A logical calculus for compact Hausdorff spaces

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joint work with G. Bezhanishvili, T. Santoli, S. Sourabh
Y. Venema

Part 1: Dualities

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We will take another route to arrive at de Vries duality.

Our approach is based on the duality used in modal logic.

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1. $\Diamond 0 = 0$,
2. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.

Continuous relations

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2. $U \in \mathsf{Clop}(X) \Rightarrow R^{-1}[U] \in \mathsf{Clop}(X)$, where
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In other words, $R^{-1} : \mathsf{Clop}(X) \to \mathsf{Clop}(X)$ is well defined.

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Theorem (Esakia, 1974) R is continuous iff $\rho : X \to VX$ defined by $\rho(x) = R[x]$ is a well-defined continuous map, where VX is the Vietoris space.

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If $(B, \Diamond)$ is a modal algebra, then (X_B, R_B) is a modal space, where X_B is the Stone dual of B and R_B is defined by

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For each closed set F both $R[F]$ and $R^{-1}[F]$ are closed.

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1. R is closed.
2. R is closed in $X \times X$.

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What axioms does this binary relation validate?

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(S1) $0 \prec a \prec 1$ for each $a \in B$;

(S2) $a \prec b, c$ implies $a \prec b \wedge c$;

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$(\mathsf{Clop}(X), \prec)$ is a Boolean algebra with a subordination.

Boolean algebras with subordinations

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Then R_B is a closed relation on X_B.

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This correspondence can be extended to dualities of appropriate categories (G.B., N.B, S.S., Y.V., 2014).

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1. R is reflexive iff $\Box p \to p$ is valid.
2. R is symmetric iff $p \to \Box\Diamond p$ is valid.
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(S5) $a \prec b$ implies $a \leq b$;

(S6) $a \prec b$ implies $\neg b \prec \neg a$;

(S7) $a \prec b$ implies there is $c \in B$ with $a \prec c \prec b$;

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So if $(B, \prec)$ validates (S1)-(S7), then in its dual (X, R) the relation R is a closed equivalence relation.

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Sahlqvist correspondence for similar languages were studied by (Balbiani and Kikot, 2012) and (Santoli, 2016).

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Definition. An onto continuous map $\pi : X \to Y$ between compact Hausdorff spaces is called irreducible if the image of a proper closed set is proper.

The Gleason cover of a compact Hausdorff space Y is a pair (X, π), where X is an extremally disconnected (ED) Stone space and $\pi : X \to Y$ is an irreducible map.

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Let $\mathcal{RO}(X)$ be the Boolean algebra of regular open subsets of X, where

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- $\bigwedge_{i \in I} U_i = \mathbf{Int} \bigcap_{i \in I} U_i$,
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If Y is a compact Hausdorff space we take its Gleason cover (X, π), and define R on X by xRy if $\pi(x) = \pi(y)$.

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Then R is irreducible iff $(B, \prec)$ satisfies (S8).

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Moreover, since X is also ED, $\mathsf{Clop}(X)$ is complete.

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A de Vries algebra is a pair $(B, \prec)$, where B is a complete Boolean algebra and $\prec$ is a compingent relation.

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This representation can be extended to a full categorical duality.

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Theorem. $(\mathcal{RO}(Y), \prec)$ is isomorphic to $(\mathsf{Clop}(X), \prec)$.

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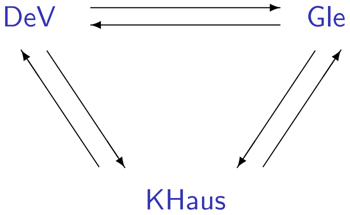
Corollary (de Vries, 1962) The category KHaus of compact Hausdorff spaces is dual to the category DeV of de Vries algebras.



Part 2: Logical calculi

Language

We will consider formulas in the following language:

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Similar semantics was considered by Esakia (1985).

A two sorted language to reason about pre-contact algebras was investigated by Balbiani, Tinchev and Vakarelov (2007).

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(I2) $(a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$;

(I3) $a \rightsquigarrow (b \wedge c) = (a \rightsquigarrow b) \wedge (a \rightsquigarrow c)$.

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(I2)-(I3) imply (S4).

Axiomatization

Other axioms can be rewritten as follows.

(I4) $a \rightsquigarrow b \leq a \to b$;

(I5) $a \rightsquigarrow b = \neg b \rightsquigarrow \neg a$;

(I6) $a \rightsquigarrow b = 1$ implies $\exists c : a \rightsquigarrow c = 1$ and $c \rightsquigarrow b = 1$;

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(I4)-(I7) correspond to (S5)-(S8)

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Corollary 2. (G.B., N.B., Santoli, Venema, 2017) The variety of strict implication algebras satisfying (I4) and (I5) is generated by BAs with subordinations satisfying (S5) and (S6).

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These are $\forall\exists$-statements.

Non-standard rules

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Such classes are called Π_2-classes.

Hierarchy

Formulas φ	$\leftrightsquigarrow$	varieties
Rules Γ/φ	$\leftrightsquigarrow$	quasi-varieties
Rules Γ/Δ	$\leftrightsquigarrow$	universal classes
?	$\leftrightsquigarrow$	Π_2-classes

Non-standard rules

A non-standard rule is one of the form:

$$(\rho) \quad \frac{F(\bar{\varphi}, \bar{p}) \to \chi}{G(\bar{\varphi}) \to \chi}$$

where χ is a formula variable, and F, G are formulas, each involving formula variables $\bar{\varphi}$, and with F involving a fresh tuple $\bar{p}$ of proposition letters.

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With the rule (ρ), we associate the first-order formula Φ_ρ, defined as:

$$\Phi_\rho \ := \quad \forall \bar{a}, b \in B \left(G(\bar{a}) \not\leq b \ \Rightarrow \ \exists \bar{c} : \ F(\bar{a}, \bar{c}) \not\leq b \right)$$

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Hierarchy

Logics	$\leftrightsquigarrow$	varieties
Consequence relations	$\leftrightsquigarrow$	quasi-varieties
Multi consequence relations	$\leftrightsquigarrow$	universal classes
Non-standard rule calculi	$\leftrightsquigarrow$	Π_2-classes

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$(\rho 6)$ corresponds to $(I6)$
$(\rho 7)$ corresponds to $(I7)$

Completeness

Theorem. (G. B., N. B., Santoli, Venema, 2017)
Let L be obtained by adding non-standard rules $\{\rho_i\}_{i\in I}$ to (I1)-(I5). Then L is sound and complete wrt the class of algebras satisfying $\{\Phi_{\rho_i}\}_{i\in I}$.

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What about topological completeness?

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Given a compingent algebra $(B, \prec)$ we take the MacNeille completion $\overline{B}$ of B.

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Theorem.

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Theorem. Let $(B, \prec)$ be a component algebra and let X be its de Vries dual. Then $(\mathcal{RO}(X), \prec)$ is isomorphic to $(\overline{B}, \lhd)$

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3. (I1)-(I5) + $(\rho 6), (\rho 7)$ is sound and complete wrt compact Hausdorff spaces.

Thank you!

Completeness of Stone spaces

Consider

(S9) $\forall a, b(a \rightsquigarrow b = 1 \Rightarrow \exists c \in B :$
$c \rightsquigarrow c = 1 \;\&\; a \rightsquigarrow c = 1 \;\&\; c \rightsquigarrow b = 1).$

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and the corresponding non-standard rule

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Compingent algebras validating (S9) are closed under MacNeille completions.

Completeness of Stone spaces

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Corollary (G. B., N. B., Santoli, Venema, 2017)

1. (I1)-(I5) + $(\rho 6), (\rho 7), (\rho 9)$ is sound and complete wrt de Vries algebras satisfying (S9).
2. (I1)-(I5) + $(\rho 6), (\rho 7), (\rho 9)$ is sound and complete wrt Stone spaces.

Propositional logic for Quasi-Priestley spaces

$(B, \prec)$ satisfies (S1)-(S5) + $(S9)$ iff its dual (X, R) is a Quasi-Priestley space.

Corollary

1. (I1)-(I5) + $(\rho 9)$ is sound and complete wrt BAs with subordinations satisfying (S5) and (S9).
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Thus our simple propositional calculus could be used to reason about Quasi-Priestley spaces (J. Haenen's Master's thesis, 2018).

Thank you!