International Workshop on Topological Methods in Logic V 2016

Tbilisi (Georgia) 13- 17 June 2016

ON THE THEORY OF PERFECT MONADIC MV-ALGEBRAS

Revaz Grigolia

(joint work with A. Di Nola and G. Lenzi)

Introduction

The predicate Lukasiewicz (infinitely valued) logic *QL* is defined in the following standard way. The existential (universal) quantfier is interpreted as supremum (infimum) in a complete *MV* –algebra. Then the valid formulas of predicate calculus are defined as all formulas having value 1 for any assignment.

The functional description of the predicate calculus is given by Rutledge. Scarpellini has proved that the set of valid formulas is not recursively enumerable.

Introduction

Let L denote the first order language based on $\cdot$, +, $\rightarrow$, $\neg$, $\exists$ and L_m denotes monadic propositional language based on $\cdot$, +, $\rightarrow$, $\neg$, $\exists$, and *Form*(L) and *Form*(L_m-) - the set of formulas of L and L_m, respectively. We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p^*(x)$ in L and define by induction a translation Ψ: *Form*(L_m) $\rightarrow$ *Form*(L) by putting:

- $\Psi(p) = p^*(x)$ if p is a propositional variable,
- $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \rightarrow$,
- $\Psi(\exists\alpha) = \exists x \Psi(\alpha)$.

Through this translation Ψ, we can identify the formulas of L_m with monadic formulas of L containing the variable x.

Introduction

Monadic MV -algebras were introduced and studied by Rutledge as an algebraic model for the predicate calculus *QL* of Lukasiewicz infinite-valued logic, in which only a single individual variable occurs.

Rutledge followed P.R. Halmos' study of monadic Boolean algebras.

MV-algebras

An ***MV-algebra*** is an algebra

$$A = (A, \oplus, \otimes, \neg, 0, 1),$$

where $(A, \oplus, 0)$ is an abelian monoid, and for all $x, y \in A$ the following identities hold:

$x \oplus 1 = 1$, $\neg\neg x = x$,

$\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$,

$x \otimes y = \neg(\neg x \oplus \neg y)$.

Lukasiewicz Logic
MV-algebras

The unit interval of real numbers [0, 1] endowed with the following operations:

$x \oplus y = \min(1, x + y)$, $x \otimes y = \max(0, x + y - 1)$,

$x^* = 1 - x$, becomes an *MV* –algebra

$S=([0, 1], \oplus, \otimes, {}^*, 0, 1)$.

For $(0 \neq) m \in \omega$ we set

$S_m = (\{0, 1/m, \ldots, m\text{-}1/m, 1\}, \oplus, \otimes, {}^*, 0, 1)$.

Lukasiewicz Logic

Lukasiewicz logic was originally defined in the early 20th-century by Jan Lukasiewicz as a three-valued logic. It was later generalized to *n*-valued (for all finite *n*) as well as infinitely-many valued variants, both propositional and first-order.

Lukasiewicz Logic

The original system of axioms for propositional infinite-valued Lukasiewicz logic used implication and negation as the primitive connectives as for classical logic:

- L_1. $(\alpha \to (\beta \to \alpha))$
- L_2. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
- L_3. $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$
- L_4. $(\neg \beta \to \neg \alpha) \to (\alpha \to \beta)$.

There is only one inference rule - *Modus Ponens*: from α and $(\alpha \to \beta)$, infer β.

Perfect *MV* -algebras

- ***Perfect*** *MV* -algebras are those *MV* –algebras generated by their infinitesimal elements or, equivalently, generated by their radical, where radical is the intersection of all maximal ideals, the radical of an MV-algebra, will be denoted by Rad(A).
- [**A. Di Nola, A. Lettieri**, *Perfect MV-algebras are Categorically Equivalent to Abelian ℓ-Groups*, Studia Logica, 88(1994), 467-490.]

Perfect *MV* -algebras

Let we have any *MV* -algebra. The least integer for which $nx = 1$ is called *the order of* x. When such an integer exists it is denoted by $ord(x)$ and say that x has finite order, otherwise we say that x has infinite order and write $ord(x) = \infty$.

An *MV*-algebra *A* is called *perfect* if for every nonzero element $x \in A$

$ord(x) = \infty$ if and only if $ord(\neg x) < \infty$.

Perfect *MV* -algebras do not form a variety and contains non-simple subdirectly irreducible *MV*–algebras. The variety generated by all perfect *MV* –algebras is also generated by a single *MV* -chain, actually the *MV* –algebra C, defined by Chang. The algebra C, with generator $c \in C$, is isomorphic to $\Gamma(Z \times_{lex} Z,(1, 0))$, with generator (0, 1). Let **MV(C)** be the variety generated by perfect algebras.

Perfect *MV* -algebras

Each perfect *MV*-algebra is associated with an abelian ℓ-group with a strong unit. Moreover,

- ***the category of perfect MV–algebras is equivalent to the category of abelian ℓ-groups.***

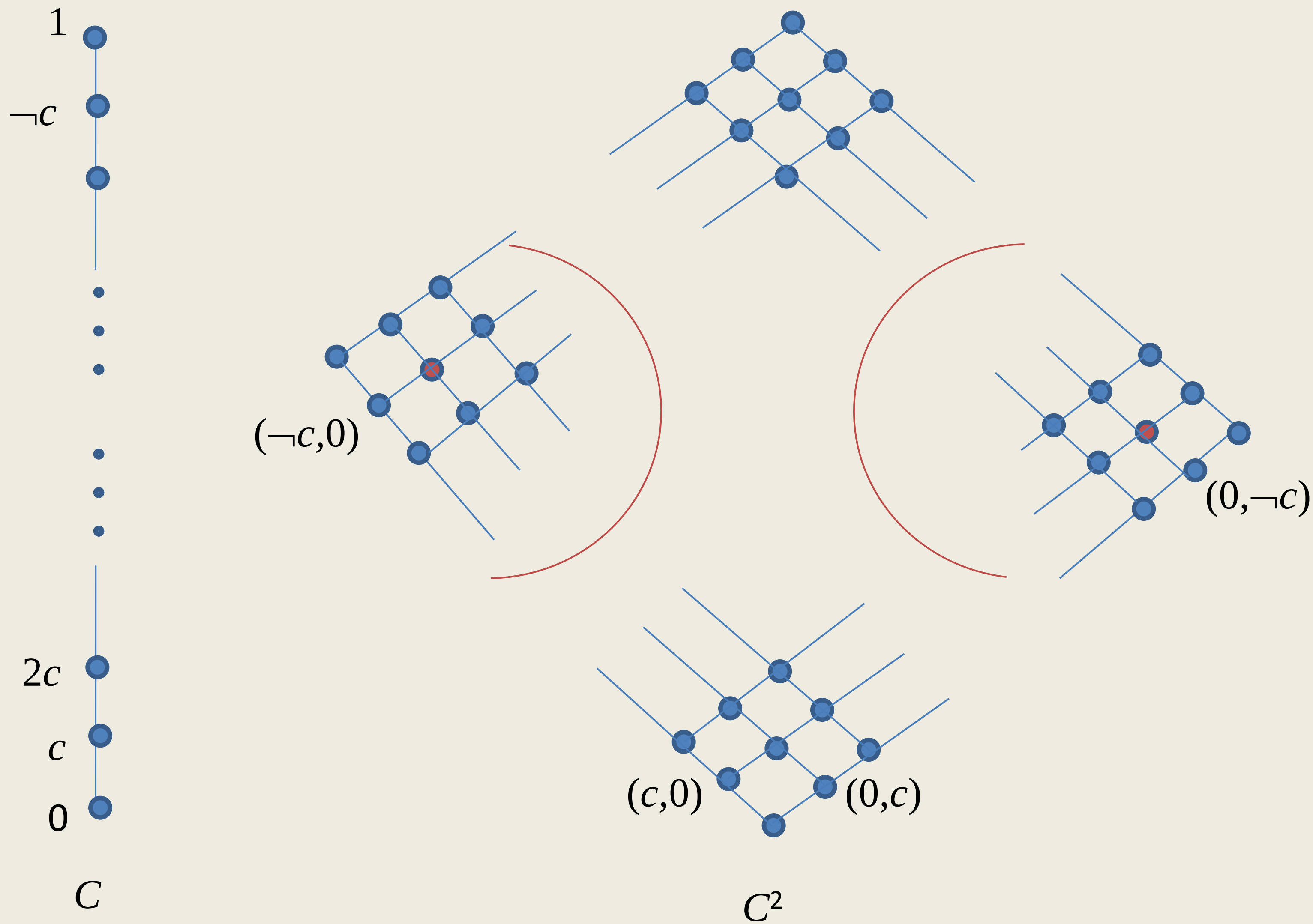
The variety generated by all perfect *MV*-algebras, denoted by ***MV(C),*** *is also generated* by a single *MV*-chain, actually the *MV*-algebra *C,* defined by Chang.

[**A. Di Nola, A. Lettieri**, *Perfect MV-algebras are Categorically Equivalent to Abelian ℓ-Groups*, Studia Logica, 88(1994), 467-490.]

Perfect *MV* -algebras

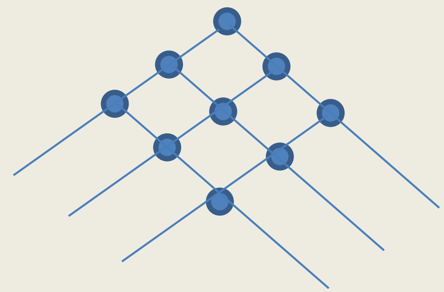
An important example of a perfect *MV*-algebra is the subalgebra *S of the Lindenbaum algebra L of the first or*der Lukasiewicz logic generated by the classes of formulas which are valid when interpreted in [0, 1] *but* non-provable.

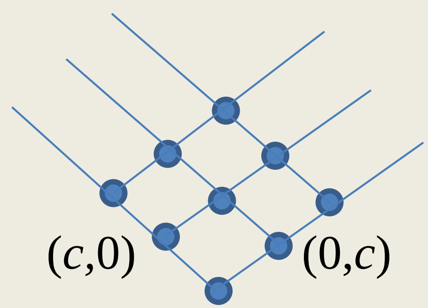
Hence perfect MV-algebras are directly connec-ted with the very important phenomenon of incompleteness in Lukasiewicz first order logic .



1
¬c
2c
c
0
C
(¬c,0)
(0,¬c)
(c,0)
(0,c)
C2

Perfect *MV* -algebras





$\mathrm{Rad}(C^2) \cup \neg\mathrm{Rad}(C^2)$

Perfect *MV* -algebras

The *MV*-algebra *C* is the subdirectly irreducible *MV*-algebra with infinitesimals. It is generated by an atom *c*, which we can interpret as

a quasi false truth value.

The negation of *c* is

***a quasi true value*.**

Now quasi truth or quasi falsehood are vague concepts.

Perfect *MV* -algebras

About quasi truth in an MV algebra, it is reasonable to accept the following propositions:

- ***there are quasi true values which are not*** 1;
- 0 ***is not quasi true***;
- ***if x is quasi true, then*** x^2 ***is quasi true***

(where x^2 denotes the MV algebraic product of x with itself).

In *C*, to satisfy these axioms it is enough to say that the quasi true values are the

co-infinitesimals.

Notice, that there is no notion of quasi truth in [0, 1] satisfying the previous axioms.

Perfect *MV* -algebras

Let L_P be the logic of perfect *MV*-algebras which coincides with the set of all Lukasiewicz formulas that are valid in all perfect *MV*-chains, or equivalently, that are valid in the *MV*-algebra *C*.

Actually, L_P is the logic obtained by adding to the axioms of Lukasiewicz sentential calculus the following axiom:

$$(x \veebar x) \,\&\, (x \veebar x) \leftrightarrow (x \,\&\, x) \veebar (x \,\&\, x)$$

[L. P. Belluce, A. Di Nola, B. Gerla, *Perfect MV -algebras and their Logic*, Applied Categorical Structures, Vol. 15, Num. 1-2 (2007), 35-151].

Perfect *MV* -algebras

Notice, that the Lindenbaum algebra of L_P is an $MV(C)$-algebra.

Perfect *MV* -algebras

An *MV-algebra* is *MV(C)-algebra* if in addition holds

$$(2x)^2 = 2x^2 .$$

[**A. Di Nola, A. Lettieri**, *Perfect MV-algebras are Categorically Equivalent to Abelian ℓ-Groups*, Studia Logica, 88(1994), 467-490.]

Monadic MV-algebras

- An algebra A $=(A, \oplus, \otimes, {}^*, \exists, 0, 1)$ (also denoted as $(A, \exists)$) is said to be *monadic MV-algebra* (for short MMV-algebra) [**A. Di Nola, R. Grigolia**, *On Monadic MV-algebras, APAL*, Vol. 128, Issues 1-3 (August 2004), pp. 125-139.]
- if $(A, \oplus, \otimes, {}^*, 0, 1)$ is an MV-algebra and in addition $\exists$ satisfies the following identities:

E1. $x \leq \exists x$,

E2. $\exists(x \vee y) = \exists x \vee \exists y$,

E3. $\exists(\exists x)^* = (\exists x)^*$,

E4. $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,

E5. $\exists(x \otimes x) = \exists x \otimes \exists x$,

E6. $\exists(x \oplus x) = \exists x \oplus \exists x$.

Monadic MV-algebras

A subalgebra A_0 of an MV-algebra A is said to be ***relatively complete*** if for every $a \in A$ the set $\{b \in A_0 : a \leq b\}$ has the least element, which is denoted by $inf\,\{b \in A_0 \leq b\}$.

The MV-algebra $\exists A$ (= $\{\exists a : a \in A\}$) is a relatively complete subalgebra of the MV-algebra $(A, \oplus, \otimes, *, 0, 1)$, and $\exists a = inf\{b \in \exists A : \mathrm{a} \leq \mathrm{b}\}$ [R].

A subalgebra A_0 of an MV-algebra A is said to be ***m-relatively complete*** [**A. Di Nola, R. Grigolia**, *On Monadic MV-algebras, APAL*, Vol. 128, Issues 1-3 (August 2004), pp. 125-139.] , if A_0 is relatively complete and two additional conditions hold:

(#) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \otimes a \Rightarrow \mathrm{v} \geq a \ \&\ v \otimes v \leq x)$,

(##) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)\ (x \geq a \oplus a \Rightarrow v \geq 0 \ \&\ v \oplus v \leq x)$.

Monadic MMV(C)-algebras

m-relatively complete subalgebra of C coincides with C but not its two-element Boolean subalgebra. In other words, $(C, \exists)$ is monadic MV(C)-algebra if $\exists x = x$.

Let we have C^n for some non-negative integer n. Then $(C^n, \exists)$ will be MMV (C)-algebra, where $\exists(a_1, \dots, a_n) = \max\{a_1, \dots, a_n\}$ and $\forall(a_1, \dots, a_n) = \min\{a_1, \dots, a_n\}$.

In this case $\exists(C^n) = \{(x, \dots, x) \in C^n : x \in C\}$.

Notice, that $(C^n, \exists)$ is subdirectly irreducible.

Monadic MMV(C)-algebras

Let

$$Alt^C{}_m = \forall(2x_1{}^2 \vee \forall(2x_1{}^2 \to 2x_2{}^2) \vee \ldots$$
$$\vee \forall(2x_1{}^2 \wedge 2x_2{}^2 \wedge \ldots \wedge 2x_m{}^2 \to 2x_{m+1}{}^2),$$

for $0 < m \in \omega$.

Let $\mathbf{K}^m$ be the subvariety of **MMV(C)** defined by the *identity* $Alt^C{}_m = 1$.

Monadic MMV(C)-algebras

Theorem 1. *There is no a variety* **V** *between the varieties* $\mathbf{K}^{m}$ and $\mathbf{K}^{m+1}$ which distinct from $\mathbf{K}^{m}$ for $0 < m \in \omega$.

Monadic MMV(C)-algebras

Theorem 2. $\mathcal{V}(\bigcup_{k\in\omega}\mathbf{K}^k) = \mathbf{MMV(C)}$.

Theorem 3. *Let us suppose that a subdirectly irreducible algebra A* $\in$ **MMV(C)**, *which is not monadic Boolean algebra, does not satisfy* $Alt^C{}_m = 1$ *for any positive integer m. Then A generate* **MMV(C)**.

Monadic MMV(C)-algebras

So we have:

$$\mathbf{K}^{1} \subset \mathbf{K}^{2} \subset \ldots \subset \mathbf{K}^{m} \subset \ldots \ \mathbf{MMV(C)}$$

Fig. 1

Monadic MMV(C)-algebras

Theorem 4. *The identity* $(\exists x)^2 \wedge (\exists x^*)^2 = 0$ *is satisfied in the subdirectly irreducible MMV(C)-algebra* $(A, \exists)$ *if and only if the MV - algebra reduct of that is perfect MV-algebra*.

From the variety **MMV(C)** we can pick out the subvariety $\mathbf{MMV(C)}_p$ by the identity $(\exists x)^2 \wedge (\exists x^*)^2 = 0$ which is generated by *MMV (C)*-algebras the *MV*-algebra reduct of which are perfect MV-algebras. Notice that this variety coincides with the variety $\mathbf{K}^1$.

Monadic MMV(C)-algebras

Let

$$\Psi_n = (\bigvee_{i=1}^{n} t(x_i) \to t(x_{i+1})) \vee (\bigvee_{i=1}^{n} t(x_{n+1}) \to t(x_i))$$

where $0 < m \in \omega$, $t(x) = (x \vee x*) \oplus \forall(x \vee x*)$.

Monadic MMV(C)-algebras

Theorem 5. *The identity* $\Psi_n = 1$ *is true in* $(C^k, \exists)$ *for* $1 < k \leq n$ *and* $\Psi_n = 1$ *does not hold in* $(C^k, \exists)$ *for* $k > n$.

Monadic MMV(C)-algebras

Let $\mathbf{K}^k{}_n = \mathbf{K}^k + \Psi_n = 1,\ k \leq n.$

Notice that $\mathbf{K}^1{}_1$ coincides with the variety of monadic MV-algebras with trivial monadic operator $\exists x = x$.

Let **MB** be the variety of monadic Boolean algebras and $\mathbf{MB}_m$ the subvariety of **MB** generated by $(\mathbf{2}^m, \exists)$ where $1 \leq m < \omega$.

Monadic MMV(C)-algebras

Theorem 6. *There is no variety* **V** *between varieties* $\mathbf{K}^k{}_n$ *and* $\mathbf{K}^k{}_{n+1}$ *which is distinct from* $\mathbf{K}^k{}_n$ *and* $\mathbf{K}^k{}_{n+1}$ *where* $k \leq n$.

Monadic MMV(C)-algebras

$$
\begin{array}{cccccccccccc}
\mathbf{K}^1 & \subset & \mathbf{K}^2 & \subset & \mathbf{K}^3 & \subset & \ldots & \subset & \mathbf{K}^m & \subset & \ldots & \mathbf{MMV(C)} \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & \\
\cup & & \cup & & \cup & & & & \cup & & & \\
\mathbf{K}^1{}_m & \subset & \mathbf{K}^2{}_m & \subset & \mathbf{K}^3{}_m & \subset & \ldots & \subset & \mathbf{K}^m{}_m & & & \\
\vdots & & \vdots & & \vdots & & & & & & & \\
\cup & & \cup & & \cup & & & & & & & \\
\mathbf{K}^1{}_3 & \subset & \mathbf{K}^2{}_3 & \subset & \mathbf{K}^3{}_3 & & & & & & & \\
\cup & & \cup & & & & & & & & & \\
\mathbf{K}^1{}_2 & \subset & \mathbf{K}^2{}_2 & & & & & & & & & \\
\cup & & \cup & & \cup & & & & \cup & & & \cup \\
\mathbf{MB}_1 & \subset & \mathbf{MB}_2 & \subset & \mathbf{MB}_3 & \subset & \ldots & \subset & \mathbf{MB}_m & \subset & \ldots & \mathbf{MB}
\end{array}
$$

If m is a positive integer, then a *partition* of m is a non-increasing sequence of positive integers $(k_1, k_2, \dots , k_r)$ whose sum is n. Each p_i is called a *part* of the partition.

For example, for the number 4: 1 + 1 + 1 + 1 = 2 + 1 + 1 = 2 + 2 = 3 + 1 = 4.

Let p(m) be the set of all partitions of the number m. Let p(m, n) be the set of all partitions of the number m with n parts. For example p(4, 2) = {(2, 2), (3, 1)}.

Theorem 7. *The variety* $\mathbf{K}^n{}_m$ *is generated by the algebra* $((R^*(C^{m_1}) \times \dots \times R^*(C^{m_n})^n), \exists)$, *where* $m = m_1 + \dots + m_n$

Monadic MMV(C)-algebras

According to the results we can define generating set of algebras for some subvarieties.

- $\mathbf{K}^1 = \mathcal{V}(\{(R^*(C_1^m), \exists)): 1 < m \in \omega\}), \ldots ,$
- $\mathbf{K}^n = \mathcal{V}(\{((R^*(C_1^m))^n, \exists)): 1 \leq m \in \omega\}, 1 < n \in \omega,$
- $\mathbf{K}^n_m = \mathcal{V}(((R^*(C^{m_1}) \times \ldots \times R^*(C^{m_n})^n), \exists)),$
- $\mathbf{MB}_m = \mathcal{V}((\mathbf{2}^m, \exists)), m \in \omega,$

where $R^*(A) = \mathrm{Rad}\, A \cup (\mathrm{Rad} A)^*$.

Monadic MMV(C)-algebras

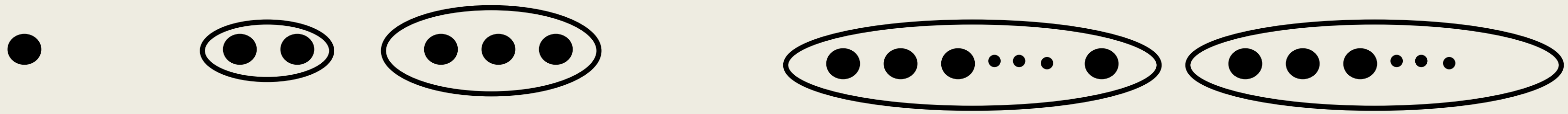
- **Theorem 8**. *Any proper subvarieties* $\mathbf{V}_1$,$\mathbf{V}_2$ *of the variety* **MMV(C)** *can be distinguished by two kind of identities* $Alt^C{}_m = 1$ and $\Psi_n=1$, $1 \leq m, n \in \omega$.

Monadic MMV(C)-algebras

- **Theorem 9**. (Main Theorem). *Let* **V** *be a proper subvariety of the variety* **MMV(C)** *of all* MMV (C)-*algebras. Then one of the following statement holds*:
- (i) **V** = **MB**;
- (ii) *there is an integer m such that* **V** = $\mathbf{MB}_m$;
- (iii) *there is a positive integer m such that* **V** = $\mathbf{K}^m$;
- (iv) *there are positive integers h, k such that* $h \leq k$ *and* **V** = $\mathbf{K}^h_k$.

Monadic MMV(C)-algebras

$$\mathbf{MB}_1 \subset \mathbf{MB}_2 \subset \mathbf{MB}_3 \subset \ldots \subset \mathbf{MB}_m \subset \ldots \qquad \mathbf{MB}$$

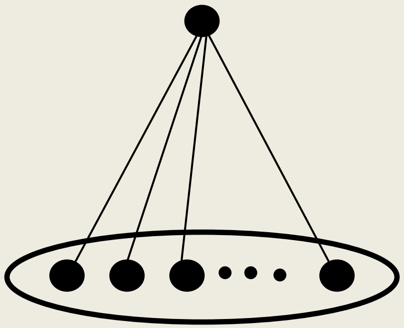


$$\mathbf{MB}_m = \mathcal{V}((\mathbf{2}^m, \exists)),\ m \in \omega$$

Monadic MMV(C)-algebras

$\mathbf{K}^1 \subset \mathbf{K}^2 \subset \mathbf{K}^3 \subset \ldots \subset \mathbf{K}^m \subset \ldots$ **MMV(C)**

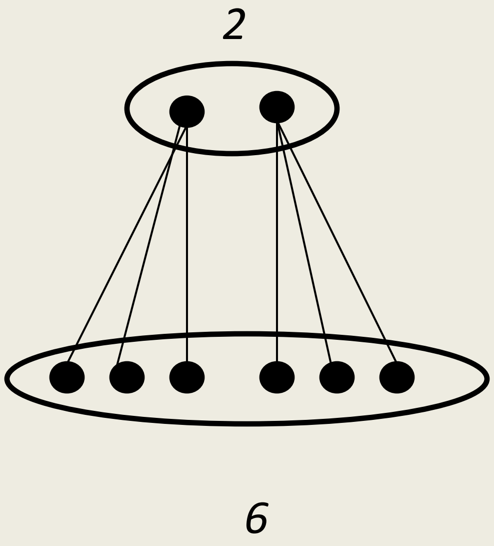
$\mathbf{K}^1 = \mathcal{V}(\{(\mathrm{R}^*(C_1{}^m), \exists)): m \in \omega\})$



$Spec((\mathrm{R}^*(C_1{}^m), \exists))$

Monadic MMV(C)-algebras

$((\mathrm{R}^*(C_1{}^3))^2, \exists) \in \mathbf{K}^{2}_{6}$



Spec $((\mathrm{R}^*(C_1{}^3))^2, \exists))$

THANK YOU