Logics for Compact Hausdorff Spaces via de Vries Duality

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• Main goal: developing a propositional calculus for compact Hausdorff spaces
Outline

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• De Vries duality: compact Hausdorff spaces and algebras
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- Language, semantics, deductive system and steps towards a completeness result
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- Main goal: developing a propositional calculus for compact Hausdorff spaces
- De Vries duality: compact Hausdorff spaces and algebras
- Language, semantics, deductive system and steps towards a completeness result
- Investigation of the theory of the introduced tools
De Vries duality

$$ (B, \prec) \mapsto X_B $$

de Vries algebra \quad space of maximal round filters of \((B, \prec)\)

$$ X \mapsto (RO(X), \prec) $$

compact Hausdorff space \quad algebra of regular open subsets of \(X\)

where \(U \prec V := \text{Cl}(U) \subseteq V\)
Definition
A de Vries algebra is a pair \((B, \prec)\) where

- \(B\) is a complete Boolean algebra
- \(\prec\) is a binary relation on \(B\) satisfying
  (Q1) \(0 \prec 0\) and \(1 \prec 1\);
  (Q2) \(a \prec b, c\) implies \(a \prec b \land c\);
  (Q3) \(a, b \prec c\) implies \(a \lor b \prec c\);
  (Q4) \(a \leq b \prec c \leq d\) implies \(a \prec d\);
  (Q5) \(a \prec b\) implies \(a \leq b\);
  (Q6) \(a \prec b\) implies \(\neg b \prec \neg a\);
  (Q7) \(a \prec b\) implies \(\exists c : a \prec c \prec b\);
  (Q8) \(a \neq 0\) implies \(\exists b \neq 0 : b \prec a\).
Boolean algebras with a binary relation
Combingent algebras

Definition
A compingent algebra is a pair \((B, \prec)\) where

- \(B\) is a Boolean algebra
- \(\prec\) is a binary relation on \(B\) satisfying
  
  (Q1) \(0 \prec 0\) and \(1 \prec 1\);
  (Q2) \(a \prec b, c\) implies \(a \prec b \wedge c\);
  (Q3) \(a, b \prec c\) implies \(a \vee b \prec c\);
  (Q4) \(a \leq b \prec c \leq d\) implies \(a \prec d\);
  (Q5) \(a \prec b\) implies \(a \leq b\);
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Boolean algebras with a binary relation

Contact algebras

Definition

A **contact algebra** is a pair \((B, \prec)\) where

- \(B\) is a Boolean algebra
- \(\prec\) is a binary relation on \(B\) satisfying
  
  (Q1) \(0 \prec 0\) and \(1 \prec 1\);
  (Q2) \(a \prec b, c\) implies \(a \prec b \land c\);
  (Q3) \(a, b \prec c\) implies \(a \lor b \prec c\);
  (Q4) \(a \leq b \prec c \leq d\) implies \(a \prec d\);
  (Q5) \(a \prec b\) implies \(a \leq b\);
  (Q6) \(a \prec b\) implies \(\neg b \prec \neg a\);
A binary relation $\prec$ on a Boolean algebra $B$ can be replaced with an operation $\succsim: B \times B \rightarrow \{0, 1\} \subseteq B$, defined as

$$a \succsim b = \begin{cases} 1 & \text{if } a \prec b \\ 0 & \text{otherwise.} \end{cases}$$
A binary relation $\prec$ on a Boolean algebra $B$ can be replaced with an operation $\rightsquigarrow: B \times B \rightarrow \{0, 1\} \subseteq B$, defined as

$$a \rightsquigarrow b = \begin{cases} 
1 & \text{if } a \prec b \\
0 & \text{otherwise.}
\end{cases}$$

We use this operation to interpret formulas of the following language into pairs $(B, \prec)$:

$$\varphi ::= p \mid T \mid \varphi \land \varphi \mid \neg \varphi \mid \varphi \rightsquigarrow \varphi$$
Boolean algebras with a binary relation
Syntax and semantics

Our language has the following property:
Boolean algebras with a binary relation
Syntax and semantics

Our language has the following property:

\( \forall \varphi \exists \varphi' \) such that for any valuation \( v \) into an algebra \((B, \prec)\):

\[
v(\varphi') = \begin{cases} 
1 & \text{if } v(\varphi) = 1 \\
0 & \text{if } v(\varphi) \neq 1.
\end{cases}
\]
Our language has the following property:

\[ \forall \varphi \exists \varphi' \text{ such that for any valuation } v \text{ into an algebra } (B, \prec) : \]

\[ v(\varphi') = \begin{cases} 
1 & \text{if } v(\varphi) = 1 \\
0 & \text{if } v(\varphi) \neq 1.
\end{cases} \]

In our case \( \varphi' := T \rightsquigarrow \varphi \).
The system $S$

Consider the deductive system axiomatised by:

- All axioms $\varphi$ of CPC

(A1) $(\bot \rightsquigarrow \varphi) \land (\varphi \rightsquigarrow \top)$

(A2) $(\varphi \rightsquigarrow \psi) \land (\varphi \rightsquigarrow \chi) \rightarrow (\varphi \rightsquigarrow \psi \land \chi)$

(A3) $(\top \rightsquigarrow \neg \varphi \lor \psi) \land (\psi \rightsquigarrow \chi) \rightarrow (\varphi \rightsquigarrow \chi)$

(A4) $(\varphi \rightsquigarrow \psi) \rightarrow (\varphi \rightarrow \psi)$

(A5) $(\varphi \rightsquigarrow \psi) \rightarrow (\chi \rightsquigarrow (\varphi \rightsquigarrow \psi))$

(A6) $(\neg (\varphi \rightsquigarrow \psi) \rightarrow (\chi \rightsquigarrow \neg (\varphi \rightsquigarrow \psi)))$

(A7) $(\varphi \rightsquigarrow \psi) \leftrightarrow (\neg \psi \rightsquigarrow \neg \varphi)$

(MP) \[ \frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \]

(R) \[ \frac{\varphi}{\top \rightsquigarrow \varphi} \]
Completeness

Theorem
The system $S$ is strongly sound and complete with respect to contact algebras:

$$\Gamma \vdash \varphi \iff \Gamma \models \varphi.$$
Completeness

Theorem
The system $\mathcal{S}$ is strongly sound and complete with respect to contact algebras:

$$\Gamma \vdash \varphi \iff \Gamma \models \varphi.$$ 

The class of contact algebras is axiomatised by (Q1)-(Q6), which are universal statements.
Completeness

Theorem
The system $S$ is strongly sound and complete with respect to contact algebras:

$$\Gamma \vdash \varphi \iff \Gamma \models \varphi.$$  

The class of contact algebras is axiomatised by (Q1)-(Q6), which are universal statements.

To deal with the $\forall \exists$ statements (Q7) and (Q8) we add non-standard rules to the system $S$. 
Π₂ — *rules*

Non-standard rules for emulating ∀∃-statements
Definition
A $\Pi_2$-rule is one of the form:

$$(\rho) \quad \frac{F(\bar{\varphi}, \bar{p}) \to \chi}{G(\bar{\varphi}) \to \chi}$$

where $F, G$ are formulas involving formula variables $\bar{\varphi}, \chi$ and fresh proposition letters $\bar{p}$. 
Π₂ — rules
Non-standard rules for emulating ∀∃-statements

Definition
A Π₂-rule is one of the form:

\[(\rho) \quad \frac{F(\varphi, \vec{p}) \rightarrow \chi}{G(\bar{\varphi}) \rightarrow \chi}\]

where \(F, G\) are formulas involving formula variables \(\varphi, \chi\) and fresh proposition letters \(\vec{p}\).

We associate such a rule \((\rho)\) with the ∀∃-statement

\[
\Phi_{\rho} := \forall \bar{x}, z \left( G(\bar{x}) \not\equiv z \Rightarrow \exists \bar{y} : F(\bar{x}, \bar{y}) \not\equiv z \right)
\]

in the signature \((\land, \neg, 1, \leadsto)\).
Logics for inductive classes of contact algebras

From logics to classes

Let $T$ be the first-order theory of contact algebras.
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Let $\{\rho_n\}_{n<\omega}$ be a set of $\Pi_2$ rules.
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**Theorem**

The system $S + \{\rho_n\}_{n<\omega}$ is strongly sound and complete with respect to $\text{Mod}(T \cup \{\Phi_{\rho_n}\}_{n<\omega})$. 
Let $T$ be the first-order theory of contact algebras.

Let $\{\rho_n\}_{n<\omega}$ be a set of $\Pi_2$ – rules.

**Theorem**

The system $S + \{\rho_n\}_{n<\omega}$ is strongly sound and complete with respect to $\text{Mod}(T \cup \{\Phi_{\rho_n}\}_{n<\omega})$.

By this theorem, extensions of $S$ with $\Pi_2$-rules are complete with respect to $\forall\exists$-definable classes of contact algebras. $\forall\exists$-definable classes are the same as inductive classes (Chang-Łos-Suszko theorem).
Vice versa, given a ∀∃-theory $T' \supseteq T$, we can find a logic which is complete with respect to $\text{Mod}(T')$. 

Proposition

Let $\Phi(\bar{x}, \bar{y})$ be a quantifier-free formula. The statement $\forall \bar{x} \exists \bar{y} \Phi(\bar{x}, \bar{y})$ is equivalent to the one associated to the $\Pi_2$-rule $(\rho \Phi)$. 

$\tilde{\Phi}(\bar{\phi}, \bar{p}) \rightarrow \chi \chi$
Vice versa, given a $\forall \exists$-theory $T' \supseteq T$, we can find a logic which is complete with respect to $\text{Mod}(T')$.

We define how to translate a quantifier-free formula $\Phi(\bar{x}, \bar{y})$ into a formula $\tilde{\Phi}(\bar{x}, \bar{y})$ of our language.
Vice versa, given a $\forall \exists$-theory $T' \supseteq T$, we can find a logic which is complete with respect to $\text{Mod}(T')$. We define how to translate a quantifier-free formula $\Phi(\bar{x}, \bar{y})$ into a formula $\tilde{\Phi}(\bar{x}, \bar{y})$ of our language.

**Proposition**

Let $\Phi(\bar{x}, \bar{y})$ be a quantifier-free formula. The statement $\forall \bar{x} \exists \bar{y} \Phi(\bar{x}, \bar{y})$ is equivalent to the one associated to the $\Pi_2$-rule

$$
\begin{array}{c}
(\rho_\Phi) \\
\tilde{\Phi}(\bar{\varphi}, \bar{p}) \rightarrow \chi
\end{array}
$$
Logics for inductive classes of contact algebras

Correspondence between logics and inductive classes

Let $T$ be the first-order theory of contact algebras.
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\[
\{\rho_n\}_{n<\omega} \mapsto T \cup \{\Phi_{\rho_n}\}_{n<\omega}
\]

set of $\Pi_2$-rules $\forall \exists$-theory extending $T$

\[
T' \mapsto \{\rho_\Phi \mid \forall \bar{x} \exists \bar{y} \Phi(\bar{x}, \bar{y}) \in T'\}
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$\forall \exists$-theory extending $T$ set of $\Pi_2$-rules
Logics for inductive classes of contact algebras

Correspondence between logics and inductive classes

Let $T$ be the first-order theory of contact algebras.

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\[ T' \longrightarrow \{\rho_{\Phi} \mid \forall\bar{x}\exists\bar{y}\Phi(\bar{x},\bar{y}) \in T'\} \]

$\forall\exists$-theory extending $T$ set of $\Pi_2$-rules

Extensions of $\mathcal{S}$ $\longleftrightarrow$ Inductive classes of contact algebras with $\Pi_2$-rules
The logic of compingent algebras

(Q7) $a \prec b$ implies $\exists c : a \prec c \prec b$;
(Q8) $a \neq 0$ implies $\exists b \neq 0 : b \prec a$. 
The logic of compingent algebras

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(Q8) $a \neq 0$ implies $\exists b \neq 0 : b \prec a$.

(Q7) and (Q8) correspond to the following rules:

\[
\begin{align*}
(\rho_7) & \quad \frac{(\varphi \rightsquigarrow p) \land (p \rightsquigarrow \psi) \rightarrow \chi}{(\varphi \rightsquigarrow \psi) \rightarrow \chi} \\
(\rho_8) & \quad \frac{p \land (p \rightsquigarrow \varphi) \rightarrow \chi}{\varphi \rightarrow \chi}
\end{align*}
\]

Thus we obtain: Corollary $S + (\rho_7) + (\rho_8)$ is strongly sound and complete with respect to compingent algebras.
The logic of compingent algebras

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(\rho_7) \quad \frac{(\varphi \rightsquigarrow p) \land (p \rightsquigarrow \psi) \rightarrow \chi}{(\varphi \rightsquigarrow \psi) \rightarrow \chi}
\]

\[
(\rho_8) \quad \frac{p \land (p \rightsquigarrow \varphi) \rightarrow \chi}{\varphi \rightarrow \chi}
\]

Thus we obtain:

**Corollary**

\( S + (\rho_7) + (\rho_8) \) is strongly sound and complete with respect to compingent algebras.
Admissibility of Π₂-rules
Admissibility of $\Pi_2$-rules

**Definition**
A $\Pi_2$-rule $(\rho)$ is **admissible** in $S$ if all the theorems of $S + (\rho)$ are provable in $S$. 
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Theorem (Criterion of admissibility)
A $\Pi_2$-rule $(\rho)$ is admissible in $S$ if and only if any contact algebra $(B, \prec)$ is a substructure of some contact algebra $(C, \prec)$ satisfying $\Phi_\rho$. 

Admissibility of $\Pi_2$-rules

**Definition**
A $\Pi_2$-rule $(\rho)$ is **admissible** in $S$ if all the theorems of $S + (\rho)$ are provable in $S$.

**Theorem (Criterion of admissibility)**
A $\Pi_2$-rule $(\rho)$ is admissibile in $S$ if and only if any contact algebra $(B, \prec)$ is a substructure of some contact algebra $(C, \prec)$ satisfying $\Phi_\rho$.

**Corollary**
$(\rho7)$ and $(\rho8)$ are admissibile in $S$. 
The logic of compact Hausdorff spaces

Definition

The MacNeille completion of a compact algebra \((B, \preceq)\) is \((B, \preceq)\), where \(B\) is the MacNeille completion of \(B\) and \(\preceq\) is defined as:

\[\alpha \preceq \beta \iff \text{there exist } a, b \in B \text{ such that } \alpha \leq a \preceq b \leq \beta.\]

Lemma

Given a compact algebra \((B, \preceq)\), its MacNeille completion \((B, \preceq)\) is a de Vries algebra.

Corollary

• \(S + (\rho_7) + (\rho_8)\) is sound and complete with respect to de Vries algebras.

• \(S + (\rho_7) + (\rho_8)\) is sound and complete with respect to compact Hausdorff spaces.
The logic of compact Hausdorff spaces

Definition
The MacNeille completion of a compingent algebra \((B, \prec)\) is \((\overline{B}, \prec)\), where \(\overline{B}\) is the MacNeille completion of \(B\) and \(\prec\) is defined as:

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\alpha \prec \beta \iff \text{there exist } a, b \in B \text{ such that } \alpha \leq a \prec b \leq \beta.
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Lemma
Given a compingent algebra \((B, \prec)\), its MacNeille completion \((\overline{B}, \prec)\) is a de Vries algebra.

Corollary
- \(S + (\rho 7) + (\rho 8)\) is sound and complete with respect to de Vries algebras.
The logic of compact Hausdorff spaces

**Definition**
The MacNeille completion of a compingent algebra \((B, \prec)\) is \((\overline{B}, \prec)\), where \(\overline{B}\) is the MacNeille completion of \(B\) and \(\prec\) is defined as:

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- \(S + (\rho 7) + (\rho 8)\) is sound and complete with respect to de Vries algebras.
- \(S + (\rho 7) + (\rho 8)\) is sound and complete with respect to compact Hausdorff spaces.
MacNeille canonicity and topological properties

Definition
An axiom or rule is MacNeille canonical if, whenever a compatible algebra \((B, \preceq)\) validates it, also its MacNeille completion \((\bar{B}, \preceq)\) does.

Corollary
Let the axiom \((A)\) and the \(\Pi_2\) rule \((\rho)\) be MacNeille canonical.

- \(S + (\rho_7) + (\rho_8) + (A)\) is sound and complete with respect to de Vries algebras validating \((A)\).
- \(S + (\rho_7) + (\rho_8) + (\rho)\) is sound and complete with respect to de Vries algebras satisfying \(\Phi_{\rho}\).

MacNeille canonical axioms and rules can be used to express topological properties.
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MacNeille canonical axioms and rules can be used to express topological properties.
MacNeille canonicity and topological properties

Examples

- Connectedness:
MacNeille canonicity and topological properties

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- **Connectedness:**
  \[(C) \ (\varphi \leadsto \varphi) \rightarrow (T \leadsto \varphi) \lor (T \leadsto \neg \varphi)\]
MacNeille canonicity and topological properties

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- **Connectedness:**
  
  \[(C) \ (\varphi \leadsto \varphi) \rightarrow (T \leadsto \varphi) \lor (T \leadsto \neg \varphi)\]

  \[S + (\rho7) + (\rho8) + (C)\] is the logic of connected compact Hausdorff spaces.
MacNeille canonicity and topological properties

Examples

- **Connectedness:**
  \[(C) \quad (\varphi \leadsto \varphi) \rightarrow (T \leadsto \varphi) \lor (T \leadsto \neg \varphi)\]

  \(S + (\rho 7) + (\rho 8) + (C)\) is the logic of *connected compact Hausdorff spaces*.

- **Zero-dimensionality:**
MacNeille canonicity and topological properties

Examples

- **Connectedness:**
  \[(C) \ (\varphi \rightsquigarrow \varphi) \rightarrow (\top \rightsquigarrow \varphi) \lor (\top \rightsquigarrow \neg \varphi)\]

  \(S + (\rho7) + (\rho8) + (C)\) is the logic of **connected compact Hausdorff spaces**.

- **Zero-dimensionality:**
  \[(\rho9) \ \frac{(\varphi \rightsquigarrow p) \land (p \rightsquigarrow \psi) \land (p \rightsquigarrow p) \rightarrow \chi}{(\varphi \rightsquigarrow \psi) \rightarrow \chi}\]
MacNeille canonicity and topological properties

Examples

- **Connectedness:**
  \[(C) \ (\varphi \leadsto \varphi) \rightarrow (T \leadsto \varphi) \lor (T \leadsto \neg \varphi)\]

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- **Zero-dimensionality:**
  \[(\rho9) \quad \frac{(\varphi \leadsto p) \land (p \leadsto \psi) \land (p \leadsto p) \rightarrow \chi}{(\varphi \leadsto \psi) \rightarrow \chi}\]

  \[S + (\rho7) + (\rho8) + (\rho9)\] is the logic of **Stone spaces**.
Related work

Our completeness result for $\Pi_2$-rules is inspired by the work of Balbiani, Tinchev and Vakarelov in *Modal Logics for Region-based Theories of Space* (2007).

\[(\text{NOR}) \quad \phi \Rightarrow (aCp \lor p^*Cb)\]

\[(\text{EXT}) \quad \phi \Rightarrow (p = 0 \lor aCp)\]

\[\phi \Rightarrow (a = 1)\]

where $p$ does not occur in $a, b, \phi$. 
Related work

Our completeness result for $\Pi_2$-rules is inspired by the work of Balbiani, Tinchev and Vakarelov in *Modal Logics for Region-based Theories of Space* (2007).

They use a first-order language without quantifiers. In this language they provide propositional calculi related to RCC (Region Connection Calculus).
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Some of these calculi involve particular non-standard rules:

**(NOR)**

\[
\varphi \Rightarrow (aCp \lor p^*Cb) \\
\varphi \Rightarrow aCb
\]

where $p$ does not occur in $a$, $b$, $\varphi$

**(EXT)**

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\varphi \Rightarrow (p = 0 \lor aCp) \\
\varphi \Rightarrow (a = 1)
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- Relational semantics based on Kripke frames;
- Topological semantics via algebras of regular closed subsets of topological spaces
Balbiani et al. consider two semantics for their language:

- Relational semantics based on Kripke frames;
- Topological semantics via algebras of regular closed subsets of topological spaces

With respect to these semantics, the authors give completeness results for the propositional calculi they introduced.
Conclusion

• We developed a finitary propositional calculus for compact Hausdorff spaces via de Vries algebras.
Conclusion

- We developed a finitary propositional calculus for compact Hausdorff spaces via de Vries algebras.
- We developed the theory of $\Pi_2$-rules, showing:
  - a correspondence between logics and inductive classes;
  - a semantic criterion for admissibility of $\Pi_2$-rules.
Conclusion

- We developed a finitary propositional calculus for compact Hausdorff spaces via de Vries algebras.
- We developed the theory of $\Pi_2$-rules, showing:
  - a correspondence between logics and inductive classes;
  - a semantic criterion for admissibility of $\Pi_2$-rules.
- We showed how MacNeille completions can be used to obtain logics for subclasses of compact Hausdorff spaces.
Thank you!