Topologies on pseudoinfinite paths

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June 14, 2016

PLAN

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2. Topological semantics
3. Products of logics
4. Neighborhood frames
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Language and logics

$$\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \Box_i \phi, \;\; i = 1, 2.$$

$\bot$, $\rightarrow$ and $\Diamond_i$ are expressible in the usual way.
Normal modal logic.
$\mathsf{K_n}$ denotes the minimal normal modal logic with n modalities and $\mathsf{K} = \mathsf{K_1}$.
$\mathsf{L_1}$ and $\mathsf{L_2}$ — two modal logics with one modality $\Box$ then the fusion of these logics is defined as

$$\mathsf{L_1} * \mathsf{L_2} = \mathsf{K_2} + \mathsf{L_1'} + \mathsf{L_2}';$$

where $\mathsf{L_i'}$ is the set of all formulas from $\mathsf{L_i}$ where in all formulas $\Box$ is replaced by $\Box_i$.

Topological semantics

We can define topology on set $X \neq \varnothing$ by specifying a closure operator $\mathbf{C} : 2^X \to 2^X$, satisfying the Kuratowski axioms:

1. $\mathbf{C}(\varnothing) = \varnothing$, $\neg\Diamond\bot$
2. $A \subseteq \mathbf{C}(A)$, for $A \subseteq X$, $p \to \Diamond p$
3. $\mathbf{C}(A \cup B) = \mathbf{C}(A) \cup \mathbf{C}(B)$, for $A, B \subseteq X$, $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$
4. $\mathbf{C}\big(\mathbf{C}(A)\big) \subseteq \mathbf{C}(A)$. $A \subseteq X$, $\Diamond\Diamond p \to \Diamond p$

Logic S4

In topological semantics for modal logic closure operator correspond to $\Diamond$.
Topological model $(\mathfrak{X}, \theta)$, where $\mathfrak{X} = (X, \mathbf{C})$ — topological space:

$$p \longmapsto \theta(p) \subseteq X$$
$$\theta(\phi \vee \psi) = \theta(\phi) \cup \theta(\psi)$$
$$\theta(\neg\phi) = X \setminus \theta(\phi)$$
$$\theta(\Diamond\phi) = \mathbf{C}(\theta(\phi)).$$

$$\mathfrak{X} \models \phi \iff \forall\theta(\theta(\phi) = X),$$
$$Log(\mathfrak{X}) = \{\phi \,|\, \mathfrak{X} \models \phi\}\,.$$

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Alexandroff topology

On a transitive reflexive Kripke frame $F = (W, R)$ we can define topology, i.e. closure operator:

$$\mathbf{C}_F(A) = R^{-1}(A).$$

It will be an Alexandroff topology (any intersection of open sets is open, all points have minimal neighborhood).

Lemma

$F \models \phi \iff (W, C_F) \models \phi$.

We define $Top(F) = (W, C_F)$

Completeness of S4 w.r.t. all Alexandroff spaces.

Many topologies are not Alexandroff: $\mathbb{R}^n$, Cantor space, $\mathbb{Q}$ or any metric space.

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Derivational semantics

We can define topological space using derivative operator $d : 2^X \to 2^X$, where $d(A)$ is the set of all limit points of A.
In a similar way we define derivational semantics:

$$\theta(\Diamond\phi) = d(\theta(\phi))$$

The logic of all topological space is wK4 $= \mathsf{K} + \Diamond\Diamond p \to \Diamond p \vee p$ (Esakia'1981).
The logic of $\mathbb{Q}$, Cantor space (or any dense-in-itself zero-dimensional metric space) is D4 $= \mathsf{K} + \Diamond\Diamond p \to \Diamond p + \Diamond\top$ (Shehtman'1990).

The product of Kripke frames

For two frames $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$

$$F_1 \times F_2 = (W_1 \times W_2, R_1^*, R_2^*),\ \text{where } (a_1, a_2) R_1^* (b_1, b_2) \Leftrightarrow a_1 R_1 b_1 \ \&\ a_2 = b_2$$
$$(a_1, a_2) R_2^* (b_1, b_2) \Leftrightarrow a_1 = b_1 \ \&\ a_2 R_2 b_2$$

For two logics L_1 and L_2

$$\mathsf{L}_1 \times \mathsf{L}_2 = Log(\{F_1 \times F_2 \mid F_1 \models \mathsf{L}_1 \ \&\ F_2 \models \mathsf{L}_2\})$$

(Shehtman, 1978)
For two classes of frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$
$$Log(\{F_1 \times F_2 \mid F_1 \in \mathfrak{F}_1 \ \&\ F_2 \in \mathfrak{F}_2\}) \supseteq Log(\mathfrak{F}_1) * Log(\mathfrak{F}_2) +$$
$$+ \Box_1 \Box_2 p \leftrightarrow \Box_1 \Box_2 p + \Diamond_1 \Box_2 p \to \Box_2 \Diamond_1 p.$$

$$\mathsf{K} \times \mathsf{K} = \mathsf{K} * \mathsf{K} + \Box_1 \Box_2 p \leftrightarrow \Box_1 \Box_2 p + \Diamond_1 \Box_2 p \to \Box_2 \Diamond_1 p$$
$$\mathsf{S4} \times \mathsf{S4} = \mathsf{S4} * \mathsf{S4} + \Box_1 \Box_2 p \leftrightarrow \Box_1 \Box_2 p + \Diamond_1 \Box_2 p \to \Box_2 \Diamond_1 p$$

$$\vdots$$

The product of topological spaces

(van Benthem et al, 2005)
For two topological space $\mathfrak{X}_1 = (X_1, \tau_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2)$

$$\mathfrak{X}_1 \times \mathfrak{X}_2 = (X_1 \times X_2, \tau_1^*, \tau_2^*), \text{ where } \tau_1^* \text{ has base } \{U_1 \times x_2 \,|\, U_1 \in \tau_1 \ \& \ x_2 \in X_2\}$$
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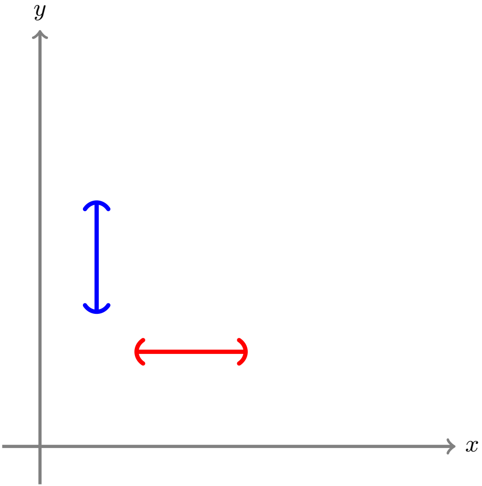

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For two logics L_1 and L_2

$$\mathsf{L}_1 \times_t \mathsf{L}_2 = Log(\{\mathfrak{X}_1 \times \mathfrak{X}_2 \,|\, \mathfrak{X}_1 \models \mathsf{L}_1 \ \& \ \mathfrak{X}_2 \models \mathsf{L}_2\})$$
$$\mathsf{S4} \times_t \mathsf{S4} = Log(\mathbb{Q} \times \mathbb{Q}) = \mathsf{S4} * \mathsf{S4} \quad \text{(van Benthem et al, 2005)}$$
$$Log(\mathbb{R} \times \mathbb{R}) \neq \mathsf{S4} * \mathsf{S4} \quad \text{(Kremer, 2010?)}$$
$$Log(Cantor \times Cantor) \neq \mathsf{S4} * \mathsf{S4}$$

d-logic of product of topological spaces was considered by L. Uridia (2011).

$$Log_{\,d}(\mathbb{Q} \times \mathbb{Q}) = \mathsf{D4} * \mathsf{D4}$$

Generalization to neighborhood frames was done by K. Sano (2011).

Known results

Theorem (2012)

Let L_1 *and* L_2 *be from the set* $\{\mathsf{D}, \mathsf{T}, \mathsf{D4}, \mathsf{S4}\}$ *then*

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = \mathsf{L}_1 * \mathsf{L}_2.$$

Not straightforward but still a

Corollary

In derivational semantics

1. $\mathsf{D4} \times_d \mathsf{D4} = \mathsf{D4} * \mathsf{D4}$.
2. *[Uridia'2011]* $Log_d(\mathbb{Q} \times \mathbb{Q}) = \mathsf{D4} * \mathsf{D4}$

Topological semantics based on closure operator or derivative operator can be generalized in the neighborhood semantics.

We can consider neighborhood function $\tau : X \to 2^{2^X}$. For $x \in X$ $\tau(x)$ is a set of neighborhoods of x. It connected with $\mathbf{C}$ is the following way:

$$A \in \tau(x) \iff x \in \mathbf{I}(A), \text{ where } \mathbf{I}(A) = X \setminus \mathbf{C}(X \setminus A).$$

And for derivational semantics

$$A \in \tau(x) \iff x \in \bar{d}(A), \text{ where } \bar{d}(A) = X \setminus d(X \setminus A).$$

Neighborhood frames

A (normal) neighborhood frame (or an n-frame) is a pair $\mathfrak{X} = (X, \tau)$, where

- $X \neq \varnothing$;
- $\tau : X \to 2^{2^X}$, such that $\tau(x)$ is a filter on X;

τ — neighborhood function of $\mathfrak{X}$,
$\tau(x)$ — neighborhoods of x.
Filter on X: nonempty $\mathcal{F} \subseteq 2^X$ such that
1) $U \in \mathcal{F}$ & $U \subseteq V \Rightarrow V \in \mathcal{F}$
2) $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$ (filter base)

The neighborhood model (n-model) is a pair $(\mathfrak{X}, V)$, where $\mathfrak{X} = (X, \tau)$ is a n-frame and $V : PV \to 2^X$ is a valuation. Similar: neighborhood 2-frame (n-2-frame) is (X, τ_1, τ_2) such that τ_i is a neighborhood function on X for each i.
Validity in model:

$$M, x \models \Box_i \psi \iff \exists V \in \tau_i(x) \forall y \in V (M, y \models \psi).$$

$M \models \varphi \quad \mathfrak{X} \models \varphi \quad \mathfrak{X} \models L \quad Log(\mathcal{C}) = \{\varphi \mid \mathfrak{X} \models \varphi$ for some $\mathfrak{X} \in \mathcal{C}\}$

$$nV(L) = \{\mathfrak{X} \mid \mathfrak{X} \text{ is an n-frame and } \mathfrak{X} \models L\}$$

Connection with Kripke frames

Definition

Let $F = (W, R)$ be a Kripke frame. We define neighborhood frame $\mathcal{N}(F) = (W, \tau)$ as follows. For any $w \in W$

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}.$$

Lemma

Let $F = (W, R)$ be a Kripke frame. Then

$$Log(\mathcal{N}(F)) = Log(F).$$

Bounded morphism for n-frames

Definition

Let $\mathfrak{X} = (X, \tau_1, \ldots)$ and $\mathcal{Y} = (Y, \sigma_1, \ldots)$ be n-frames. Then function $f : X \to Y$ is a bounded morphism if

1. f is surjective;
2. for any $x \in X$ and $U \in \tau_i(x)$ $f(U) \in \sigma_i(f(x))$;
3. for any $x \in X$ and $V \in \sigma_i(f(x))$ there exists $U \in \tau_i(x)$, such that $f(U) \subseteq V$.

In notation $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$.

Lemma

If $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$ then $Log(\mathcal{Y}) \subseteq Log(\mathfrak{X})$.

Not always fusion

It is not the case for logic K!

Lemma

For any two n-frames $\mathfrak{X}_1$ and $\mathfrak{X}_2$

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models \Box_1\bot \to \Box_2\Box_1\bot.$$

And even more, for any closed $\Box_1$-free formula ϕ and any closed $\Box_2$-free formula ψ

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models \phi \to \Box_1\phi, \qquad \mathfrak{X}_1 \times \mathfrak{X}_2 \models \psi \to \Box_2\psi.$$

Proof.

$$\begin{aligned} \mathfrak{X}_1 \times \mathfrak{X}_2, (x,y) \models \Box_1\bot \iff \varnothing \in \tau_1'(x,y) \iff \\ \varnothing \in \tau_1(x) \iff \forall y' \in X_2\ (\varnothing \in \tau_1'(x,y')) \iff \\ \forall y' \in X_2\ (\mathfrak{X}_1 \times \mathfrak{X}_2, (x,y') \models \Box_1\bot) \implies \mathfrak{X}_1 \times \mathfrak{X}_2, (x,y) \models \Box_2\Box_1\bot. \end{aligned}$$

Hence, $\mathfrak{X}_1 \times \mathfrak{X}_2 \models \Box_1\bot \to \Box_2\Box_1\bot$. □

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Proof.

Since ψ does not contain neither $\Box_2$, nor variables, its value does not depend on the second coordinate. Let $F = \mathfrak{X}_1 \times \mathfrak{X}_2$. So $F, (x, y) \models \psi$, then $\forall y'(F, (x, y') \models \psi)$, hence, $F, (x, y) \models \Box_2\psi$. □

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Definition

For two unimodal logics L_1 and L_2, we define

$$\langle L_1, L_2\rangle = L_1 * L_2 + \Delta, \text{ where}$$

$$\Delta = \{\phi \to \Box_2\phi \mid \phi \text{ is closed and } \Box_2\text{-free}\} \cup \{\psi \to \Box_1\psi \mid \psi \text{ is closed and } \Box_1\text{-free}\}.$$

Lemma

For any two normal modal logics L_1 and L_2 $\langle L_1, L_2\rangle \subseteq L_1 \times_n L_2$.

Note that if $\Diamond\top \in L_1 \cap L_2$ then $L_1 * L_2 \models \Delta$.

Cantor space and infinite paths

Standart construction: Cantor space as the set on infinite paths on infinite binary tree T_2.
The base of topology is the sets of the following type:

$$U_m(\alpha) = \{\beta \,|\, a_1 = b_1, \ldots a_m = b_m\}.$$

where α and β are two infinite paths:

$$\alpha = a_1 a_2 a_3 \ldots, \qquad \beta = b_1 b_2 b_3 \ldots.$$

In order to proof completeness of S4 w.r.t. Cantor space we need to construct p-morphism from Cantor space to arbitrary finite S4-frame. [Mints, 1998]

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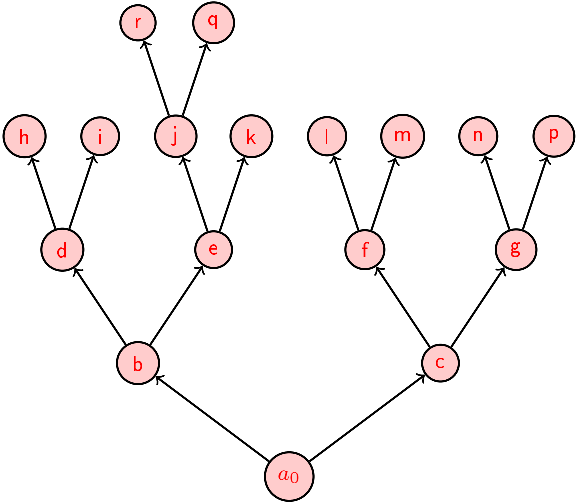
Constructing “dense” topologies from Kripke frames

We need to construct a “dense” topological space based on a Kripke frame. This becomes important in studying of products of topological spaces.

Example



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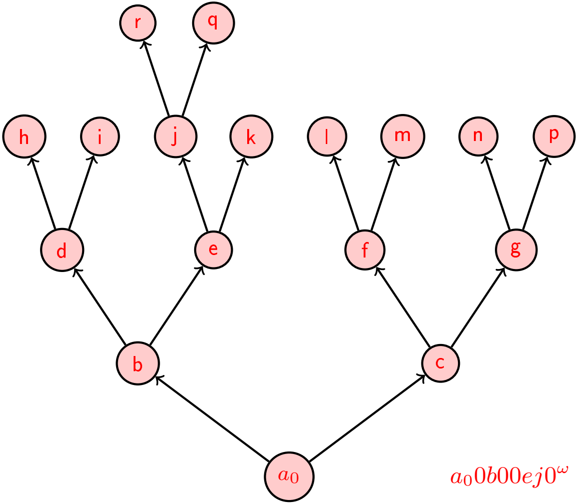

$a_0 0 b 0 0 e j 0^\omega$

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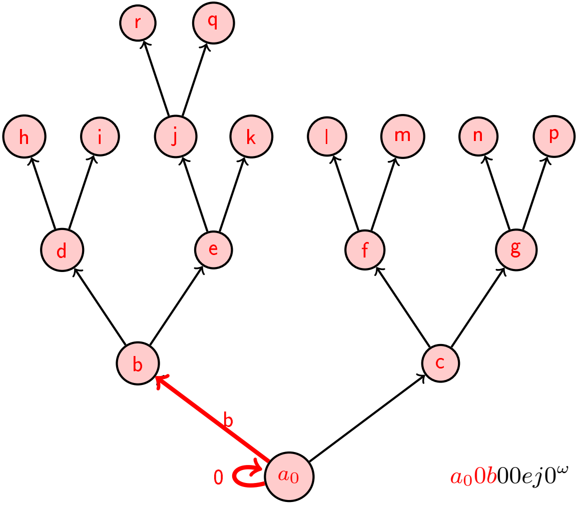

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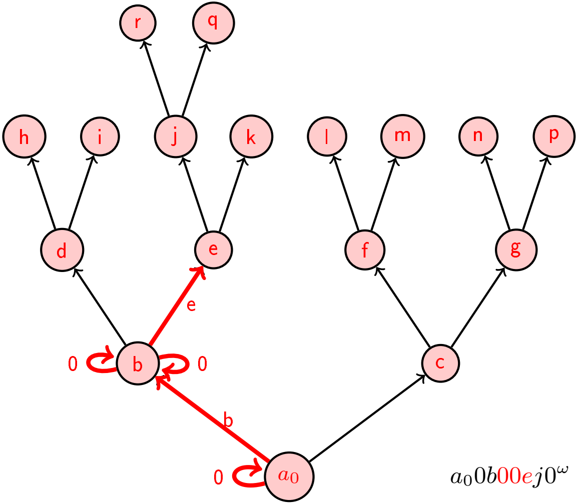

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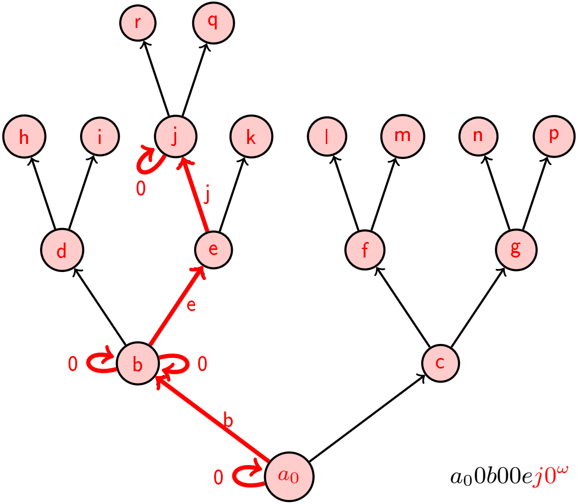

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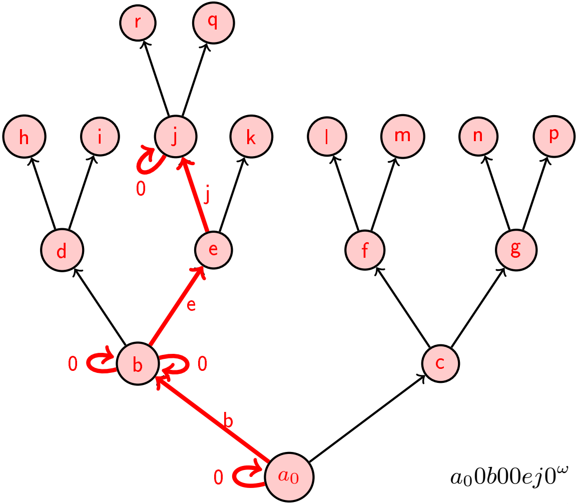

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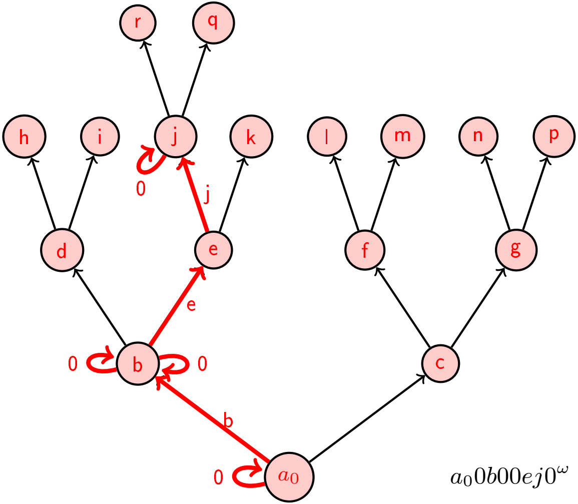

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$a_0 \;\; b \quad ej$

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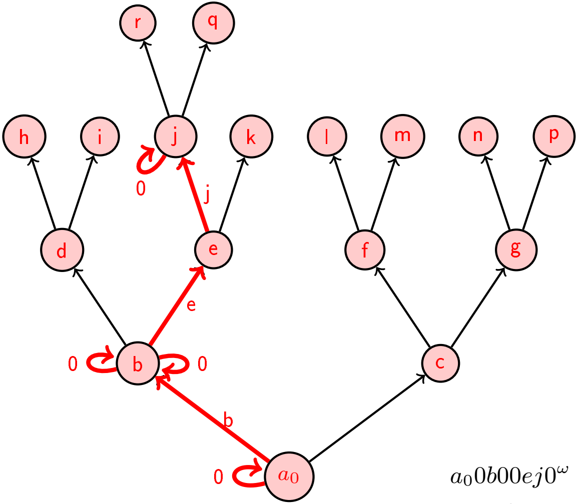

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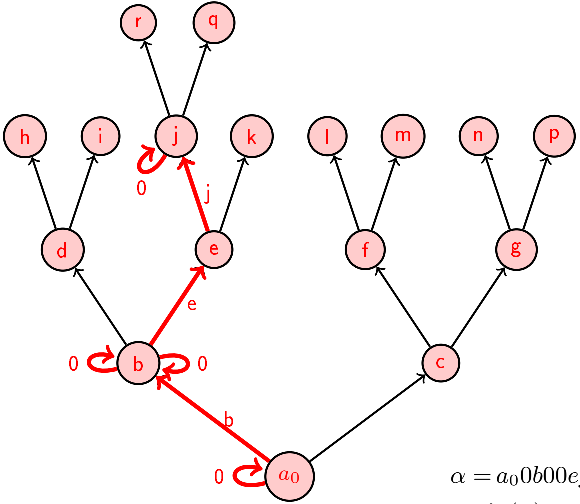

$$\alpha = a_0 0 b 0 0 e j 0^\omega$$

$$f_F(\alpha) = j$$

$N_\omega(F)$

Definition
Sets $U_n(\alpha)$ form a filter base. So we can define

$$\tau(\alpha) - \text{the filter with base } \{U_n(\alpha) \,|\, n \in \mathbb{N}\};$$
$$\mathcal{N}_\omega(F) = (W_\omega, \tau) \text{ — is a dense n-frame based on } F.$$

Frame $\mathcal{N}_\omega(F)$ is dense in a sense that the intersection of all neighborhoods of a point is empty. So, there are no minimal neighborhoods unlike in $Top(F)$.

Lemma
Let $F = (W, R)$ be a Kripke frame with root a_0, then

$$f_F : \mathcal{N}_\omega(F) \twoheadrightarrow \mathcal{N}(F).$$

Corollary
For any frame F $Log(\mathcal{N}_\omega(F)) \subseteq Log(\mathcal{N}(F)) = Log(F)$.

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Counterexample

It possible that $Log(\mathcal{N}_\omega(F)) \neq Log(F)$. Consider:

$$G = (\{1\}^*, S),\ 1^n S 1^m \iff m = n + 1.$$

Obviously $G \models \Diamond p \to \Box p$.

Lemma

$\mathcal{N}_\omega(G) \not\models \Diamond p \to \Box p$

Proof.

Consider valuation $\theta(p) = \left\{0^{2n}10^\omega \mid n \in \mathbb{N}\right\}$. Then in any neighbourhood of point 0^ω there are points where p is true and there are points where p is false. Hence,

$$\mathcal{N}_\omega(G) \models \Diamond p \wedge \Diamond \neg p.$$

□

Completeness results

Theorem (2014)

$\mathsf{K} \times_n \mathsf{K} = \langle \mathsf{K}, \mathsf{K} \rangle$.

Theorem

If logics L_1 *and* L_2 *are axiomatizable by closed formulas and by axioms like* $\Diamond^k p \to \Diamond p$ *then* $\mathsf{L}_1 \times_n \mathsf{L}_2 = \langle \mathsf{L}_1, \mathsf{L}_2 \rangle$.

Corollary

$\mathsf{K4} \times_d \mathsf{K4} = \langle \mathsf{K4}, \mathsf{K4} \rangle$.

Logic S5

We put

$$\begin{aligned} \Delta_1 &= \{\phi \to \Box_2\phi \,|\, \phi \text{ is closed and } \Box_2\text{-free}\}\,, \\ com_{12} &= \Box_1\Box_2 p \to \Box_2\Box_1 p, \\ com_{21} &= \Box_2\Box_1 p \to \Box_1\Box_2 p, \\ chr &= \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p. \end{aligned}$$

Theorem

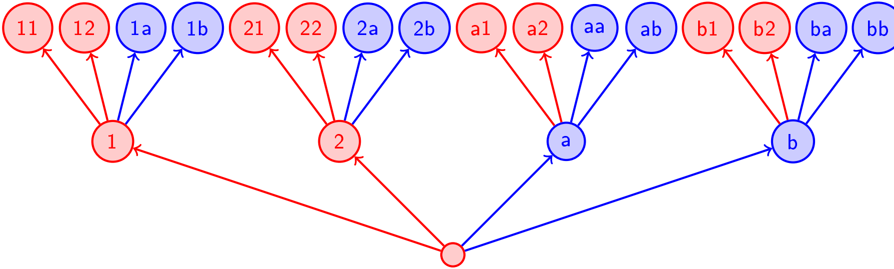
If logic L *is axiomatizable by closed formulas and by axioms like* $\Diamond^k p \to \Diamond p$ *then* $\mathsf{L} \times_n \mathsf{S5} = \mathsf{L} * \mathsf{S5} + \Delta_1 + com_{12} + chr$.

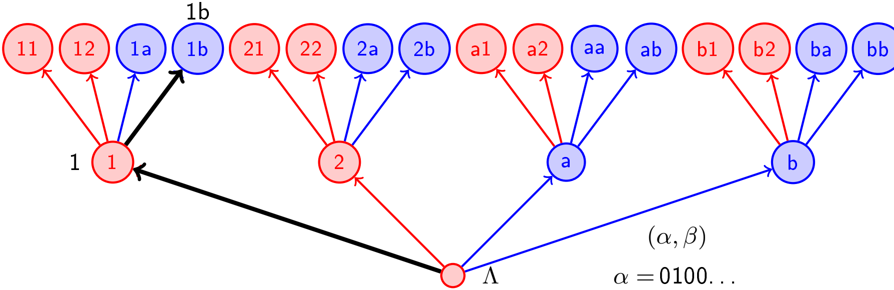
How to prove

PLAN

We have two loogic L_1 and L_2

$$\text{Canonicity of the logic } \langle \mathsf{L}_1, \mathsf{L}_2 \rangle.$$
$$\Downarrow$$
$$\text{Construct } F_1 \models \mathsf{L}_1 \text{ and } F_2 \models \mathsf{L}_2 \text{ and } \langle F_1, F_2 \rangle \twoheadrightarrow \mathcal{F}_{\langle \mathsf{L}_1, \mathsf{L}_2 \rangle}.$$
$$\Downarrow$$
$$\text{Construct } \mathcal{N}_\omega{}^{\Gamma_1}(F_1) \times \mathcal{N}_\omega{}^{\Gamma_2}(F_2) \twoheadrightarrow \mathcal{N}\left(\langle F_1, F_2 \rangle^{\Gamma_1 \cup \Gamma_2}\right).$$
$$\Downarrow$$
$$\text{Check that } \mathcal{N}_\omega{}^{\Gamma_1}(F_1) \models \mathsf{L}_1 \text{ and } \mathcal{N}_\omega{}^{\Gamma_2}(F_2) \models \mathsf{L}_2$$

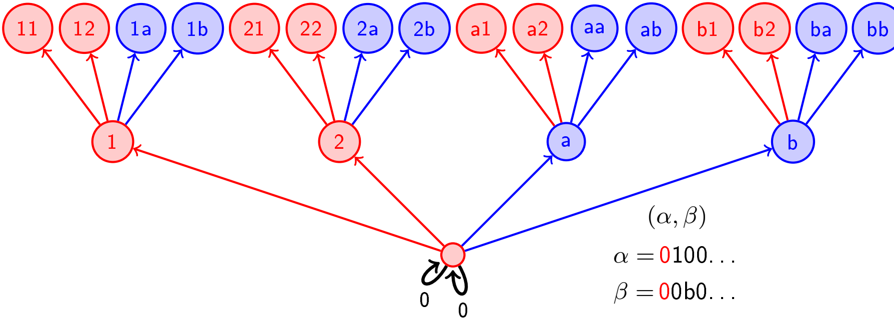

11
12
1a
1b
21
22
2a
2b
a1
a2
aa
ab
b1
b2
ba
bb
1
2
a
b



(α, β)

$\alpha = 0100\ldots$

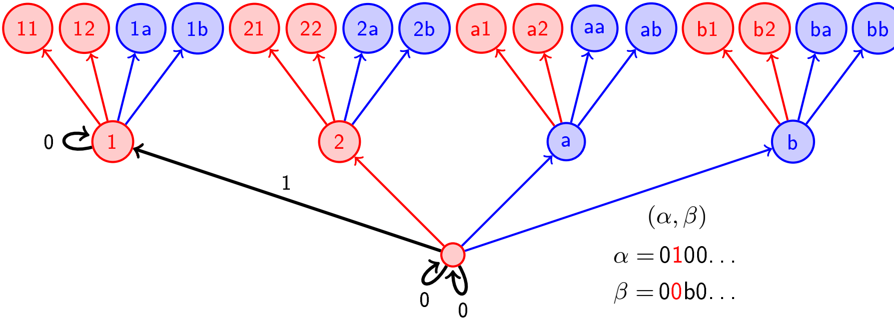
$\beta = 00b0\ldots$



(α, β)

$\alpha = 0100\ldots$

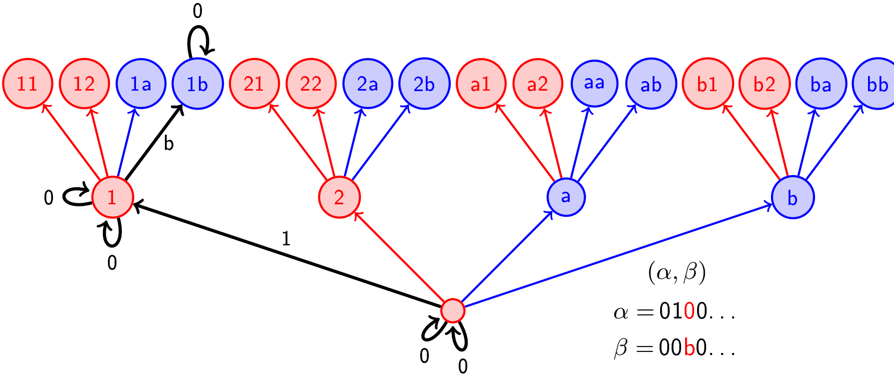
$\beta = 00b0\ldots$



(α, β)

$\alpha = 0100\ldots$

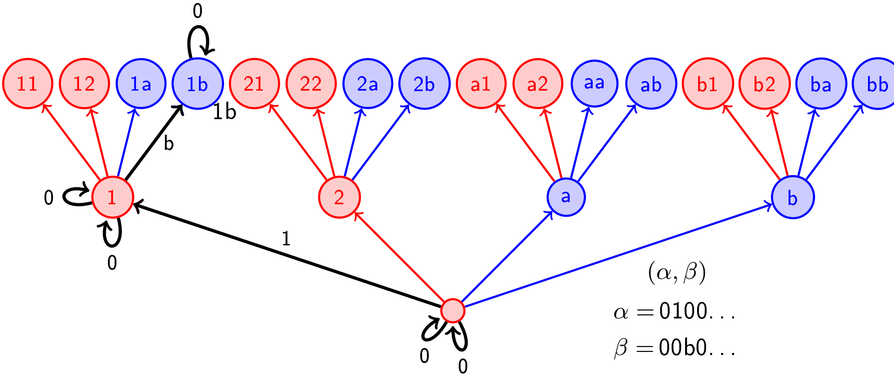
$\beta = 00b0\ldots$



(α, β)

$\alpha = 0100\ldots$

$\beta = 00b0\ldots$



(α, β)

$\alpha = 0100\ldots$

$\beta = 00b0\ldots$

How to prove for S5

PLAN
We have two loogic L and S5

$$\text{Canonicity of the logic } \langle \mathsf{L}, \mathsf{S5}].$$
$$\Downarrow$$
$$\text{Construct } F_1 \models \mathsf{L} \text{ and } F_2 = (\mathbb{R}, \mathbb{R}^2) \text{ and } \langle F_1, F_2 \rangle \twoheadrightarrow \mathcal{F}_{\langle \mathsf{L}, \mathsf{S5}]}.$$
$$\Downarrow$$
$$\text{Construct } {\mathcal{N}_\omega}^\Gamma(F_1) \times \mathcal{N}_\omega(F_2) \twoheadrightarrow \mathcal{N}\left(\langle F_1, F_2]^\Gamma\right).$$
$$\Downarrow$$
$$\text{Check that } {\mathcal{N}_\omega}^\Gamma(F_1) \models \mathsf{L}$$

We define

$$C_{12} = \{ab \mapsto ba \,|\, a \in W_1,\ b \in W_2\}$$

We also define three Kripke frames:

$$\langle F_1, F_2\rangle = (F_1 \between F_2, R_1^{<}, R_2^{<})$$

$$\langle F_1, F_2] = (F_1 \between F_2, R_1^{<}, R_2^{\lhd})$$

$$\vec{a} R_1^{<} \vec{b} \iff \exists u \in W_1 (\vec{b} = \vec{a}u)$$

$$\vec{a} R_2^{<} \vec{b} \iff \exists v \in W_2 (\vec{b} = \vec{a}v)$$

$$\vec{a} R_2^{\lhd} \vec{b} \iff \exists \vec{b}' (\vec{a} R_2^{<} \vec{b}' \ \&\ \vec{b}' \underset{C_{12}}{\Longrightarrow} \vec{b})$$

Lemma

For F_1 and F_2 defined above

$$\langle F_1, F_2] \models com_{12}, chr\Delta_1.$$

THANK YOU!