Bisimulation games and formula depth

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Introduction

- **Ehrenfeucht-Fraïssé games** are well known in classical model theory, they are used to study elementary equivalence.
- **Bisimulation games** are their analogues for Kripke models and modal and intermediate logic.

Bisimulation games have been used
- for constructing bisimulations
- for study of expressive power of logical languages
- for completeness proofs in modal logic
- for proofs of local finiteness in modal and intermediate logics (and more exactly, for classifying formulas).
Modal propositional language

*N-modal formulas* are built from a countable set of proposition letters $\{p_1, p_2, \ldots\}$ using boolean connectives and unary modal connectives $\square_1, \ldots, \square_N$; as usual $\Diamond_i = \neg \square_i \neg$

If $N=1$ we denote the modalities just by $\square$ and $\Diamond$.

*The modal depth* $md(A)$ is defined by induction:

- $md(p_i) = 0$, $md(\neg A) = md(A)$,
- $md(A \lor B) = md(A \land B) = \max(md(A), md(B))$,
- $md(\square_i A) = md(A) + 1$
Intuitionistic propositional language

*Intuitionistic formulas* are built from $\text{PL} = \{p_1, p_2, \ldots\}$ and the connectives $\wedge, \vee, \rightarrow, \bot$.

$\top A := A \rightarrow \bot$

*The implication depth* $\text{di}(A)$ is defined by induction:

$\text{di}(p_i) = \text{di}(\bot) = 0,$

$\text{di}(A \lor B) = \text{di}(A \land B) = \max(\text{di}(A), \text{di}(B)),$

$\text{di}(A \rightarrow B) = \max(\text{di}(A), \text{di}(B)) + 1.$
Logics-1

An *N-modal logic* is a set of N-modal formulas L such that:

- L contains all boolean tautologies
- L is closed under Modus Ponens: if A, A → B ∈ L, then B ∈ L.
- L is closed under Substitution:
  
  if A(p₁,...,pₙ) ∈ L, then A(B₁,...,Bₙ) (for any formulas B₁,...,Bₙ)
  
- if A ∈ L, then □ᵢA ∈ L
- □ᵢ(A → B) → (□ᵢA → □ᵢB) ∈ L

The *minimal logic* $K_N$ is the smallest such set; $K$ denotes $K_1$. 
An *intermediate logic* is a set of intuitionistic formulas $L$ such that:

- $L$ contains all intuitionistic axioms
- $L$ is closed under Modus Ponens: if $A, A \rightarrow B \in L$, then $B \in L$.
- $L$ is closed under Substitution:
  
  if $A(p_1, \ldots, p_n) \in L$, then $A(B_1, \ldots, B_n)$ (for any formulas $B_1, \ldots, B_n$)
- $L$ is consistent

The smallest intermediate logic is intuitionistic ($\mathbf{H}$), the largest is classical ($\mathbf{CL}$).
L[k] denotes the restriction of a logic L to formulas in variables $p_1, \ldots, p_k$. The sets $L[k]$ are called weak logics.

The modal depth of a formula $A$ in a (maybe weak) modal logic $L$:

$$\text{md}_L(A) := \text{min}\{\text{md}(B) | L \vdash A \leftrightarrow B\}$$

The implication depth of a formula $A$ in an intermediate logic $L$:

$$\text{di}_L(A) := \text{min}\{\text{di}(B) | L \vdash A \leftrightarrow B\}$$

The modal / implication depth of a logic $L$:

$$\text{md}(L) := \max\{\text{md}_L(A) | A \text{ is in the language of } L\}$$

$$\text{di}(L) := \max\{\text{di}_L(A) | A \text{ is an intuitionistic formula}\}$$
Trivial examples:
\( \text{di}(H) = \infty, \text{md}(K) = \infty \)
\( \text{di}(CL) = 1 \)
\( \text{md}(K + \Box \bot) = \text{md}(K + p \leftrightarrow \Box p) = 0. \)

A nontrivial (well-known) example:
\( \text{md}(S5) = 1 \)
Kripke frames and models-1

An $N$-modal Kripke frame is a nonempty set with $N$ binary relations $F = (W, R_1, \ldots, R_N)$.

An intuitionistic Kripke frame is a poset $F = (W, \leq)$.

A valuation in $F$ is a function $\theta : PL \rightarrow 2^W$ (so $\theta(p_i) \subseteq W$).

$(F, \theta)$ is a Kripke model over $F$.

In intuitionistic Kripke models $\theta(p_i)$ should be $\leq$-stable:

$x \in \theta(p_i) \& x \leq y \Rightarrow y \in \theta(p_i)$

In $k$-weak Kripke models only $p_1, \ldots, p_k$ are evaluated.
The inductive truth definition for the modal case \((M,x \vDash A)\)

- \(M,x \vDash p_i \text{ iff } x \in \theta(p_i)\)
- \(M,x \vDash \Box_i A \text{ iff } \forall y(xR_i y \Rightarrow M,y \vDash A)\)
- \(M,x \vDash \Diamond_i A \text{ iff } \exists y(xR_i y \& M,y \vDash A)\)

A formula \(A\) is **valid** in a frame \(F\) (in symbols, \(F \vDash A\)) if \(A\) is true at all points in every Kripke model over \(F\).
Kripke frames and models-3

The inductive truth definition for the intuitionistic case \((M,x \vDash A)\)

- \(M,x \vDash p_i \text{ iff } x \in \theta(p_i)\)
- \(M,x \vDash A \lor B \text{ iff } (M,x \vDash A \text{ or } M,x \vDash B)\)
- \(M,x \vDash A \land B \text{ iff } (M,x \vDash A \text{ and } M,x \vDash B)\)
- \(M,x \vDash A \rightarrow B \text{ iff } \forall y \geq x (M,y \vDash A \Rightarrow M,y \vDash B)\)

Then

- \(M,x \vDash \neg A \text{ iff } \forall y \geq x M,y \not\vDash A\)

A formula \(A\) is valid in a frame \(F\) (in symbols, \(F \vDash A\)) if \(A\) is true at all points in every intuitionistic Kripke model over \(F\).
Kripke frames and models-4

Canonical model theorem

For any modal or intermediate logic $L$ (weak or not) there exists the **canonical model** $M_L$ such that

- for any $A$ in the language of $L$

  $$M_L \models (\models) A \text{ iff } L \vdash A$$

- $M_L$ is **distinguishable**:
  
  two points $x,y$ satisfy the same formulas iff $x=y$. 
Tabularity and FMP

*Kripke complete* logics

\[ L(F) := \{ A \mid F \models A \} \] (the *logic of a frame* \( F \)).

\[ L(C) := \bigcap \{ L(F) \mid F \in C \} \] (the *logic of a class of frames* \( C \)).

- If \( F \) is finite, \( L(F) \) is called *tabular* (or *finite*).
- If \( C \) consists of finite frames, \( L(C) \) has the *finite model property* (FMP). Or:

\[ L \text{ has the FMP iff } L \text{ is an intersection of tabular logics.} \]

**Proposition** ('Harrop's theorem') If \( L \) is finitely axiomatizable and has the FMP, then \( L \) is decidable.
Bisimulation games-1

n-bisimulations by Johan Van Benthem (1989) <<
n-equivalence by Kit Fine (1974)

**Def** For a k-weak Kripke model $M=(W,R_1,\ldots,R_N,\theta)$
consider the $0$-equivalence relation between points

$$x \equiv_0 y := \forall j \leq k \ (M,x \models p_j \iff M,y \models p_j)$$

Given $M$ and two points $x_0 \equiv_0 y_0$ we can play the $r$-round
bisimulation game $BG_r(M,x_0,y_0)$.

Players: Spoiler (Abelard) vs Duplicator (Éloïse).

**Remark** More generally, bisimulation games can be defined
for two Kripke models $M,M'$ and points $x_0 \in M$, $y_0 \in M'$. We do
not need this in our talk.
Bisimulation games-2

The initial position in $BG_r(M, x_0, y_0)$ is $(x_0, y_0)$.

Round $(n+1)$
- **Spoiler** chooses $i$, $x_{n+1}$ [or $y_{n+1}$] such that $x_n R_i x_{n+1} [y_n R_i y_{n+1}]$
- **Duplicator** chooses $y_{n+1}$ [$x_{n+1}$] such that $y_n R_i y_{n+1}$ [$x_n R_i x_{n+1}$] and $x_{n+1} \equiv_0 y_{n+1}$

A player loses if he/she cannot move.
- **Duplicator** wins after $r$ rounds.
Bisimulation games-3

**Def** Formula and game *n-equivalence* relations (on M)

- \( x \equiv_n y \) := for any \( A(p_1,\ldots,p_k) \) of modal depth \( \leq n \)
  \[
  M, x \models A \iff M, y \models A
  \]
- \( x \sim_n y \) := Duplicator has a winning strategy in \( BG_n(M, x, y) \)

**Main Theorem on finite bisimulation games** (Stirling, 1995)

\[
\equiv_n = \sim_n
\]

- The same theorem holds for the intuitionistic case.
Local tabularity-1

Def A logic $L$ is *locally tabular* (or *locally finite*) if for any $k$ there are finitely many formulas in $p_1, \ldots, p_k$ up to equivalence in $L$.

Equivalent definitions:

- $L$ is locally tabular if all its weak fragments $L[k]$ are tabular.
- The variety of $L$-algebras is *locally finite*: every finitely generated $L$-algebra is finite.
- For every finite $k$, the free $k$-generated $L$-algebra (the *Lindenbaum algebra* of $L[k]$) is finite.
- Every weak canonical model $M_{L[k]}$ is finite.
Local tabularity-2

Finite modal (implication) depth ⇒

local tabularity ⇒ fmp

• The first implication is easy: there are finitely many $k$-formulas of bounded depth up to equivalence in the basic modal or intuitionistic logic.

• The second one is well-known: a locally tabular logic is complete w.r.t. its weak canonical frames

The second implication is not revertible: plenty of examples (K, S4, H etc.)

PROBLEM. Does every locally tabular modal or intermediate logic have a finite formula depth?

The problem seems difficult. Conjecture: no.
**Formula depth and games-1**

In every Kripke model there is a decreasing sequence 

\[ \equiv_0 \supseteq \equiv_1 \supseteq \ldots \quad \text{Put} \quad \equiv_\infty := \bigcap_n \equiv_n \]

**Lemma 1** In a weak Kripke model every relation \( \equiv_n \) induces a finite partition (\( W/\equiv_n \) is finite).

**Lemma 2** \( x \equiv_\infty y \) iff for any \( A(p_1,\ldots,p_k) \) \( (M,x \vdash A \iff M,y \vdash A) \)

**Lemma 3** (distinguishability) In canonical models:

\[ x \equiv_\infty y \text{ iff } x = y. \]

**Stabilization lemma (modal case)**

If \( \equiv_n = \equiv_{n+1} \) in every \( M_{L\mid_k} \) (bisimulation games *stabilize at round* \( n \)), then \( \text{md}(L) \leq n \).

**Stabilization lemma (intuitionistic case)** If \( \equiv_n = \equiv_{n+1} \) in every \( M_{L\mid_k} \), then \( \text{di}(L) \leq n+1 \).
Proof of modal Stabilization lemma

For every \( x \) in \( M_{L[k]} \), put

\[
B_x := \bigwedge \{ C \mid x \models C, \text{md}(C) \leq n \}
\]

Then \( B_x \) defines \( x \). So for any \( k \)-formula \( A \)

\[
M_{L[k]} \models A \iff \bigvee \{ B_x \mid x \models A \},
\]

and the disjunction is actually finite.

By Canonical model theorem

\[
L \models A \iff \bigvee \{ B_x \mid x \models A \}. \text{ QED}
\]
Stabilization lemma (intuitionistic case) If $\equiv_n = \equiv_{n+1}$ in every $M_{L\lceil k}$, then $\text{di}(L\lceil k) \leq n+1$.

Proof. Similar to the modal case, but now we need

$$B_x := \bigwedge \{ D \mid x \Vdash D, \text{di}(D) \leq n \},$$

$$C_x := \bigvee \{ D \mid x \not\Vdash D, \text{di}(D) \leq n \}.$$

Then $y \not\Vdash B_x \rightarrow C_x$ iff $y \leq x$. So for any $k$-formula $A$

$$M_{L\lceil k} \models A \iff \bigwedge \{ B_x \rightarrow C_x \mid x \not\Vdash A \}.$$ 

Hence $L \vdash A \iff \bigwedge \{ B_x \rightarrow C_x \mid x \not\Vdash A \}$. QED
Normal forms in intuitionistic logic

The previous proof allows us to present every intuitionistic formula in the normal form, as a conjunction of `characteristic formulas' (cf. [Ghilardi, 1992]). This is an analogue to Hintikka theorem for classical FOL.

**Depth 1** Characteristic k-formulas are $B_j \rightarrow C_j$, where

$$B_j := \bigwedge \{ p_i \mid i \in J \}, \quad C_j := \bigvee \{ p_i \mid i \notin J \},$$

for $J \subseteq \{1, \ldots, k\}$.

**Depth n+1** Characteristic k-formulas are $B_j \rightarrow C_j$, where

$$B_j := \bigwedge \{ D_i \mid i \in J \}, \quad C_j := \bigvee \{ D_i \mid i \notin J \},$$

where $D_1, \ldots, D_m$ are all characteristic formulas of depth n, $J \subseteq \{1, \ldots, m\}$.
Lemma on repeating positions Suppose in a Kripke model $M$ $x \equiv_n y$ and the Duplicator has a winning strategy $s$ in $BG_n(M,x,y)$ such that every play controlled by $s$ has at least two repeating positions. Then $x \equiv_{n+1} y$. 
**Formula depth and games-6**

**tabularity ⇒ finite formula depth**

**Theorem** If $F$ is finite, then $\text{md}(L(F)) \leq |F|^2 + 1$.

**Proof:** The Pigeonhole principle gives repeating positions.

**Remark** In many cases we have a better (linear) upper bound.
Examples of finite depth-1

\[ \text{md}(K + \Box^n \bot) = n-1 \]

and more generally,

\[ \text{md}(K_N + \Box^n \bot) = n-1 \]

where

\[ \Box A := \Box_1 A \land \ldots \land \Box_N A. \]

The axiom \( \Box^n \bot \) forbids paths of length \( n \) in Kripke frames:

\[ x_1 R x_2 \ldots R x_n, \text{ where } R = R_1 \cup \ldots \cup R_N \]

Proof. For the upper bound: every play of a bisimulation game contains at most \((n-1)\) rounds. For the lower bound:

\[ \text{md}_L(\Box^{n-1} \bot) = n-1. \]

An earlier result: \( K_N + \Box^n \bot \) is locally tabular (Gabbay & Sh, 1998; a routine proof by induction).
Examples of finite depth-2

\[ \text{md}(S_5) = 1 \text{ (a well-known fact)} \]

Proof. If Duplicator can win the 1-game, she can win the 2-game.
Examples of finite depth-3

\[ \text{md}(\text{DL}) = 2 \]

**DL** is the *difference logic*

\[ \text{DL} = K + \Diamond \Box p \rightarrow p + \Diamond \Diamond \Diamond p \rightarrow p \vee \Diamond p \]

- **DL** is complete w.r.t inequality frames \((W, \neq_W)\).
- Arbitrary **DL**-frames are obtained from **S5**-frames (equivalence frames) by making some points irreflexive.
- Proof (for the lower bound):
  - \( x \models \Diamond^2 p \)
  - \( y \not\models \Diamond^2 p \)
  - \( t \models p \)
  - \( z \models p \)
  - \( x \equiv_1 y \)
Examples of finite depth-4

For the upper bound we have to examine games in canonical models

**Lemma** In $M_{DL|k}$ $x \equiv_0 y$ & $xRy$ implies $x \equiv_1 y$.

Proof. Duplicator's responses for the moves of Spoiler are:

S: (x,z) (with $z \neq x,y$)  D: (y,z)

S: (x,x)  D: (y,x)

S: (x,y)  D: (y,x)

They lead to 0-equivalent points. QED
Examples of finite depth-5

Now in the general case suppose $x \equiv_2 y$ in $M_{DL[k]}$. We have to show that $x \equiv_3 y$. Let us start playing a 2-round game, so we have $x' \equiv_1 y'$, and we have to show $x' \equiv_2 y'$. 

\[ \text{Diagram showing arrows from } x \text{ to } x'' \text{ and from } y \text{ to } y'' \text{ with intermediate } x' \text{ and } y' \]
Examples of finite depth-6

Consider the next Spoiler's move \((x', x'')\).

(a) \(x'' = x\). The Duplicator responds with \(y'' = y\).

(b) \(x'' \neq x, x'' \not\equiv_0 x'\). Then \(xR_0 x''\), and \((x, x'')\) can be regarded as the first move in the 2-round game. For the response \((y, y'')\) we have \(y'R_0 y''\) (since \(y' \neq y''\), otherwise \(x'' \equiv_0 x'\)) and \(x'' \equiv_1 y''\).
Examples of finite depth-7

(c) $x'' \neq x$, $x'' \equiv_0 x'$. There is a response $(y', y'')$, with $x'' \equiv_0 y''$. So $y'' \equiv_0 y'$ by the transitivity of $\equiv_0$.

Now by Lemma $x'' \equiv_1 x'$ and $y'' \equiv_1 y'$; thus $x'' \equiv_1 y''$ by the transitivity of $\equiv_1$. QED.
Examples of finite depth-8

\[ \text{di}(H+i\text{bd}_n) \leq 2n-1 \]

In posets \( i\text{bd}_n \) forbids *chains of length* \( n+1 \): \( x_1 < x_2 \ldots < x_{n+1} \).

\[ i\text{bd}_1 = p_1 \lor \neg p_1, \]

\[ i\text{bd}_{n+1} = p_{n+1} \lor (p_{n+1} \rightarrow i\text{bd}_n). \]

**Def** Intermediate logics of finite transitive depth: extensions of \( H+i\text{bd}_n \) are of depth \( \leq n-1 \) (or of height \( \leq n \)).

**Theorem** (Kuznetsov – Komori) These logics are locally tabular.

Proof of the upper bound: by induction we show that \( x \equiv_k y \) implies \( x \equiv_{k+1} y \) whenever \( \text{depth}(x) + \text{depth}(y) \leq k \). So the bisimulation game stabilizes at \( 2n-2 \).
Examples of finite depth-9

\[
\text{md(Grz}+\text{bd}_n) \leq 2n-2, \\
\text{md(Grz3}+\text{bd}_n) = n-1
\]

\textbf{Grz} is the logic of finite partial orders, \\
\textbf{Grz3} is the logic of finite chains.

In transitive Kripke frames \text{bd}_n forbids \textit{chains of clusters of length } n+1: x_1Rx_2...Rx_{n+1}, \text{ where}

\[\neg x_iRx_{i+1} \text{ for each } i.\]

\[
\text{bd}_n = \neg \Diamond (Q_1 \wedge \Diamond (Q_2 \wedge ... \wedge \Diamond Q_{n+1})),
\]

\[Q_i = p_i \wedge \bigwedge \{ \neg \Diamond p_j \mid 1 \leq j < i \}.
\]

\textbf{Grz3} + \text{bd}_n = L(n\text{-element chain})
Examples of finite depth-10

$\text{di}(\text{LC}) = 2$, where $\text{LC} = H+ (p \rightarrow q) \lor (q \rightarrow p)$ is the intermediate logic of arbitrary chains.

Proof. $x \equiv_1 y$ implies $x \equiv_2 y$, since $x' \equiv_0 y'$ implies $x' \equiv_1 y'$: we can ignore the first move. If the 1-round game response for $(x,x'')$ is $(y,y'')$ with $y'' < y$, then $x'' \equiv_0 y''$, and $y'' \equiv_0 y'$ as the model in intuitionistic. So $(y',y')$ can be the response for $(x',x'')$.
Examples of finite depth-11

\[ \text{md}(\text{Grz} + \text{bd}_2) = 2 \]

(0, 1 show the truth values of p)

Here \( x \equiv_1 y \), but \( x \not\equiv_2 y \): Duplicator wins after 1 round. Spoiler wins after 2 rounds.

A distinguishing formula is \( \Box \Diamond p \). So it has depth 2 in \( \text{Grz} + \text{bd}_2 \)

But note that \( \text{md}(\text{Grz}_3 + \text{bd}_2) = 1 \) and

\[ \text{Grz}_3 + \text{bd}_2 \vdash \Box \Diamond p \iff (\Box p \lor (\neg p \land \Diamond p)) \].
Examples of finite depth-12

di(\textit{LC+ibd}_2) = di(\textit{LC})= 2, while \ di(\textit{H+ibd}_2) =3:

As in the modal case, \(x \equiv_1 y\), but \(x \not\equiv_2 y\):

\(x \models \neg p, y \models p\)

Note that \(di(\neg p \rightarrow p)=3\) in \(\textit{H+ibd}_2\)

But \(di(\neg p \rightarrow p)=1\) in \(\textit{LC+ibd}_2\) : it is equivalent to \((p \lor \neg p)\).
Examples of finite depth-13

\[ \text{md}(K4 + \text{bd}_n) \leq 4n - 3 \]

**Theorem** (Segerberg 1971; Maksimova 1975) For \( L \supseteq K4 \)

\( L \) is locally tabular iff \( L \) is of finite transitive depth.

**Def** \( L \) is of **finite transitive depth** if \( L \vdash \text{bd}_n \) for some \( n \).

**Corollary** For extensions of \( K4 \) local tabularity is equivalent to finite modal depth.

**PROBLEM** (Chagrov) Find a description of local tabularity for extensions of \( K \).
Examples of finite depth-14

If \( \text{md}(L) = m \), then \( \text{md}(\{\mathbf{K} + \square^n \perp, L\}) \leq (m+1)n-1 \)

Def. The commutative join (commutator)

\[
[L_1, L_2] := L_1 \ast L_2 \text{ (the fusion)} +
\]

\[\blacklozenge_j \square_i p \leftrightarrow \square_i \blacklozenge_j p \text{ (commutation axioms)}\]

\[\blacklozenge_j \square_i p \rightarrow \blacklozenge_j \blacklozenge_i p \text{ (Church-Rosser axioms)}\]
Tabularity criterion-1

**Theorem** (Chagrov 1994)

L is tabular iff \( L \vdash \alpha_n \land \text{Alt}_n \) for some \( n \).

The formulas \( \alpha_n \), \( \text{Alt}_n \) correspond to universal conditions on frames:

- \( \alpha_n \) forbids simple paths of length \( n \):
  \[ x_1Rx_2...Rx_n, \text{ where all the } x_i \text{ are different.} \]

- \( \text{Alt}_n \) forbids \( n \)-branching: \( xRx_1,...,xRx_n, \), where all the \( x_i \) are different.
Tabularity criterion-2

\[ \alpha_n = \neg \Box (P_1 \land \Box (P_2 \land \ldots \land (P_{n-1} \land \Box P_n) \ldots )) , \]

\[ \text{Alt}_n = \neg (\Box P_1 \land \Box P_2 \land \ldots \land \Box P_n) , \]

where

\[ P_i = \neg p_i \land \{ p_j \mid 1 \leq j \leq n, j \neq i \} . \]
Theorems on local tabularity-1

1. Every logic $\mathbf{K}_n + \alpha_n$ (Chagrov's formula) is locally tabular.

(This theorem was conjectured in 1994 by Chagrov.)

The proof does not give the FMD. To reach a repeating position, Duplicator should keep track of all possible returns.

So she plays her own stronger game:

at the position $(x,y)$ at every stage not only $x \equiv_0 y$, but for any $m<n$, $i \leq N$

there is a return $m$ steps back from $x$ along $R_i$ iff there is a return $m$ steps back from $y$ along $R_i$.

This is actually a bisimulation game in another model.

As it stabilizes at $n$, we obtain the local tabularity.
Theorems on local tabularity-2

2. The logics \([K_N + \alpha_n, K_N' + \Box^n \bot]\), \([K_N + \alpha_n, S5]\) are locally tabular.

Remark. In general products and commutative joins do not preserve local tabularity, a counterexample is \([S5, S5] = S5^2\) (Tarski).

Theorem [N. Bezhanishvili, 2002] \(S5^2\) is pre-locally tabular. Probably, there exists a game-theoretic proof.
THANK YOU!
References-1


Logics

- **K** = **L** (all frames)
- **K4** := **K** + ◇◇p → ◇p = **L** (all transitive frames)
- **S4** := **K4** + p → ◇p = **L** (all transitive reflexive frames)
  = **L** (all partial orders)
- **Grz** := **S4** + Ʌ(p ∧ □(p → ◇(Ʌp ∧ ◇p)))
  = **L** (all finite partial orders)
- **Grz3** := **Grz** + ◇p ∧ ◇q → ◇(p ∧ ◇q) ∨ ◇(q ∧ ◇p)
  = **L** (all finite chains)
- **S5** := **S4** + ◇□p → p = **L** (all equivalence frames)
  = **L** (all universal frames [clusters])

All these logics have the FMP, so they are decidable.