Monadic second order logic as the model companion of temporal logic

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We first review basic information on model completeness; then we relate model completeness to bisimulation quantifiers in propositional intuitionistic and (some) modal logics. Finally, we show how model completeness connects linear and braching time temporal logics with monadic second order logic.

Outline

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    - Infinite words and LTL
    - The 'Fair CTL' logic
    - Binary trees
    - Arbitrarily branching trees

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4. Conclusions

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Equivalently,

(α) All embeddings between T^*-models are elementary;

(β) Every T-model embeds in some T^*-model and vice versa.

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- *In particular, any theory T has at most one model companion, and the model companion of T exists iff the class of existentially closed T-models is elementary.*
- *A model companion T^* of T is a model completion iff the class of T-models has amalgamation iff T^* has quantifier elimination.*

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(All of these examples are in fact model completions.)

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We point out that, in principle, task (I) is quite independant from the other two. For instance, one can accomplish task (i) just proving that the class of existentially closed models is elementary.

Model companion: tasks

Another method for task (I), is to find a mechanism producing for each single existential φ a universal 'solvability condition' ψ for it.
More precisely, we assign to every existential φ a universal ψ and propose

$$\forall \underline{x}\ (\varphi(\underline{x}) \leftrightarrow \psi(\underline{x})) \tag{1}$$

as axioms for T^*. For this to work, we need two conditions:

(i) $\forall \underline{x}\ (\varphi(\underline{x}) \to \psi(\underline{x}))$ must be T-valid (because this is a universal sentence);

(ii) $\forall \underline{x}\ (\psi(\underline{x}) \to \varphi(\underline{x}))$ must be true in all existentially closed models.

Notice also that, by taking negations, one could produce out of a universal ψ an existential φ satisfying the same conditions.

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- Task (I): model companion exists (actually, it is a model completion);
- Task (II): the axiomatization is simple: just one axiom saying that there are no atoms;
- Task (III): any free algebra on infinitely many generators is a model of the model companion.

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Theorem (A.M. Pitts)

For each propositional variable x and for each formula t of IpC, there exist formulas $\exists^x t$ and $\forall^x t$ of IpC (effectively computable from t) containing only variables not equal to x which occur in t, and such that for any formula u not involving x, we have

$$\vdash_{IpC} \exists^x t \to u \qquad \textit{iff} \qquad \vdash_{IpC} t \to u$$

and

$$\vdash_{IpC} u \to \forall^x t \qquad \textit{iff} \qquad \vdash_{IpC} u \to t.$$

The case of Heyting Algebras

From Pitts' theorem, it turns out that the system of equations and inequations with parameters $\underline{a}$ from a Heyting algebra H

$$t(\underline{a}, x) = 1 \;\&\; u_1(\underline{a}, x) \neq 1 \;\&\; \ldots \;\&\; u_m(\underline{a}, x) \neq 1 \tag{2}$$

is solvable in an extension of H iff the quantifier-free formula

$$(\exists^x t)(\underline{a}) = 1 \;\&\; (\forall^x (t \to u_1))(\underline{a}) \neq 1 \;\&\; \ldots$$

$$\ldots \;\&\; (\forall^x (t \to u_m))(\underline{a}) \neq 1 \tag{3}$$

is true in H (for proof details see S.Ghilardi, M.Zawadowski's book "Sheaves, Games and Model Completions").

The case of Heyting Algebras

But this means that we can axiomatize the model companion T^* by the formulas

$$\exists x \text{ (2)} \leftrightarrow \text{(3)} \tag{4}$$

This fits the schema (1); it is even better, because (3) is quantifier-free, so that (4) is in fact a quantifier-elimination procedure (quantifiers can be eliminated one by one) and the model companion is actually a model completion. The quantifier-elimination procedure is effective because terms like $\exists^x t$ and $\forall^x t$ can be effectively computed.

Thus Task (I) is done for the Heyting algebras case.

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The curious fact is that there are a lot of existentially closed Heyting algebras (every Heyting algebra embeds into an existentially closed one by general facts) and, via the algorithm supplied by (4), we can get all first order information we want to have concerning them. Yet we are unable to supply any concrete example and, even worse, we know that obvious candidate examples (free algebras, open sets of a topological space, etc.) are not good.

The case of Heyting Algebras

Exercise

In any existentially closed non degenerate Heyting algebra H we have that:

(i) *the relation $<$ is dense;*

(ii) *for every element $y \neq 0$ there is a complemented element x such that $0 < x < y$ (consequently, regular and complemented elements of H form atomless Boolean algebras);*

(iii) *the dual lattice of H has no non-zero join-irreducible elements;*

(iv) *the difference of x and y exists iff $x \leq y \vee \neg y$.*

The case of Modal Algebras

The existence of a model completion and a suitable version of Pitts theorem are equivalent under certain hypothesis from universal algebras (see G.Z. book mentioned above); these hypotheses apply to $K4$-based varieties but not to K-based varieties (and in fact, Pitts theorem holds for K but modal algebras do not have a model completion).

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It follows that GL (i.e. diagonalizable) and Grz-algebras have model completions, whereas $S4$-algebras do not (since they enjoy amalgamation, the same negative result applies to the existence of a model companion.).

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For *GL* and *Grz*-algebras the situation is the same as for Heyting algebras (tasks (II)-(III) are open).

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- If one converts a formula ψ into an automaton, and then back into a formula, one obtains an equivalent formula φ which is 'almost existential'.
- Recall that first-order theories in which every formula is equivalent to an existential (equivalently, to a universal) one are precisely the *model complete* ones.

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- ... in a sense we make precise by introducing a suitable class of temporal algebras.

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The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \mathrm{F}, \mathrm{X}, \mathbf{I})$, where $\mathrm{F}a := {\downarrow}a$, $\mathrm{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

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    4. $\mathrm{X}\mathbf{I} = 0$.

Model companion of $\mathrm{LTL_I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\sqcup$, unary operations $-$, F, and X, and constant symbols 0 and **I**.
Let T be the $\mathcal{L}$-theory of $\mathrm{LTL_I}$-algebras. We have:

Theorem

The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras.

The proof of this result needs two steps.

Model companion of $\mathrm{LTL_I}$-algebras

First step. The following two results can be used to show that every $\mathrm{LTL_I}$-algebra embeds into a model which is elementarily equivalent to the $\mathcal{L}$-structure $\mathcal{P}(\omega)$:

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Lemma

Any quantifier-free $\mathcal{L}$-formula is T-provably equivalent to an $\mathcal{L}$-equation.

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First step. The following two results can be used to show that every $\mathrm{LTL_I}$-algebra embeds into a model which is elementarily equivalent to the $\mathcal{L}$-structure $\mathcal{P}(\omega)$:

Lemma

Any quantifier-free $\mathcal{L}$-formula is T-provably equivalent to an $\mathcal{L}$-equation.

Theorem (Completeness of $\mathrm{LTL_I}$ with respect to ω)

For any $\mathcal{L}$-term t, if $\mathcal{P}(\omega) \models t = 0$, then for any $\mathrm{LTL_I}$-algebra $\mathbb{A}$, $\mathbb{A} \models t = 0$.

Model companion of $\mathrm{LTL_I}$-algebras

Second step. To show that the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is model-complete we use automata.

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The second step is the result of the following three facts exhibithing, starting from any (universal) $\mathcal{L}$-formula ψ, an existential formula φ equivalent to it in the $\mathrm{LTL_I}$-structure $\mathcal{P}(\omega)$.

Model companion of $\mathrm{LTL_I}$-algebras

First fact is a variant of standard translation:

Proposition

For any $\mathcal{L}$-formula $\psi(\underline{x})$, there exists an $\mathrm{S1S}$-formula $\Psi(\underline{x})$ such that, for any colouring $\sigma : \underline{x} \to \mathcal{P}(\omega)$, we have

$$\mathbb{P}(\omega), V_\sigma \models_{\mathrm{FO}} \psi(\underline{x}) \iff \omega, \sigma \models_{\mathrm{S1S}} \Psi(\underline{x}).$$

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Second fact is Büchi theorem:

Theorem

Let $\Psi(\underline{x})$ be a formula of $\mathrm{S1S}$. There exists a Büchi automaton $\mathcal{A}$ over the alphabet $\Sigma := \mathcal{P}(\underline{x})$ such that, for any $\sigma : \omega \to \mathcal{P}(\underline{x})$,

$$\omega, \sigma \models_{\mathrm{S1S}} \Phi \iff \mathcal{A} \text{ accepts } \sigma.$$

Model companion of $\mathrm{LTL_I}$-algebras

Third fact is that Büchi acceptance is expressible using $\mathrm{LTL_I}$-operations:

Proposition

For any Büchi automaton $\mathcal{A} = (\underline{q}, q_I, \Delta, F)$ over $\Sigma := \mathcal{P}(\underline{x})$ with set of states $\underline{q}$, there exists an $\mathcal{L}$-term $\mathrm{acc}_{\mathcal{A}}(\underline{x}, \underline{q})$ such that for any colouring $\sigma : \omega \longrightarrow \mathcal{P}(\underline{x})$, we have

$$\mathcal{A} \textit{ accepts } (\omega, \sigma) \iff \mathcal{P}(\omega), V_\sigma \models \exists \underline{q}\, \mathrm{acc}_{\mathcal{A}}(\underline{x}, \underline{q}) = 1.$$

Model companion of $\mathrm{LTL_I}$-algebras

Indeed, the required $\mathcal{L}$-term $\mathrm{acc}_{\mathcal{A}}(\underline{x}, \underline{q})$ is taken to be $\mathrm{acc}_1 \sqcap \mathrm{acc}_2 \sqcap \mathrm{acc}_3$, where

$$\mathrm{acc}_1(\underline{x}, \underline{q}) := -\mathbf{I} \sqcup q_0,$$

$$\mathrm{acc}_2(\underline{x}, \underline{q}) := \bigvee_{q \in \underline{q}} \left(\begin{array}{l} q \sqcap \bigwedge\limits_{q' \in \underline{q} \setminus \{q\}} -q' \sqcap \\ \bigvee \left\{ \mathrm{X}\, q' \sqcap \odot\alpha \mid \alpha \in \mathcal{P}(\underline{x}), q' \in \delta(q, \alpha) \right\} \end{array} \right),$$

$$\mathrm{acc}_3(\underline{x}, \underline{q}) := \bigvee_{q_f \in F} \mathrm{G}\,\mathrm{F}\, q_f$$

We used the definition

$$\odot\alpha := \bigwedge_{x \in \alpha} x \sqcap \bigwedge_{x \notin \alpha} -x \quad .$$

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For $\mathrm{LTL_I}$-algebras ($\sim$ linear temporal logic), the situation is the following.

- Task (I): model companion exists;
- Task (II): to be done, an axiomatization should be similar to known axiomatizations of $S1S$ (see e.g. Riba 2012);
- Task (III): we have a good example of a model of the model companion (namely, the $\mathcal{L}$-structure $\mathcal{P}(\omega)$).

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CTL with fairness

If we move from words to trees, the obvious candidate temporal logic is CTL. However, since Büchi acceptance condition is inadequate for tree automata, we need more than CTL, because we must be able to express fairness conditions.

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In the model ckecking literature, it is well-known that the main limitation of CTL is the impossibility of expressing fairness constraints. That's why such fairness constraints are included in the specifications, although not in the logic formalism itself (see Clarke's book or the NuSMV tool).

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What we need is more, we need built-in fairness constraints. Usually CTL has operators EX (which we write as $\Diamond$), EG, EU (other ones can be derived).

CTL with fairness

We just make EG binary; its semantics is now

- $s \models \mathrm{EG}(\psi_1, \psi_2)$ *iff there exists an infinite R-path* $s = s_0, s_1, \ldots$ *such that* $s_t \models \psi_1$ *for all* t *and there exist infinitely many* t *with* $s_t \models \psi_2$.

In words: $\mathrm{EG}(\psi_1, \psi_2)$ holds iff there is an infinite ψ_2-fair path where ψ_1 holds everywhere.

It can be shown that $\mathrm{EG}(\psi_1, \psi_2)$ corresponds to the greatest fixpoint of

$$y \mapsto \psi_1 \wedge \Diamond \mathrm{EU}(\psi_2 \wedge y, \psi_1)$$

Axiomatization

As a set of axioms for our logic CTL^f we take the following ones:

- a standard set of axioms and rules for the system **KD**;
- fixpoint axioms and rules:

$$\varphi \vee (\psi \wedge \Diamond \mathrm{EU}(\varphi, \psi)) \to \mathrm{EU}(\varphi, \psi) \tag{$\mathrm{EU_{fix}}$}$$

$$\frac{\varphi \vee (\psi \wedge \Diamond \chi) \to \chi}{\mathrm{EU}(\varphi, \psi) \to \chi} \tag{$\mathrm{EU_{min}}$}$$

$$\mathrm{EG}(\varphi, \psi) \to \varphi \wedge \Diamond \mathrm{EU}(\psi \wedge \mathrm{EG}(\varphi, \psi), \varphi) \tag{$\mathrm{EG_{fix}}$}$$

$$\frac{\chi \to \varphi \wedge \Diamond \mathrm{EU}(\psi \wedge \chi, \varphi)}{\chi \to \mathrm{EG}(\varphi, \psi)} \tag{$\mathrm{EG_{max}}$}$$

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An analogous theorem holds for the variant of CTL^f where we have two additional deterministic modalities $\mathrm{X}_l, \mathrm{X}_r$ and the axiom

$$\Diamond\varphi \leftrightarrow (\mathrm{X}_l\varphi \vee \mathrm{X}_r\varphi)$$

and we restrict models to the models whose underlying frame is the full infinite binary tree.

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For binary fair CTL, our plans go similar to the CTL case.

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The complex algebra $\mathcal{P}(2^*)$ of the full binary tree has a natural structure of $\mathrm{bCTL}^{f}_{\mathrm{I}}$-algebra; we have

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One side of the theorem is again proved via completeness of binary fair CTL with respect to the full binary tree. For the other case, one still uses automata, more specifically one uses the correspondence between formulas of *S*2*S* and parity binary tree automata.

$\mathrm{bCTL}_{\mathrm{I}}^{f}$-algebras

Parity acceptance condition of binary tree automata is encoded via the term $\mathsf{acc}_1 \sqcap \mathsf{acc}_2 \sqcap \mathsf{acc}_3$:

$$\mathsf{acc}_1(\underline{x}, \underline{q}) := -\mathbf{I} \sqcup q_I,$$

$$\mathsf{acc}_2(\underline{x}, \underline{q}) := \bigvee_{q \in \underline{q}} \begin{pmatrix} q \sqcap \bigwedge_{q' \in \underline{q} \setminus \{q\}} -q' \sqcap \\ \bigvee \{ \bullet\vartheta \mid (q, \vartheta) \in \Delta \} \end{pmatrix},$$

$$\mathsf{acc}_3(\underline{x}, \underline{q}) := \bigwedge \left\{ \mathrm{AF} \left(\bigvee_{\Omega(q') < n} q', \bigwedge_{\Omega(q) = n} -q \right) \right\},$$

where the last big-meet is taken over the set of the odd numbers n that belongs to the range of Ω and where, for a triple $\vartheta = (\alpha, q_0, q_1)$ (with $\alpha \in \mathcal{P}(\underline{x}), q_0, q_1 \in \underline{q}$), we write $\bullet\vartheta$ for

$$\mathrm{X}_0(q_0) \sqcap \mathrm{X}_1(q_1) \sqcap \bigwedge_{x \in \alpha} x \sqcap \bigwedge_{x \notin \alpha} -x \; .$$

The case of $\mathrm{bCTL}^f_\mathrm{I}$-algebras

For $\mathrm{bCTL}^f_\mathrm{I}$-algebras ($\sim$ two-successors branching time temporal logic), the situation is the following.

The case of $\mathrm{bCTL}_{\mathrm{I}}^{f}$-algebras

For $\mathrm{bCTL}_{\mathrm{I}}^{f}$-algebras ($\sim$ two-successors branching time temporal logic), the situation is the following.

- Task (I): model companion exists;
- Task (II): still open (work that might be useful here: Geerbrandt-Ten Cate);
- Task (III): we have a good example of a model of the model companion (namely, the structure $\mathcal{P}(2^*)$).

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4. Conclusions

$\mathrm{CTL}_\mathrm{I}^f$-algebras

The last case to analyze is *branching time temporal logic over infinite trees (with no width bound)*. Here we do not have a reference structure and the situation is similar to the Heyting algebras case: a model companion exists, but no undestandable axiomatization and no specific concrete example of algebraically closed structure is known.

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To show that a model companion exists, if we let T to be the universal theory of $\mathrm{CTL}_{\mathrm{I}}^{f}$-algebras (see below), *we shall associate with every universal formula ψ an existential formula φ and axiomatize T^* by all the formulas $\psi \leftrightarrow \varphi$ obtained in this way*. Such a T^* will be of course model-complete, the challenge will be to show that T and T^* have the same universal consequences (equivalently, that every model of T embeds into a model of T^*).

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1. $(A, \sqcup, -, 0)$ is a Boolean algebra;
2. $\lozenge : A \to A$ is a modal operator on A, i.e., preserves 0 and $\sqcup$;
3. EU and EG are binary operations on A such that, for any $a, b \in A$,
   - $\mathrm{EU}(a, b)$ is the least pre-fixpoint of the function $x \mapsto a \sqcup (b \sqcap \lozenge x)$,
   - $\mathrm{EG}(a, b)$ is the greatest post-fixpoint of the function $y \mapsto a \sqcap \lozenge\mathrm{EU}(b \sqcap y, a)$.
4. for any $a \in A$, the following conditions hold:
   - if $a \neq 0$ then $\mathbf{I} \sqsubseteq \mathrm{EU}(a, 1)$,
   - $\mathrm{EU}(\mathbf{I}, 1) = 0$,

$\mathrm{CTL}^f_\mathrm{I}$-algebras

Definition

An *$\mathrm{CTL}^f_\mathrm{I}$-algebra* is a tuple $(A, \sqcup, -, 0, \mathrm{EU}, \mathrm{EG}, \Diamond, \mathbf{I})$, where

1. $(A, \sqcup, -, 0)$ is a Boolean algebra;
2. $\Diamond : A \to A$ is a modal operator on A, i.e., preserves 0 and $\sqcup$;
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   - $\mathrm{EU}(a, b)$ is the least pre-fixpoint of the function $x \mapsto a \sqcup (b \sqcap \Diamond x)$,
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   - $\Diamond 1 = 1$.

The case of $\mathrm{CTL}_{\mathrm{I}}^{f}$-algebras

Theorem

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In principle, the situation is not so different than the case of infinite words and of the binary trees. However, here we do not have a reference structure like $\mathcal{P}(\omega)$ or $\mathcal{P}(2^*)$; the surrogate of such reference structure is the well known construction of ω-expansions. We just need ω-expansions of Kripke models (= colourings) over trees.

The case of $\mathrm{CTL}^f_{\mathrm{I}}$-algebras

Definition

Let (S, R) be a tree with root s_0 and let $\sigma : S \longrightarrow \mathcal{P}(\underline{x})$ be a $\underline{x}$-colouring of it. The *ω-expansion*, $(S_\omega, R_\omega, \sigma_\omega)$, of (S, R, σ) is defined as follows:

$$
\begin{aligned}
S_\omega &:= \{(k_1, s_1)\dots(k_n, s_n) \in (\omega \times S)^* \mid s_i R s_{i+1}\ (0 \le i < n)\},\\
R_\omega[(k_1, s_1)\cdots(k_n, s_n)] &:= \{(k_1, s_1)\cdots(k_n, s_n)(k_{n+1}, s_{n+1}) \ : \\
&\qquad : \ k_{n+1} \in \omega, s_n R s_{n+1}\},\\
\sigma_\omega(\epsilon) &:= \sigma(s_0),\\
\sigma_\omega((k_1, s_1)\dots(k_n, s_n)) &:= \sigma(s_n).
\end{aligned}
$$

It is easy to see that $(S_\omega, R_\omega, \sigma_\omega)$ and (S, R, σ) are bisimilar.

The case of $\mathrm{CTL}_{\mathrm{I}}^{f}$-algebras

Let us pick our universal formula ψ in the language $\mathcal{L}$ of $\mathrm{CTL}_{\mathrm{I}}^{f}$-algebras.
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Proposition

For any first-order $\mathcal{L}$-formula $\psi(\underline{x})$, there exists a monadic second order formula $\Psi(\underline{x})$ such that, for any $\underline{x}$-coloured tree (S, R, σ),

$$\mathcal{P}(S), V_\sigma \models_{FO} \psi(\underline{x}) \iff S, R, \sigma \models_{MSO} \Psi(\underline{x}).$$

The case of $\mathrm{CTL}^f_{\mathrm{I}}$-algebras

As to automata, we now make use of nondeterministic modal automata (these are the automata corresponding to formulas of the modal μ-calculus); the following result comes from Janin-Wałukiewicz:

Proposition

For any monadic second order formula $\Psi(\underline{x})$, there exists a non-deterministic modal automaton $\mathcal{A}_\Psi$ over $\underline{x}$ such that, for any $\underline{x}$-coloured tree (S, R, σ),

$$(S_\omega, R_\omega, \sigma_\omega) \models \Psi(\underline{x}) \iff \mathcal{A}_\Psi \text{ accepts } (S_\omega, R_\omega, \sigma_\omega).$$

The case of $\mathrm{CTL}_{\mathrm{I}}^{f}$-algebras

We just now come back to the first-order language of $\mathrm{CTL}_{\mathrm{I}}^{f}$-algebras; the following Proposition is proved analogously to the binary case (we only have to change the acc_2 term because the transition relation of μ-automata is different than the transition relation of binary tree automata, but the modification is easy to imagine):

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Proposition

For any non-deterministic modal automaton $\mathcal{A}$ over $\underline{x}$ with set of states $\underline{q}$, there exists an $\mathcal{L}$-term $\mathsf{acc}_{\mathcal{A}}(\underline{x}, \underline{q})$ such that for any $\underline{x}$-coloured tree (S, R, σ), we have

$$\mathcal{A} \textit{ accepts } (S_\omega, R_\omega, \sigma_\omega) \iff \mathcal{P}(S_\omega), V_{\sigma_\omega} \models \exists \underline{q}\, \mathsf{acc}_{\mathcal{A}}(\underline{x}, \underline{q}) = \top.$$

The case of $\mathrm{CTL}^{f}_{\mathrm{I}}$-algebras

The formula $\exists \underline{q}\, \mathsf{acc}_{\mathcal{A}_\Psi}(\underline{x}, \underline{q}) = \top$ is existential, so it is of the desired shape. Using the last three propositions we can actually get an existential formula out of a universal one; the information supplied by these propositions and the completeness theorem for fair CTL are the ingredients for the proof that T^* so defined is actually the model companion of T.

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The formula $\exists \underline{q}\ \mathrm{acc}_{\mathcal{A}_\psi}(\underline{x}, \underline{q}) = \top$ is existential, so it is of the desired shape. Using the last three propositions we can actually get an existential formula out of a universal one; the information supplied by these propositions and the completeness theorem for fair CTL are the ingredients for the proof that T^* so defined is actually the model companion of T.

Of course, T^* is a quite mysterious theory. It is not the theory of any frame-based model: in fact, it is easily seen from the above characterization that the existentially closed $\mathrm{CTL}^f_\mathrm{I}$-algebras (i.e. the models of T^*) are almost atomless - in fact **I** is the only atom they have. This is because if we start with the universal formula $\psi \equiv \forall y(y \sqsubseteq x \to 0 = y \vee x = y)$ we get $x \sqsubseteq \mathbf{I}$ as the resulting existential (in this case also quantifier-free) formula φ.

The case of $\mathrm{CTL}_{\mathrm{I}}^{f}$-algebras

Thus, for $\mathrm{CTL}_{\mathrm{I}}^{f}$-algebras, the situation is the following.

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Thus, for $\mathrm{CTL}_{\mathrm{I}}^{f}$-algebras, the situation is the following.

- Task (I): model companion exists;
- Task (II): still open;
- Task (III): still open.

1 Review of Model Completeness

2 Model Completeness and Bisimulation Quantifiers

3 Model Completeness and Monadic Second Order Logic
- Infinite words and LTL
- The 'Fair CTL' logic
- Binary trees
- Arbitrarily branching trees

4 Conclusions

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THANKS FOR ATTENTION !