Spectra of compact regular frames

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= the set of prime filters of L

with enough additional structure to recover L back

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Priestley space $\mathrm{Spec}(L)$.
Sets of the form $\varphi(a)$ — precisely those clopens
which are upsets (w. r. t. $\leqslant$).

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$$\bigvee_{i \in I} a_i = 1 \;\Rightarrow\; a_{i_1} \vee \cdots \vee a_{i_n} = 1 \text{ for some } i_1, \ldots, i_n \in I$$

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(since $\neg b$ is the complement of the closure of b)).

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In fact, *every compact regular frame is isomorphic to the frame of all open sets of some compact Hausdorff space* (Isbell duality).

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Main thing for that: it has *enough points*.

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(i. e. its complement is a principal ideal)
$\iff$ the filter $\mathbf{p}$ is *completely prime* –

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(lattice homomorphism preserving all joins).

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iff $\downarrow \mathfrak{p}$ *is clopen*
(this clopen downset is the complement of $\varphi(p)$ for the corresponding prime element $p \in L$).

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of L, and its join is determined by

$$\varphi(\bigvee \mathfrak{I}_U) = \mathbf{C}\,U.$$

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Converse goes through another equivalent condition —
L does not have any nontrivial dense open upsets.

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It also follows that the image of $\mathrm{pt}(L) \hookrightarrow \mathrm{Spec}(L)$ lies in $\mathrm{Min}(L)$.

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Putting the two together:
For L compact, $\mathrm{pt}(L) \hookrightarrow \mathrm{Spec}(L)$ is inside $\mathrm{Min}(L)$;
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Thus for compact regular frames one may identify $\mathrm{pt}(L)$ with $\mathrm{Min}(L)$.

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More is true: for $L = \mathscr{O}(X)$ with a compact Hausdorff X,
the composite $X \approx \mathrm{pt}(\mathscr{O}(X)) \approx \mathrm{Min}(L) \subseteq \mathbf{S}\mathrm{pec}(\mathscr{O}(X))$
is a homeomorphism onto $\mathrm{Min}(L)$ with the subspace topology.

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The reverse map $(\neg\neg)^{-1} : \mathrm{Spec}(L^{\neg\neg}) \hookrightarrow \mathrm{Spec}(L)$ is a homeomorphism onto $\mathrm{Max}(L) \subseteq \mathrm{Spec}(L)$.

Max(L) and the Gleason cover

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Thus $\tilde{X}$ is homeomorphic to $\mathrm{Max}(\mathscr{O}(X))$;
and we saw that X itself is homeomorphic to $\mathrm{Min}(\mathscr{O}(X))$.
The map γ_X can be also naturally realized in these terms.

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A space X is normal iff $\mathscr{O}(X)$ is normal in this sense
(given disjoint closed sets, let a and b be their complements,
then a' and b' will be the required separating disjoint opens).

Normality in terms of Spec

From II.3.7 of Johnstone's "Stone Spaces" one finds:
a distributive lattice L is normal iff for any $\mathfrak{p} \in \mathrm{Spec}(L)$ there is a *unique* $\mathfrak{m} \in \mathrm{Min}(L)$ with $\mathfrak{p} \geqslant \mathfrak{m}$.

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for the inclusion $\mathrm{Min}(L) \hookrightarrow \mathrm{Spec}(L)$ which is actually continuous.

In particular, we get a well-defined continuous map

$$\pi_X : \mathrm{Max}(L) \hookrightarrow \mathrm{Spec}(L) \twoheadrightarrow \mathrm{Min}(L).$$

Gleason cover in terms of Spec

Using uniqueness involved in the definition of π_X one shows easily that for $L = \mathscr{O}(X)$ the diagram

$$\begin{tikzcd}
\mathrm{Spec}(L^{\neg\neg}) \arrow[r, "\approx"] \arrow[d, "\gamma_X"'] & \mathrm{Max}(L) \arrow[d, "\pi_X"] \\
\mathrm{pt}(L) \arrow[r, "\approx"] & \mathrm{Min}(L)
\end{tikzcd}$$

commutes.

Zero-dimensionality in terms of Spec

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For any clopen upset U of $\mathrm{Spec}(L)$, let

$$Z_U := \bigcup \{\, \varphi(c) \subseteq U \mid c \in L \text{ complemented} \,\}\,.$$

In other words, Z_U is the union of all clopen bisets contained in U.

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In other words, Z_U is the union of all clopen bisets contained in U.

It is then straightforward to show that L is zero-dimensional iff $Z_{\varphi(a)}$ is dense in $\varphi(a)$ for every $a \in L$.

Extremal disconnectedness

Call a frame L *extremally disconnected* if $\neg a \vee \neg\neg a = 1$ for every $a \in L$.
Equivalently, if every regular element is complemented.
It is then more or less clear that $\mathscr{O}(X)$
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A compact regular frame L is extremally disconnected
iff for every $\mathfrak{p} \in \mathrm{Spec}(L)$ *there is a unique* $\mathfrak{q} \in \mathrm{Max}(L)$ *with* $\mathfrak{q} \geqslant \mathfrak{p}$.

Scatteredness

Recall that a space X is *scattered* if every nonempty subspace of X contains an isolated point.

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A T_0 space X is scattered iff $\mathscr{O}(X)$ is scattered in this sense.

Scatteredness in terms of Spec

A frame L is scattered
iff the maximum of any clopen downset of $\mathrm{Spec}(L)$ *is clopen,*
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The key observation here is that $b \in D_a$ is equivalent to

$$\mathrm{Max}(X \setminus \varphi(a)) \subseteq \varphi(b).$$

Rank and height

A scattered space X has finite *Cantor-Bendixson rank n* if
$\delta^{n+1}(X) = \varnothing$ and $\delta^{n}(X) \neq \varnothing$
(equivalently, $\tau^{n+1}(0) = 1$ and $\tau^{n}(0) \neq 1$ in $L = \mathscr{O}(X)$).

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A scattered frame L has rank n iff $\mathrm{Spec}(L)$ *is of height n*

(that is, maximal length of a chain in $(\mathrm{Spec}(L), \leqslant)$ is n).

Infinite height

In fact for any compact regular frame L which is not scattered, $\mathrm{Spec}(L)$ has infinite height.

Essentially this boils down to the fact that
a compact Hausdorff space X is not scattered
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It is easy to show that $\mathrm{Spec}(\mathscr{O}([0,1]))$ has infinite height.
Then one uses the fact that for a continuous surjection $X \twoheadrightarrow Y$
between compact Hausdorff spaces
height of $\mathrm{Spec}(\mathscr{O}(X))$ is no less than that of $\mathrm{Spec}(\mathscr{O}(Y))$.

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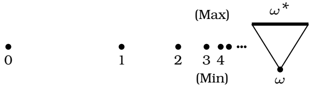
This in turn depends on the "pointless" version of the fact that
any continuous map $X \to Y$ with X compact and Y regular is closed.

Morphisms

For any frame homomorphism $h : L \to M$ with L regular and M compact, the induced map $h^{-1} : \mathrm{Spec}(M) \to \mathrm{Spec}(L)$ is a co-p-morphism.

Pictures

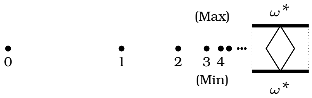
$\omega + 1$



$\mathrm{Spec}(\mathscr{O}(\omega + 1))$

Pictures

$\beta\omega$



$\mathrm{Spec}(\mathscr{O}(\beta\omega))$

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THANK YOU!