Logics for Compact Hausdorff Spaces via de Vries Duality

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Outline

- Main goal: developing a propositional calculus for compact Hausdorff spaces

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- De Vries duality: compact Hausdorff spaces and algebras

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- Language, semantics, deductive system and steps towards a completeness result

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- De Vries duality: compact Hausdorff spaces and algebras
- Language, semantics, deductive system and steps towards a completeness result
- Investigation of the theory of the introduced tools

De Vries duality

$$\begin{tikzcd} \mathbf{deV} \arrow[r, bend left] & \mathbf{KHaus} \arrow[l, bend left] \end{tikzcd}$$

$$(B, \prec) \longmapsto X_B$$

de Vries algebra — space of maximal round filters of $(B, \prec)$

$$X \longmapsto (RO(X), \prec)$$

compact Hausdorff space — algebra of regular open subsets of X

where $U \prec V := \mathbf{Cl}(U) \subseteq V$

Definition

A de Vries algebra is a pair $(B, \prec)$ where

- B is a complete Boolean algebra
- $\prec$ is a binary relation on B satisfying

(Q1) $0 \prec 0$ and $1 \prec 1$;
(Q2) $a \prec b, c$ implies $a \prec b \wedge c$;
(Q3) $a, b \prec c$ implies $a \vee b \prec c$;
(Q4) $a \leq b \prec c \leq d$ implies $a \prec d$;
(Q5) $a \prec b$ implies $a \leq b$;
(Q6) $a \prec b$ implies $\neg b \prec \neg a$;
(Q7) $a \prec b$ implies $\exists c : \; a \prec c \prec b$;
(Q8) $a \neq 0$ implies $\exists b \neq 0 : \; b \prec a$.

Definition

A compingent algebra is a pair $(B, \prec)$ where

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(Q7) $a \prec b$ implies $\exists c: \ a \prec c \prec b$;
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Definition

A contact algebra is a pair $(B, \prec)$ where

- B is a Boolean algebra
- $\prec$ is a binary relation on B satisfying

(Q1) $0 \prec 0$ and $1 \prec 1$;
(Q2) $a \prec b, c$ implies $a \prec b \wedge c$;
(Q3) $a, b \prec c$ implies $a \vee b \prec c$;
(Q4) $a \leq b \prec c \leq d$ implies $a \prec d$;
(Q5) $a \prec b$ implies $a \leq b$;
(Q6) $a \prec b$ implies $\neg b \prec \neg a$;

Boolean algebras with a binary relation

Syntax and semantics

A binary relation $\prec$ on a Boolean algebra B can be replaced with an operation $\rightsquigarrow: B \times B \to \{0, 1\} \subseteq B$, defined as

$$a \rightsquigarrow b = \begin{cases} 1 & \text{if } a \prec b \\ 0 & \text{otherwise.} \end{cases}$$

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We use this operation to interpret formulas of the following language into pairs $(B, \prec)$:

$$\varphi \ := \ p \mid \top \mid \varphi \wedge \varphi \mid \neg\varphi \mid \varphi \rightsquigarrow \varphi$$

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$\forall \varphi \; \exists \varphi'$ such that for any valuation v into an algebra $(B, \prec)$:

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In our case $\varphi' := \top \rightsquigarrow \varphi$.

The system $\mathcal{S}$

Consider the deductive system axiomatised by:

- All axioms φ of **CPC**

(A1) $(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top)$

(A2) $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \psi \wedge \chi)$

(A3) $(\top \rightsquigarrow \neg\varphi \vee \psi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \chi)$

(A4) $(\varphi \rightsquigarrow \psi) \to (\varphi \to \psi)$

(A5) $(\varphi \rightsquigarrow \psi) \to (\chi \rightsquigarrow (\varphi \rightsquigarrow \psi))$

(A6) $\neg(\varphi \rightsquigarrow \psi) \to (\chi \rightsquigarrow \neg(\varphi \rightsquigarrow \psi))$

(A7) $(\varphi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\varphi)$

(MP) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

(R) $\dfrac{\varphi}{\top \rightsquigarrow \varphi}$

Completeness

Theorem

The system $\mathcal{S}$ is strongly sound and complete with respect to contact algebras:

$$\Gamma \vdash \varphi \quad \Leftrightarrow \quad \Gamma \models \varphi.$$

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To deal with the $\forall\exists$ statements (Q7) and (Q8) we add non-standard rules to the system $\mathcal{S}$.

$\Pi_2 - rules$

Non-standard rules for emulating ∀∃-statements

Π_2 – *rules*

Non-standard rules for emulating $\forall\exists$-statements

Definition

A Π_2-rule is one of the form:

$$(\rho) \quad \frac{F(\bar{\varphi}, \bar{p}) \to \chi}{G(\bar{\varphi}) \to \chi}$$

where F, G are formulas involving formula variables $\bar{\varphi}, \chi$ and fresh proposition letters $\bar{p}$.

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where F, G are formulas involving formula variables $\bar{\varphi}, \chi$ and fresh proposition letters $\bar{p}$.

We associate such a rule (ρ) with the $\forall\exists$-statement

$$\Phi_\rho \;:=\; \forall \bar{x}, z \left(G(\bar{x}) \not\leq z \;\;\underset{\rightarrow}{\;}\;\; \exists \bar{y} : \; F(\bar{x}, \bar{y}) \not\leq z \right)$$

in the signature $(\wedge, \neg, 1, \rightsquigarrow)$.

Let T be the first-order theory of contact algebras.

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Let $\{\rho_n\}_{n<\omega}$ be a set of $\Pi_2 - \mathit{rules}$.

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Theorem

The system $\mathcal{S} + \{\rho_n\}_{n<\omega}$ is strongly sound and complete with respect to $\mathbf{Mod}(T \cup \{\Phi_{\rho_n}\}_{n<\omega})$.

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By this theorem, extensions of $\mathcal{S}$ with Π_2-rules are complete with respect to $\forall\exists$-definable classes of contact algebras. $\forall\exists$-definable classes are the same as inductive classes (Chang-Łos-Suszko theorem).

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We define how to translate a quantifier-free formula $\Phi(\bar{x}, \bar{y})$ into a formula $\tilde{\Phi}(\bar{x}, \bar{y})$ of our language.

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Proposition

Let $\Phi(\bar{x}, \bar{y})$ be a quantifier-free formula.
The statement $\forall\bar{x}\exists\bar{y}\Phi(\bar{x}, \bar{y})$ is equivalent to the one associated to the Π_2-rule

$$(\rho_\Phi) \quad \frac{\tilde{\Phi}(\bar{\varphi}, \bar{p}) \to \chi}{\chi}$$

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$$\underset{\text{set of }\Pi_2\text{-rules}}{\{\rho_n\}_{n<\omega}} \longmapsto \underset{\forall\exists\text{-theory extending }T}{T \cup \{\Phi_{\rho_n}\}_{n<\omega}}$$

$$\underset{\forall\exists\text{-theory extending }T}{T'} \longmapsto \underset{\text{set of }\Pi_2\text{-rules}}{\{\rho_\Phi \mid \forall\bar{x}\exists\bar{y}\Phi(\bar{x},\bar{y}) \in T'\}}$$

Logics for inductive classes of contact algebras

Correspondence between logics and inductive classes

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Extensions of $\mathcal{S}$ with Π_2-rules	$\longleftrightarrow$	Inductive classes of contact algebras

The logic of compingent algebras

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(Q7) and (Q8) correspond to the following rules:

$$(\rho 7) \quad \frac{(\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

$$(\rho 8) \quad \frac{p \wedge (p \rightsquigarrow \varphi) \to \chi}{\varphi \to \chi}$$

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Thus we obtain:

Corollary

$\mathcal{S} + (\rho 7) + (\rho 8)$ is strongly sound and complete with respect to compingent algebras.

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Theorem (Criterion of admissibility)

A Π_2-rule (ρ) is admissibile in $\mathcal{S}$ if and only if any contact algebra $(B, \prec)$ is a substructure of some contact algebra $(C, \prec)$ satisfying Φ_ρ.

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Corollary

$(\rho 7)$ and $(\rho 8)$ are admissibile in $\mathcal{S}$.

The logic of compact Hausdorff spaces

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Definition

The MacNeille completion of a compingent algebra $(B, \prec)$ is $(\overline{B}, \prec)$, where $\overline{B}$ is the MacNeille completion of B and $\prec$ is defined as:

$$\alpha \prec \beta \;\Leftrightarrow\; \text{there exist } a, b \in B \text{ such that } \alpha \leq a \prec b \leq \beta.$$

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Lemma

Given a compingent algebra $(B, \prec)$, its MacNeille completion $(\overline{B}, \prec)$ is a de Vries algebra.

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Lemma

Given a compingent algebra $(B, \prec)$, its MacNeille completion $(\overline{B}, \prec)$ is a de Vries algebra.

Corollary

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Corollary

- $\mathcal{S} + (\rho 7) + (\rho 8)$ is sound and complete with respect to de Vries algebras.
- $\mathcal{S} + (\rho 7) + (\rho 8)$ is sound and complete with respect to compact Hausdorff spaces.

MacNeille canonicity and topological properties

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Corollary

Let the axiom (A) and the Π_2-rule (ρ) be MacNeille canonical.

- $\mathcal{S} + (\rho 7) + (\rho 8) + (\mathsf{A})$ is sound and complete with respect to de Vries algebras validating (A).
- $\mathcal{S} + (\rho 7) + (\rho 8) + (\rho)$ is sound and complete with respect to de Vries algebras satisfying Φ_ρ.

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Let the axiom (A) and the Π_2-rule (ρ) be MacNeille canonical.

- $\mathcal{S} + (\rho 7) + (\rho 8) + (\mathrm{A})$ is sound and complete with respect to de Vries algebras validating (A).
- $\mathcal{S} + (\rho 7) + (\rho 8) + (\rho)$ is sound and complete with respect to de Vries algebras satisfying Φ_ρ.

MacNeille canonical axioms and rules can be used to express topological properties.

- Connectedness:

MacNeille canonicity and topological properties

Examples

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  (C) $(\varphi \rightsquigarrow \varphi) \to (\top \rightsquigarrow \varphi) \vee (\top \rightsquigarrow \neg\varphi)$

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- Zero-dimensionality:

$$(\rho 9) \quad \frac{(\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \wedge (p \rightsquigarrow p) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

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  $\mathcal{S} + (\rho 7) + (\rho 8) + (\rho 9)$ is the logic of Stone spaces.

Related work

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In this language they provide propositional calculi related to RCC (Region Connection Calculus).

Some of these calculi involve particular non-standard rules:

$$\text{(NOR)} \quad \frac{\varphi \Rightarrow (aCp \vee p^* Cb)}{\varphi \Rightarrow aCb} \quad \text{where } p \text{ does not occurr in } a, b, \varphi$$

$$\text{(EXT)} \quad \frac{\varphi \Rightarrow (p = 0 \vee aCp)}{\varphi \Rightarrow (a = 1)} \quad \text{where } p \text{ does not occurr in } a, \varphi$$

Related work

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- Relational semantics based on Kripke frames;
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With respect to these semantics, the authors give completeness results for the propositional calculi they introduced.

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Conclusion

- We developed a finitary propositional calculus for compact Hausdorff spaces via de Vries algebras.
- We developed the theory of Π_2-rules, showing:
  - a correspondence between logics and inductive classes;
  - a semantic criterion for admissibility of Π_2-rules.
- We showed how MacNeille completions can be used to obtain logics for subclasses of compact Hausdorff spaces.

Thank you!