A logical calculus for compact Hausdorff spaces

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Compact Hausdorff spaces are one of the most studied classes of topological spaces. There are many different approaches to duality theory for compact Hausdorff spaces. To name a few:

(i) through rings of (real-valued) continuous functions (Gelfand, Kolmogoroff, Stone),

(ii) through the frame of opens (Isbell),

(iii) through the σ-frame of cozero sets (Banaschewski),

(iv) through the Boolean frame of regular opens equipped with a proximity-like relation (de Vries).

Our guiding example will be the latter. In recent years there has been a renewed interest in the modal logic community toward Boolean algebras equipped with binary relations. The study of such relations and their representation theory has a long history, and is related to the study of pointfree geometry, pointfree topology, and region based theory of space. Our primary examples of Boolean algebras with relations will be de Vries algebras (Boolean frames equipped with a proximity-like relation). Our main goal is to use the methods of modal logic and universal algebra to investigate the logical calculi of Boolean algebras with binary relations. This will lead, via de Vries duality, to simple propositional calculi for compact Hausdorff spaces, Stone spaces, etc.

As was shown by Dimov and Vakarelov, pre-contact relations on Boolean algebras dually correspond to closed relations on Stone spaces. Earlier Celani developed a duality theory for Stone spaces equipped with closed relations by means of quasi-modal operators on Boolean algebras. Instead of working with quasi-modal operators or pre-contact relations, we prefer to work with subordinations, which are dual to pre-contact relations. This allows us to provide an alternative approach to de Vries duality. Based on this perspective, we utilize the non-standard rule approach of Balbiani, Tinchev, and Vakarelov to build a simple propositional calculus for compact Hausdorff spaces. We use MacNeille completions to show that this calculus is sound and complete for compact Hausdorff spaces. We will also discuss other completeness results with respect to interesting subclasses of the class of compact Hausdorff spaces, as well as similarities and differences with related work.