Monadic second order logic as the model companion of temporal logic

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The aim of this talk is to transfer basic ideas from Robinson’s style model-theoretic algebra to the realm of algebraic logic.
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We first review basic information on model completeness; then we relate model completeness to bisimulation quantifiers in propositional intuitionistic and (some) modal logics. Finally, we show how model completeness connects linear and branching time temporal logics with monadic second order logic.
Outline

1 Review of Model Completeness
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2. Model Completeness and Bisimulation Quantifiers
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3. Model Completeness and Monadic Second Order Logic
   - Infinite words and LTL
   - The ‘Fair CTL’ logic
   - Binary trees
   - Arbitrarily branching trees
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Model companion: definition

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Let $T \subseteq T^*$ be theories in a first-order language $\mathcal{L}$ (with $T$ universal).
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1. $T^*$ is model complete, i.e., for any $\mathcal{L}$-formula $\phi$ there is a universal $\mathcal{L}$-formula $\psi$ such that $T \vdash \phi \iff \psi$;
2. $T^*$ is a companion of $T$, i.e., for any universal $\mathcal{L}$-formula $\phi$, $T^* \vdash \phi$ if, and only if, $T \vdash \phi$.

Equivalently,

1. All embeddings between $T^*$-models are elementary;
2. Every $T$-model embeds in some $T^*$-model and vice versa.
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Fact

A theory $T^*$ is a model companion of $T$ iff the class of $T^*$-models coincides with the class of existentially closed $T$-models.
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In particular, any theory $T$ has at most one model companion, and the model companion of $T$ exists iff the class of existentially closed $T$-models is elementary.
Model companion: basic facts

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- In particular, any theory $T$ has at most one model companion, and the model companion of $T$ exists iff the class of existentially closed $T$-models is elementary.
- A model companion $T^*$ of $T$ is a model completion iff the class of $T$-models has amalgamation iff $T^*$ has quantifier elimination.
# Model companion: examples

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<th>Model companion</th>
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<tbody>
<tr>
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<tr>
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Model companion: tasks

Concerning a given $T$, we are left with three basic tasks:

(I) showing that $T^*$ exists;

(II) axiomatizing $T^*$ in an understandable way;

(III) producing some concrete model of $T^*$. 

Non trivial mathematics can be involved in that. We point out that, in principle, task (I) is quite independent from the other two. For instance, one can accomplish task (I) just proving that the class of existentially closed models is elementary.
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We point out that, in principle, task (I) is quite independant from the other two. For instance, one can accomplish task (i) just proving that the class of existentially closed models is elementary.
Another method for task (I), is to find a mechanism producing for each single existential $\varphi$ a universal ‘solvability condition’ $\psi$ for it. More precisely, we assign to every existential $\varphi$ a universal $\psi$ and propose

$$\forall x (\varphi(x) \leftrightarrow \psi(x))$$

(1)

as axioms for $T^*$. For this to work, we need two conditions:

(i) $\forall x (\varphi(x) \rightarrow \psi(x))$ must be $T$-valid (because this is a universal sentence);

(ii) $\forall x (\psi(x) \rightarrow \varphi(x))$ must be true in all existentially closed models.

Notice also that, by taking negations, one could produce out of a universal $\psi$ an existential $\varphi$ satisfying the same conditions.
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The case of Boolean Algebras

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For Boolean algebras (= classical logic), the solution is easy and well known.

- Task (I): model companion exists (actually, it is a model completion);
- Task (II): the axiomatization is simple: just one axiom saying that there are no atoms;
- Task (III): any free algebra on infinitely many generators is a model of the model companion.
The case of Heyting Algebras

For Heyting algebras (= intuitionistic logic), the situation is more complex and it is related to the so-called bisimulation quantifiers. We review Pitts’ theorem.

\[\begin{align*}
\text{Theorem (A.M. Pitts)} \\
\text{For each propositional variable } x \text{ and for each formula } t \text{ of IpC, there exist formulas } \\
\exists x t \text{ and } \forall x t \text{ of IpC (effectively computable from } t) \\
\text{containing only variables not equal to } x \text{ which occur in } t, \text{ and such that} \\
\vdash \text{IpC } \exists x t \rightarrow u \text{ iff } \vdash \text{IpC } t \rightarrow u \\
\text{and } \vdash \text{IpC } u \rightarrow \forall x t \text{ iff } \vdash \text{IpC } u \rightarrow t.
\end{align*}\]
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**Theorem (A.M. Pitts)**

For each propositional variable $x$ and for each formula $t$ of $IpC$, there exist formulas $\exists^x t$ and $\forall^x t$ of $IpC$ (effectively computable from $t$) containing only variables not equal to $x$ which occur in $t$, and such that for any formula $u$ not involving $x$, we have

$$\vdash_{IpC} \exists^x t \rightarrow u \quad \text{iff} \quad \vdash_{IpC} t \rightarrow u$$

and

$$\vdash_{IpC} u \rightarrow \forall^x t \quad \text{iff} \quad \vdash_{IpC} u \rightarrow t.$$
The case of Heyting Algebras

From Pitts’ theorem, it turns out that the system of equations and inequations with parameters \( a \) from a Heyting algebra \( H \)

\[
t(a, x) = 1 \& u_1(a, x) \neq 1 \& \ldots \& u_m(a, x) \neq 1
\]  

(2)

is solvable in an extension of \( H \) iff the quantifier-free formula

\[
(\exists^x t)(a) = 1 \& (\forall^x (t \rightarrow u_1))(a) \neq 1 \& \ldots
\]

\[
\ldots \& (\forall^x (t \rightarrow u_m))(a) \neq 1
\]  

(3)

is true in \( H \) (for proof details see S.Ghilardi, M.Zawadowski’s book “Sheaves, Games and Model Completions”).
The case of Heyting Algebras

But this means that we can axiomatize the model companion $T^*$ by the formulas

$$\exists x \ (2) \leftrightarrow (3)$$

(4)

This fits the schema (1); it is even better, because (3) is quantifier-free, so that (4) is in fact a quantifier-elimination procedure (quantifiers can be eliminated one by one) and the model companion is actually a model completion. The quantifier-elimination procedure is effective because terms like $\exists^x t$ and $\forall^x t$ can be effectively computed.

Thus Task (1) is done for the Heyting algebras case.
Unfortunately, **Tasks (II)-(III) are still open problems** (but recently Darnière and Juncker announced a solution for Task (II) for the locally finite amalgamable varieties of Heyting algebras).
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The curious fact is that there are a lot of existentially closed Heyting algebras (every Heyting algebra embeds into an existentially closed one by general facts) and, via the algorithm supplied by (4), we can get all first order information we want to have concerning them. Yet we are unable to supply any concrete example and, even worse, we know that obvious candidate examples (free algebras, open sets of a topological space, etc.) are not good.
The case of Heyting Algebras

Exercise

In any existentially closed non degenerate Heyting algebra $H$ we have that:

(i) the relation $<$ is dense;

(ii) for every element $y \neq 0$ there is a complemented element $x$ such that $0 < x < y$ (consequently, regular and complemented elements of $H$ form atomless Boolean algebras);

(iii) the dual lattice of $H$ has no non-zero join-irreducible elements;

(iv) the difference of $x$ and $y$ exists iff $x \leq y \lor \neg y$. 
The case of Modal Algebras

The existence of a model completion and a suitable version of Pitts theorem are equivalent under certain hypothesis from universal algebras (see G.Z. book mentioned above); these hypotheses apply to $K_4$-based varieties but not to $K$-based varieties (and in fact, Pitts theorem holds for $K$ but modal algebras do not have a model completion).

It follows that $GL$ (i.e. diagonalizable) and $Grz$-algebras have model completions, whereas $S_4$-algebras do not (since they enjoy amalgamation, the same negative result applies to the existence of a model companion).

For $GL$ and $Grz$-algebras the situation is the same as for Heyting algebras (tasks (II)-(III) are open).
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**MSOL on infinite words**

- **$S1S$** (second-order logic of one successor) is a monadic second-order logic interpreted in the structure $(\omega, \leq, S, 0)$. 

Büchi (1962) proved that $S1S$ is decidable. His proof uses a back-and-forth conversion between second-order formulas and automata on infinite words.

If one converts a formula $\psi$ into an automaton, and then back into a formula, one obtains an equivalent formula $\phi$ which is 'almost existential'.

Recall that first-order theories in which every formula is equivalent to an existential (equivalently, to a universal) one are precisely the model complete ones.
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- ... in a sense we make precise by introducing a suitable class of temporal algebras.
**LTL$_1$-algebras**

LTL$_1$-algebras are the universal class of BAO’s corresponding to linear temporal logic without until, enriched with an ‘initial atom’, $I$. 

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**Example**

The complex algebra of $(\omega, \leq, 0, S)$ is an LTL$_1$-algebra $(P(\omega), \cup, -, \emptyset, F, X, I)$, where $F_a := \downarrow a$, $X_a := S^{-1}(a)$, and $I := \{0\}$. 

**Definition**

An LTL$_1$-algebra is a tuple $(A, \cup, -, 0, F, X, I)$, where

1. $(A, \cup, -, 0)$ is a Boolean algebra;
2. $F : A \rightarrow A$ is a modal operator on $A$, i.e., preserves $0$ and $\cup$;
3. $X : A \rightarrow A$ is a Boolean endomorphism on $A$;
4. for any $a \in A$, the following conditions hold:
   1. $F_a = a \cup XF_a$,
   2. if $X_a \subseteq a$ then $F_a \subseteq a$,
   3. if $a \neq 0$ then $I \subseteq F_a$.
   4. $X_I = 0$. 

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S. Ghilardi & S. J. v. Gool

MSOL as model companion 22 / 51
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LTL$_I$-algebras are the universal class of BAO’s corresponding to linear temporal logic without until, enriched with an ‘initial atom’, $I$.

**Example**

The complex algebra of $(\omega, \leq, 0, S)$ is an LTL$_I$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, F, X, I)$, where $Fa := \downarrow a$, $Xa := S^{-1}(a)$, and $I := \{0\}$.

**Definition**

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1. $(A, \sqcup, -, 0)$ is a Boolean algebra;
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S. Ghilardi & S. J. v. Gool

MSOL as model companion
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   1. \(Fa = a \sqcup XFa\),
   2. if \(Xa \sqsubseteq a\) then \(Fa \sqsubseteq a\),
   3. if \(a \neq 0\) then \(I \sqsubseteq Fa\),
   4. \(XI = 0\).
Let $\mathcal{L}$ be the first-order language with binary operation $\Box$, unary operations $\neg$, $F$, and $X$, and constant symbols 0 and $I$.
Let $T$ be the $\mathcal{L}$-theory of $\text{LTL}_1$-algebras. We have:

**Theorem**

The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\text{LTL}_1$-algebras.

The proof of this result needs two steps.
Model companion of $\text{LTL}_I$-algebras

*First step.* The following two results can be used to show that every $\text{LTL}_I$-algebra embeds into a model which is elementarily equivalent to the $\mathcal{L}$-structure $\mathcal{P}(\omega)$:

**Lemma** Any quantifier-free $\mathcal{L}$-formula is $\mathcal{T}$-provably equivalent to an $\mathcal{L}$-equation.

**Theorem (Completeness of $\text{LTL}_I$ with respect to $\omega$)** For any $\mathcal{L}$-term $t$, if $\mathcal{P}(\omega) \models t = 0$, then for any $\text{LTL}_I$-algebra $A$, $A \models t = 0$. 
Model companion of LTL₁-algebras

**First step.** The following two results can be used to show that every LTL₁-algebra embeds into a model which is elementarily equivalent to the \( \mathcal{L} \)-structure \( \mathcal{P}(\omega) \):

**Lemma**

*Any quantifier-free \( \mathcal{L} \)-formula is \( T \)-provably equivalent to an \( \mathcal{L} \)-equation.*
First step. The following two results can be used to show that every LTL₁-algebra embeds into a model which is elementarily equivalent to the $\mathcal{L}$-structure $\mathcal{P}(\omega)$:

**Lemma**

Any quantifier-free $\mathcal{L}$-formula is $T$-provably equivalent to an $\mathcal{L}$-equation.

**Theorem (Completeness of LTL₁ with respect to $\omega$)**

For any $\mathcal{L}$-term $t$, if $\mathcal{P}(\omega) \models t = 0$, then for any LTL₁-algebra $\mathbb{A}$, $\mathbb{A} \models t = 0$. 
Model companion of $\text{LTL}_1$-algebras

*Second step*. To show that the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is model-complete we use automata.
**Second step.** To show that the theory of the \( \mathcal{L} \)-structure \( \mathcal{P}(\omega) \) is model-complete we use automata.

Recall that for a finite set of variables \( \underline{x} \), there is a bijective correspondence between colourings \( \sigma : \omega \rightarrow \mathcal{P}(\underline{x}) \) and valuations \( V : \underline{x} \rightarrow \mathcal{P}(\omega) \).
Model companion of $\text{LTL}_1$-algebras

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We use $V_\sigma$ for the valuation corresponding to $\sigma$ and $\sigma_V$ for the colouring corresponding to $V$. 
Second step. To show that the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is model-complete we use automata.

Recall that for a finite set of variables $\underline{x}$, there is a bijective correspondence between colourings $\sigma : \omega \rightarrow \mathcal{P}(\underline{x})$ and valuations $V : \underline{x} \rightarrow \mathcal{P}(\omega)$.

We use $V_\sigma$ for the valuation corresponding to $\sigma$ and $\sigma_V$ for the colouring corresponding to $V$.

The second step is the result of the following three facts exhibiting, starting from any (universal) $\mathcal{L}$-formula $\psi$, an existential formula $\varphi$ equivalent to it in the $\text{LTL}_1$-structure $\mathcal{P}(\omega)$. 
Model companion of $\text{LTL}_1$-algebras

First fact is a variant of standard translation:

**Proposition**

*For any $\mathcal{L}$-formula $\psi(x)$, there exists an $S1S$-formula $\Psi(x)$ such that, for any colouring $\sigma : x \to \mathcal{P}(\omega)$, we have*

$$\mathcal{P}(\omega), V_\sigma \models_{\text{FO}} \psi(x) \iff \omega, \sigma \models_{S1S} \Psi(x).$$
Model companion of \( \text{LTL}_1 \)-algebras

First fact is a variant of standard translation:

**Proposition**

For any \( \mathcal{L} \)-formula \( \psi(x) \), there exists an \( S1S \)-formula \( \Psi(x) \) such that, for any colouring \( \sigma : x \to \mathcal{P}(\omega) \), we have

\[
P(\omega), V_\sigma \models_{\text{FO}} \psi(x) \iff \omega, \sigma \models_{S1S} \Psi(x).
\]

Second fact is Büchi theorem:

**Theorem**

Let \( \psi(x) \) be a formula of \( S1S \). There exists a Büchi automaton \( A \) over the alphabet \( \Sigma := \mathcal{P}(x) \) such that, for any \( \sigma : \omega \to \mathcal{P}(x) \),

\[
\omega, \sigma \models_{S1S} \Phi \iff A \text{ accepts } \sigma.
\]
Third fact is that Büchi acceptance is expressible using LTL\textsubscript{I}-operations:

**Proposition**

For any Büchi automaton $A = (q, q_I, \Delta, F)$ over $\Sigma := \mathcal{P}(x)$ with set of states $q$, there exists an $\mathcal{L}$-term $\text{acc}_A(x, q)$ such that for any colouring $\sigma : \omega \rightarrow \mathcal{P}(x)$, we have

$$A \text{ accepts } (\omega, \sigma) \iff \mathcal{P}(\omega), V_{\sigma} \models \exists q \text{ acc}_A(x, q) = 1.$$
Model companion of LTL₁-algebras

Indeed, the required $\mathcal{L}$-term $\text{acc}_A(x, q)$ is taken to be $\text{acc}_1 \sqcap \text{acc}_2 \sqcap \text{acc}_3$, where

\[
\text{acc}_1(x, q) := -1 \sqcup q_0,
\]
\[
\text{acc}_2(x, q) := \bigvee_{q \in q} \left( q \sqcap \bigwedge_{q' \in q \setminus \{q\}} -q' \sqcap \bigvee \{X q' \sqcap \odot \alpha \mid \alpha \in \mathcal{P}(x), q' \in \delta(q, \alpha)\} \right),
\]
\[
\text{acc}_3(x, q) := \bigvee_{q_f \in F} G F q_f
\]

We used the definition

\[
\odot \alpha := \bigwedge_{x \in \alpha} x \sqcap \bigwedge_{x \notin \alpha} -x.
\]
The case of $\text{LTL}_1$-algebras

For $\text{LTL}_1$-algebras ($\sim$ linear temporal logic), the situation is the following.
The case of $\text{LTL}_1$-algebras

For $\text{LTL}_1$-algebras ($\sim$ linear temporal logic), the situation is the following.

- **Task (I):** model companion exists;
- **Task (II):** to be done, an axiomatization should be similar to known axiomatizations of $S1S$ (see e.g. Riba 2012);
- **Task (III):** we have a good example of a model of the model companion (namely, the $\mathcal{L}$-structure $\mathcal{P}(\omega)$).
Review of Model Completeness

Model Completeness and Bisimulation Quantifiers

Model Completeness and Monadic Second Order Logic
- Infinite words and LTL
- The ‘Fair CTL’ logic
- Binary trees
- Arbitrarily branching trees

Conclusions
If we move from words to trees, the obvious candidate temporal logic is CTL. However, since Büchi acceptance condition is inadequate for tree automata, we need more than CTL, because we must be able to express **fairness** conditions.
CTL with fairness

If we move from words to trees, the obvious candidate temporal logic is CTL. However, since Büchi acceptance condition is inadequate for tree automata, we need more than CTL, because we must be able to express fairness conditions.

In the model checking literature, it is well-known that the main limitation of CTL is the impossibility of expressing fairness constraints. That’s why such fairness constraints are included in the specifications, although not in the logic formalism itself (see Clarke’s book or the NuSMV tool).
If we move from words to trees, the obvious candidate temporal logic is CTL. However, since Büchi acceptance condition is inadequate for tree automata, we need more than CTL, because we must be able to express fairness conditions.

In the model checking literature, it is well-known that the main limitation of CTL is the impossibility of expressing fairness constraints. That’s why such fairness constraints are included in the specifications, although not in the logic formalism itself (see Clarke’s book or the NuSMV tool).

What we need is more, we need built-in fairness constraints. Usually CTL has operators EX (which we write as ♦), EG, EU (other ones can be derived).
CTL with fairness

We just make $\text{EG}$ binary; its semantics is now

$$s \models \text{EG}(\psi_1, \psi_2) \iff \text{there exists an infinite } R\text{-path } s = s_0, s_1, \ldots \text{ such that } s_t \models \psi_1 \text{ for all } t \text{ and there exist infinitely many } t \text{ with } s_t \models \psi_2.$$  

In words: $\text{EG}(\psi_1, \psi_2)$ holds iff there is an infinite $\psi_2$-fair path where $\psi_1$ holds everywhere.

It can be shown that $\text{EG}(\psi_1, \psi_2)$ corresponds to the greatest fixpoint of

$$y \mapsto \psi_1 \land \Diamond \text{EU}(\psi_2 \land y, \psi_1).$$
Axiomatization

As a set of axioms for our logic $\text{CTL}^f$ we take the following ones:
- a standard set of axioms and rules for the system $\text{KD}$;
- fixpoint axioms and rules:

\[
\begin{align*}
\varphi \lor (\psi \land \Diamond \text{EU}(\varphi, \psi)) & \rightarrow \text{EU}(\varphi, \psi) & \text{(EU}_\text{fix}) \\
\varphi \lor (\psi \land \Diamond \chi) & \rightarrow \chi \\
\frac{\text{EU}(\varphi, \psi) \rightarrow \chi}{\text{EU}(\varphi, \psi) \rightarrow \chi} & \text{(EU}_\text{min}) \\
\text{EG}(\varphi, \psi) & \rightarrow \varphi \land \Diamond \text{EU}(\psi \land \text{EG}(\varphi, \psi), \varphi) & \text{(EG}_\text{fix}) \\
\chi & \rightarrow \varphi \land \Diamond \text{EU}(\psi \land \chi, \varphi) \\
\frac{\chi \rightarrow \varphi \land \Diamond \text{EU}(\psi \land \chi, \varphi)}{\chi \rightarrow \text{EG}(\varphi, \psi)} & \text{(EG}_\text{max})
\end{align*}
\]
Completeness

A (non trivial indeed) tableaux construction ensures the following:
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**Theorem**

$\textit{CTL}^\dagger$ is complete with respect to infinite (serial) trees.
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**Theorem**

\( \text{CTL}^f \) is complete with respect to infinite (serial) trees.

An analogous theorem holds for the variant of \( \text{CTL}^f \) where we have two additional deterministic modalities \( X_l, X_r \) and the axiom

\[ \Diamond \varphi \leftrightarrow (X_l \varphi \lor X_r \varphi) \]

and we restrict models to the models whose underlying frame is the full infinite binary tree.
Review of Model Completeness

Model Completeness and Bisimulation Quantifiers

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Conclusions
For binary fair CTL, our plans go similar to the CTL case.
\textbf{bCTL}^f_1\textbf{-algebras}

For binary fair CTL, our plans go similar to the CTL case.

\textbf{Definition}

An \textbf{bCTL}^f_1\textbf{-algebra} is a tuple \((A, \sqcup, -, 0, \text{EU}, \text{EG}, X_l, X_r, \lozenge, I)\), where

5. for any \(a \in A\), the following conditions hold:
   \[ \text{EU} \circ \lozenge a = X_l a \sqcup X_r a, \]
   \[ \text{if } a \neq 0 \text{ then } I \preceq \text{EU}(a, 1), \]
   \[ \text{EU}(I, 1) = 0. \]
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An **bCTL^f\_I-algebra** is a tuple \((A, \sqcup, -, 0, EU, EG, X_l, X_r, \lozenge, I)\), where

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4. \(EU\) and \(EG\) are binary operations on \(A\) such that, for any \(a, b \in A\),

\[
\begin{align*}
EU(I, 1) &= 0, \\
\Diamond a &= X_l a \sqcup X_r a, \\
\text{if } a \neq 0 \text{ then } I \sqsubseteq EU(a, 1).
\end{align*}
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**Definition**

An \( \text{bCTL}_I^f \)-algebra is a tuple \((A, \square, -, 0, \text{EU}, \text{EG}, X_l, X_r, \Diamond, I)\), where

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bCTL$_I^f$-algebras

For binary fair CTL, our plans go similar to the CTL case.

Definition

An $bCTL_I^f$-algebra is a tuple $(A, \sqcup, -, 0, EU, EG, X_l, X_r, \Diamond, I)$, where

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5. for any $a \in A$, the following conditions hold:
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Definition

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S. Ghilardi & S. J. v. Gool

MSOL as model companion
The complex algebra \( \mathcal{P}(2^*) \) of the full binary tree has a natural structure of \( b\text{CTL}_1^f \)-algebra; we have
The complex algebra $\mathcal{P}(2^*)$ of the full binary tree has a natural structure of $\text{bCTL}^f_1$-algebra; we have

**Theorem**

*The theory of the structure $\mathcal{P}(2^*)$ is the model companion of the theory of $\text{bCTL}^f_1$-algebras.*
The complex algebra $\mathcal{P}(2^*)$ of the full binary tree has a natural structure of $b\text{CTL}^f_1$-algebra; we have

**Theorem**

*The theory of the structure $\mathcal{P}(2^*)$ is the model companion of the theory of $b\text{CTL}^f_1$-algebras.*

One side of the theorem is again proved via completeness of binary fair CTL with respect to the full binary tree. For the other case, one still uses automata, more specifically one uses the correspondence between formulas of $S2S$ and parity binary tree automata.
bCTL$_1^f$-algebras

Parity acceptance condition of binary tree automata is encoded via the term $\text{acc}_1 \sqcap \text{acc}_2 \sqcap \text{acc}_3$:

\[
\text{acc}_1(x, q) := -I \sqcup q,
\]

\[
\text{acc}_2(x, q) := \bigvee_{q \in q} \left( q \sqcap \bigwedge_{q' \in q \setminus \{q\}} -q' \sqcap \bigvee \{ \bullet \vartheta \mid (q, \vartheta) \in \Delta \} \right),
\]

\[
\text{acc}_3(x, q) := \bigwedge \left\{ \text{AF} \left( \bigvee_{\Omega(q') < n} q', \bigwedge_{\Omega(q) = n} -q \right) \right\},
\]

where the last big-meet is taken over the set of the odd numbers $n$ that belongs to the range of $\Omega$ and where, for a triple $\vartheta = (\alpha, q_0, q_1)$ (with $\alpha \in \mathcal{P}(x)$, $q_0, q_1 \in q$), we write $\bullet \vartheta$ for

\[
X_0(q_0) \sqcap X_1(q_1) \sqcap \bigwedge_{x \in \alpha} x \sqcap \bigwedge_{x \notin \alpha} -x.
\]
The case of $b\text{CTL}^f_1$-algebras

For $b\text{CTL}^f_1$-algebras ($\sim$ two-successors branching time temporal logic), the situation is the following.
The case of $\mathbf{bCTL}^f_1$-algebras

For $\mathbf{bCTL}^f_1$-algebras ($\sim$ two-successors branching time temporal logic), the situation is the following.

- **Task (I):** model companion exists;
- **Task (II):** still open (work that might be useful here: Geerbrantd-Ten Cate);
- **Task (III):** we have a good example of a model of the model companion (namely, the structure $\mathcal{P}(2^*)$).
Review of Model Completeness

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Conclusions
The last case to analyze is *branching time temporal logic over infinite trees (with no width bound)*. Here we do not have a reference structure and the situation is similar to the Heyting algebras case: a model companion exists, but no understandable axiomatization and no specific concrete example of algebraically closed structure is known.
The last case to analyze is *branching time temporal logic over infinite trees (with no width bound)*. Here we do not have a reference structure and the situation is similar to the Heyting algebras case: a model companion exists, but no understandable axiomatization and no specific concrete example of algebraically closed structure is known.

To show that a model companion exists, if we let $T$ to be the universal theory of $\text{CTL}_I^f$-algebras (see below), *we shall associate with every universal formula $\psi$ an existential formula $\varphi$ and axiomatize $T^*$ by all the formulas $\psi \leftrightarrow \varphi$ obtained in this way*. Such a $T^*$ will be of course model-complete, the challenge will be to show that $T$ and $T^*$ have the same universal consequences (equivalently, that every model of $T$ embeds into a model of $T^*$).
**Definition**

An **CTL\textsubscript{I}^f-algebra** is a tuple \((A, \Box, \neg, 0, \text{EU}, \text{EG}, \Diamond, I)\), where

1. \((A, \Box, \neg, 0)\) is a Boolean algebra;
2. \(\Diamond: A \rightarrow A\) is a modal operator on \(A\), i.e., preserves 0 and \(\Box\);
3. \(\text{EU}\) and \(\text{EG}\) are binary operations on \(A\) such that, for any \(a, b \in A\),
   - \(\text{EU}(a, b)\) is the least pre-fixpoint of the function \(x \mapsto a \Box (b \neg \Diamond x)\),
   - \(\text{EG}(a, b)\) is the greatest post-fixpoint of the function \(y \mapsto a \neg \Diamond \text{EU}(b \neg y, a)\).
4. For any \(a \in A\), the following conditions hold:
   - if \(a \neq 0\) then \(I \sqsubseteq \text{EU}(a, 1)\),
   - \(\text{EU}(I, 1) = 0\),
   - \(\Diamond 1 = 1\).
Definition

An \(\text{CTL}^f_1\)-\textit{algebra} is a tuple \((A, \sqcup, -, 0, \text{EU}, \text{EG}, \lozenge, \mathbf{I})\), where
1. \((A, \sqcup, -, 0)\) is a Boolean algebra;
**Definition**

An **CTL\(_I^f\)**-algebra is a tuple \((A, \sqcap, -, 0, \text{EU}, \text{EG}, \lozenge, I)\), where

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**Definition**

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**CTL\(^f\)_I-algebras**

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   - EU\((a, b)\) is the least pre-fixpoint of the function \(x \mapsto a \sqcup (b \cap \diamond x)\),
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S. Ghilardi & S. J. v. Gool

MSOL as model companion
The case of $\mathsf{CTL}_I^f$-algebras

**Theorem**

*The theory $T$ of $\mathsf{CTL}_I^f$-algebras has a model-companion $T^*$.***
The case of $\text{CTL}_I^f$-algebras

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In principle, the situation is not so different than the case of infinite words and of the binary trees. However, here we do not have a reference structure like $P(\omega)$ or $P(2^*)$; the surrogate of such reference structure is the well known construction of $\omega$-expansions. We just need $\omega$-expansions of Kripke models (= colourings) over trees.
The case of $\text{CTL}_I^f$-algebras

**Definition**

Let $(S, R)$ be a tree with root $s_0$ and let $\sigma : S \rightarrow \mathcal{P}(x)$ be a $x$-colouring of it. The $\omega$-expansion, $(S_\omega, R_\omega, \sigma_\omega)$, of $(S, R, \sigma)$ is defined as follows:

- $S_\omega \coloneqq \{(k_1, s_1) \ldots (k_n, s_n) \in (\omega \times S)^* \mid s_i R s_{i+1} (0 \leq i < n)\}$,
- $R_\omega[(k_1, s_1) \cdots (k_n, s_n)] \coloneqq \{(k_1, s_1) \cdots (k_n, s_n)(k_{n+1}, s_{n+1}) \mid k_{n+1} \in \omega, s_n R s_{n+1}\}$,
- $\sigma_\omega(\epsilon) \coloneqq \sigma(s_0)$,
- $\sigma_\omega((k_1, s_1) \ldots (k_n, s_n)) \coloneqq \sigma(s_n)$.

It is easy to see that $(S_\omega, R_\omega, \sigma_\omega)$ and $(S, R, \sigma)$ are bisimilar.
The case of $\text{CTL}_I^f$-algebras

Let us pick our universal formula $\psi$ in the language $\mathcal{L}$ of $\text{CTL}_I^f$-algebras. First fact we use is again a variant of standard translation:
The case of $\text{CTL}_I^f$-algebras

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**Proposition**

For any first-order $\mathcal{L}$-formula $\psi(x)$, there exists a monadic second order formula $\Psi(x)$ such that, for any $x$-coloured tree $(S, R, \sigma)$,

$$\mathcal{P}(S), V_\sigma \models_{\text{FO}} \psi(x) \iff S, R, \sigma \models_{\text{MSO}} \Psi(x).$$
The case of $\text{CTL}_I^f$-algebras

As to automata, we now make use of nondeterministic modal automata (these are the automata corresponding to formulas of the modal $\mu$-calculus); the following result comes from Janin-Wałukiewicz:

**Proposition**

For any monadic second order formula $\Psi(x)$, there exists a non-deterministic modal automaton $A_\Psi$ over $x$ such that, for any $x$-coloured tree $(S, R, \sigma)$,

$$(S_\omega, R_\omega, \sigma_\omega) \models \Psi(x) \iff A_\Psi \text{ accepts } (S_\omega, R_\omega, \sigma_\omega).$$
The case of $\text{CTL}_I^f$-algebras

We just now come back to the first-order language of $\text{CTL}_I^f$-algebras; the following Proposition is proved analogously to the binary case (we only have to change the acc$_2$ term because the transition relation of $\mu$-automata is different than the transition relation of binary tree automata, but the modification is easy to imagine):
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**Proposition**

For any non-deterministic modal automaton $A$ over $x$ with set of states $q$, there exists an $\mathcal{L}$-term $\text{acc}_A(x, q)$ such that for any $x$-coloured tree $(S, R, \sigma)$, we have

$$A \text{ accepts } (S_\omega, R_\omega, \sigma_\omega) \iff \mathcal{P}(S_\omega), V_{\sigma_\omega} \models \exists q \text{ acc}_A(x, q) = \top.$$
The case of $\text{CTL}^f_I$-algebras

The formula $\exists q \text{ acc}_{A^\psi}(x, q) = \top$ is existential, so it is of the desired shape. Using the last three propositions we can actually get an existential formula out of a universal one; the information supplied by these propositions and the completeness theorem for fair CTL are the ingredients for the proof that $T^*$ so defined is actually the model companion of $T$. 

Of course, $T^*$ is a quite mysterious theory. It is not the theory of any frame-based model: in fact, it is easily seen from the above characterization that the existentially closed $\text{CTL}^f_I$-algebras (i.e. the models of $T^*$) are almost atomless - in fact $I$ is the only atom they have. This is because if we start with the universal formula $\psi \equiv \forall y (y \sqsubseteq x \rightarrow 0 = y \lor x = y)$ we get $x \sqsubseteq I$ as the resulting existential (in this case also quantifier-free) formula $\varphi$. 

S. Ghilardi & S. J. v. Gool 
MSOL as model companion 48 / 51
The case of $\text{CTL}_I^f$-algebras

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The case of $\text{CTL}_I^f$-algebras

Thus, for $\text{CTL}_I^f$-algebras, the situation is the following.
Thus, for $\text{CTL}^f_1$-algebras, the situation is the following.

- Task (I): model companion exists;
- Task (II): still open;
- Task (III): still open.
1. Review of Model Completeness

2. Model Completeness and Bisimulation Quantifiers

3. Model Completeness and Monadic Second Order Logic
   - Infinite words and LTL
   - The ‘Fair CTL’ logic
   - Binary trees
   - Arbitrarily branching trees

4. Conclusions
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Probably more problems were raised than problems were solved...
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THANKS FOR ATTENTION!