# Complete Additivity and Modal Incompleteness

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based on a joint work with Wesley Holliday (UC Berkeley)

Complete additi$\mathcal{V}$ity, together with lattice-$\mathcal{C}$ompleteness and $\mathcal{A}$tomicity, is one of the three defining properties of dual algebras of Kripke frames. Unlike the other two properties, it can hold even in the countably generated free algebra in the variety associated with a given logic, so it might seem harmless from the point of view of theoremhood and validity. Surprisingly, this turns out to be an illusion. We demonstrate the existence of $\mathcal{V}$-incomplete unimodal logics, i.e., logics whose associated variety of Boolean Algebras with Operators (BAOs) is not HSP-generated by its completely additive members. In the bimodal signature, the result can be strengthened to show there are non-degenerate varieties of BAOs containing no completely additive algebras. Pushing the boundary still further, we prove that the Blok Dichotomy obtains for this notion of completeness, i.e., even those logics which happen to be $\mathcal{V}$-complete tend to be surrounded by uncountably many $\mathcal{V}$-incomplete ones, which are sound wrt precisely the same class of $\mathcal{V}$-BAOs. This solves an open question posed more than a decade ago in my PhD Thesis and its strengthening posed in the *Algebras and coalgebras* chapter of the *Handbook of Modal Logic*. Furthermore, we also discovered that, despite its ostensibly second-order formulation (which obscured its real nature and made the issue of $\mathcal{V}$-(in)completeness particularly confusing), $\mathcal{V}$ is actually an elementary property, definable by a $\forall\exists\forall$-sentence. Holliday used this insight in his work on *possibility semantics* to provide elegant presentations of duals of categories of $\mathcal{V}$-BAOs. Moreover, our characterization of complete additivity allows syntactic criteria for $\mathcal{V}$-completeness inspired by characterizations of $\mathcal{A}$- and $\mathcal{AV}$-completeness in terms of conservativity of minimal hybrid extensions with various non-orthodox rules or $\mathcal{T}$-completeness in terms of conservativity of minimal tense extensions (unsurprisingly, all these properties are also $\forall\exists\forall$). Time permitting, I will describe how the presently investigated language of *Global Quantificational Modalities* can provide an uniform perspective on such syntactic characterizations.