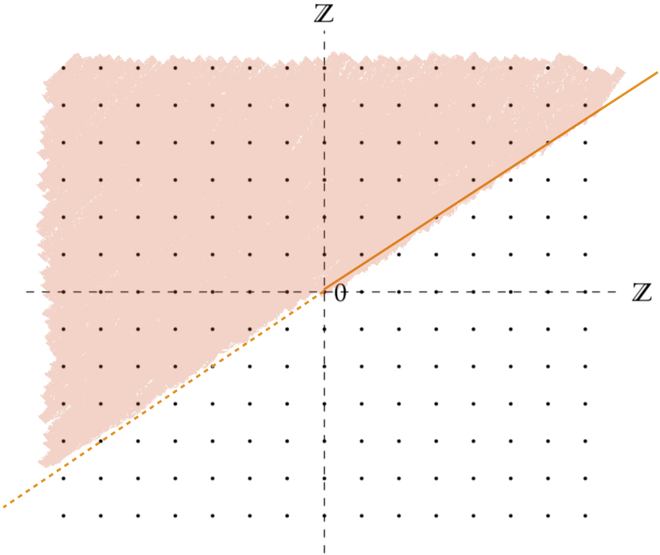

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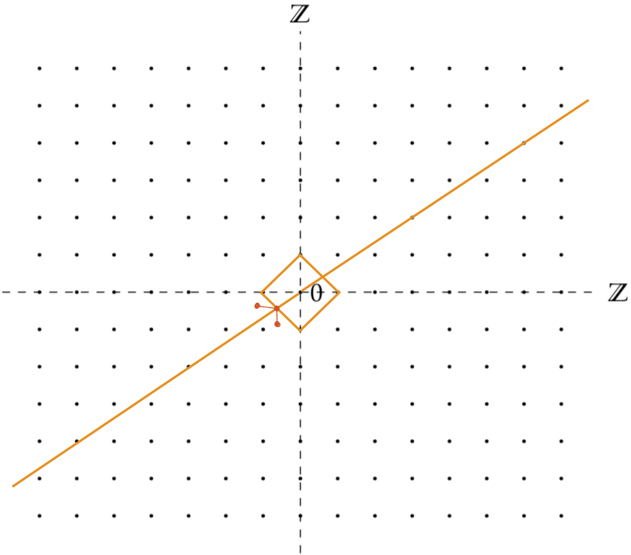

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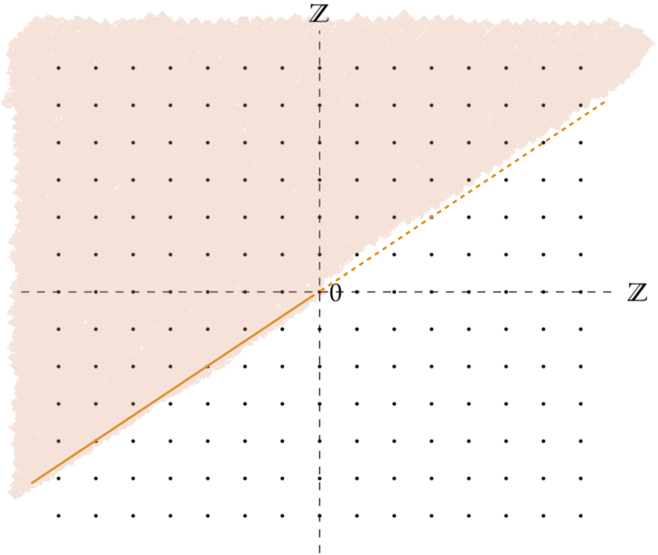

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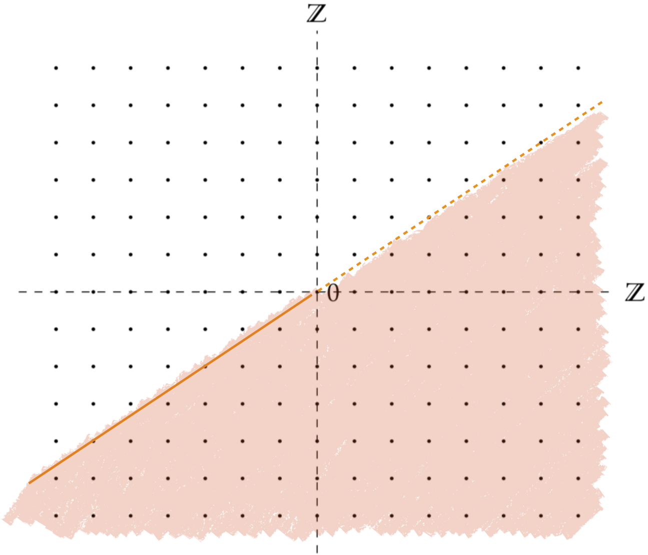

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The Non-Commutative Case

An Intermediate Step

Given an orderable group G, it is possible to define a topological space of orders on G (Sikora, 2004).

Question. Can we find an ℓ-group H_G that can be represented via the topological space of orders on G?

- Conrad's *'Free Lattice-Ordered Groups'* (1969) suggests that we should consider the free representable ℓ-group over G.
- In a representable ℓ-group, there are *enough* minimal prime ideals.

Conjecture. Given an orderable group G, the free representable ℓ-group $F(G)$ over G can be represented via $\mathscr{O}(G)$.

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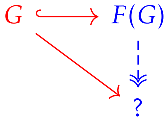

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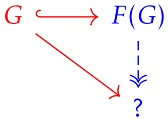

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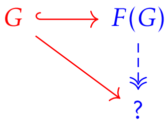

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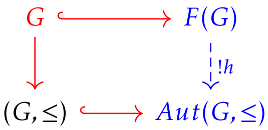

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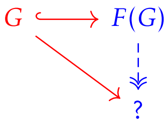

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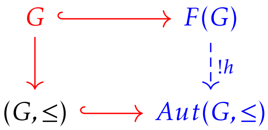

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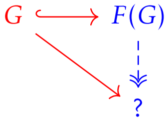
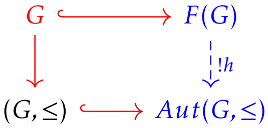


Conrad's construction of the free ℓ-group over a right-orderable group uses right-ordered copies $(G, \leq)$ of G indirectly, by using the ℓ-groups $Aut(G, \leq)$.



Which congruences of $F(G)$ correspond to $\ker h$?
Is it possible given $Aut(G, \leq)$ to 'canonically recover' the right order $\leq$ on G?