Instantial neighborhood semantics with an application to game logic

Johan van Benthem Nick Bezhanishvili Sebastian Enqvist

Institute for Logic, Language and Computation
University of Amsterdam

June 14, 2016

Part I: Instantial Neighborhood Semantics

Modal logic in topology

- Box as interior:

$$[\![\Box\varphi]\!]^{\mathcal{T}}_{V} := \mathcal{I}([\![\varphi]\!]^{\mathcal{T}}_{V})$$

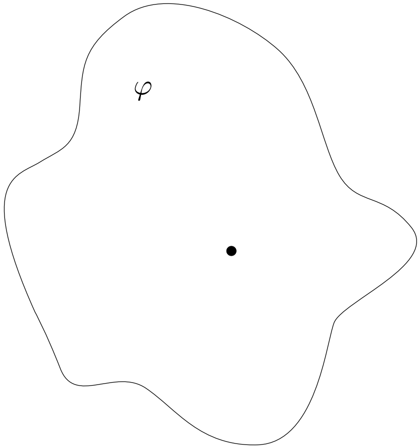
- *Globally valid* formulas $\Leftrightarrow$ equational theory of a space:

$$\begin{array}{lcl} \mathcal{I}(x \cap y) = \mathcal{I}x \cap \mathcal{I}y & \mapsto & \Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q \\ \Box p \to p & \mapsto & -\mathcal{I}(x) \cup x = 1 \end{array}$$

Theorem (McKinsey-Tarski)

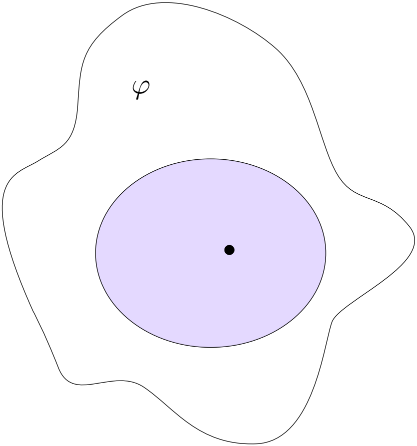
S4 *is the modal logic of the real line.*

Local satisfaction relation:



Local satisfaction relation:

$\Box\varphi$ true



Neighborhood semantics

Definition

A neighborhood frame is a pair (X, R) where $R \subseteq X \times \mathcal{P}X$. A neighborhood model is a frame with a valuation.

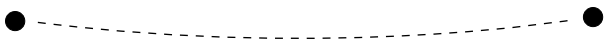
$$s \Vdash \Box\varphi \Leftrightarrow \exists Z : (s, Z) \in R \;\&\; \forall v \in Z : v \Vdash \varphi$$

- Spaces to frames: $(u, Z) \in R_\tau \Leftrightarrow Z \in \tau \;\&\; u \in Z$.
- Monotone modal logics:

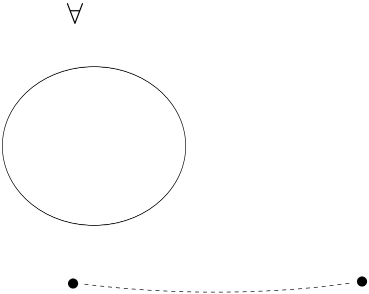
$$\Box p \to \Box(p \vee q) \checkmark \qquad \Box p \wedge \Box q \to \Box(p \wedge q) \text{ X}$$

- Game logic.

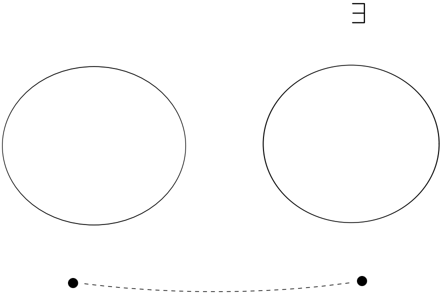
Neighborhood bisimulations



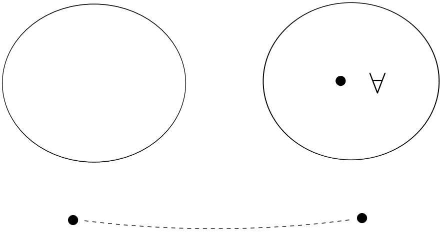
Neighborhood bisimulations



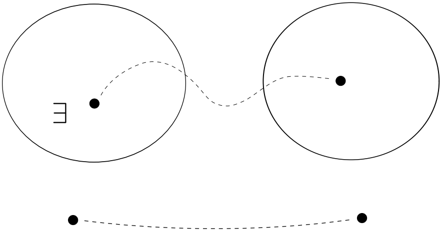
Neighborhood bisimulations



Neighborhood bisimulations



Neighborhood bisimulations



Instantial neighborhood logic

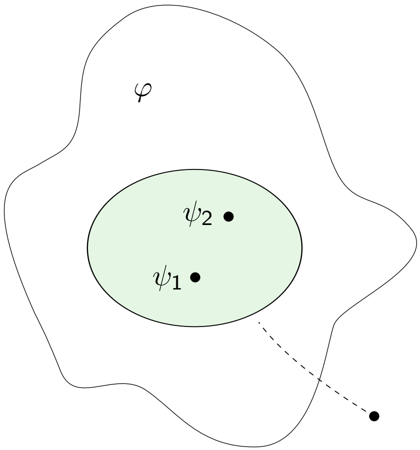
- Box has a quantifier pattern of the form $\exists\forall$: only universal quantifiers over individual neighborhoods
- Idea: allow existential quantification over neighborhoods!

Grammar:

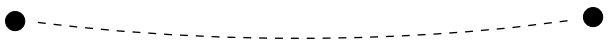
$$\varphi := p \mid \top \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Box(\psi_1, \ldots, \psi_n; \varphi)$$

Semantics

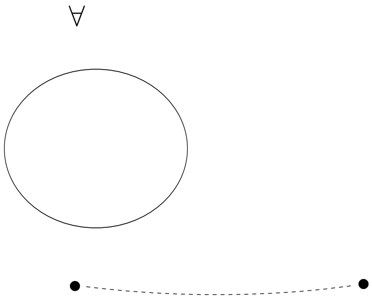
$\Box(\psi_1, \psi_2; \varphi)$ true



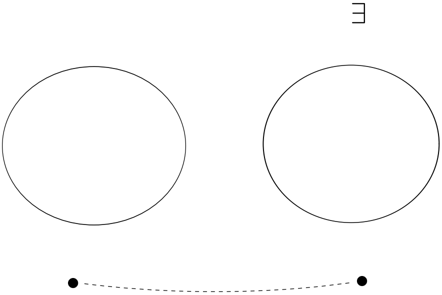
Instantial neighborhood bisimulations



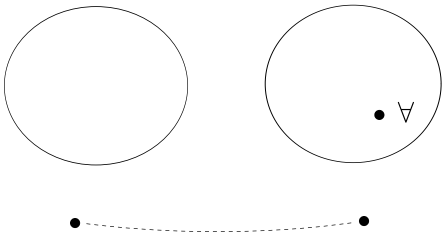
Instantial neighborhood bisimulations



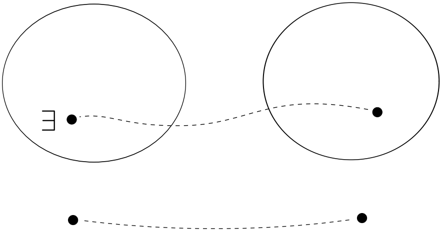
Instantial neighborhood bisimulations



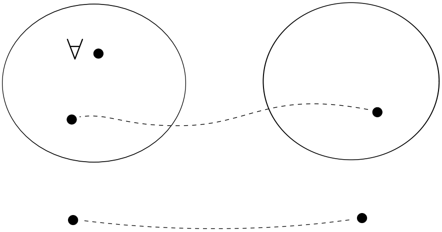
Instantial neighborhood bisimulations



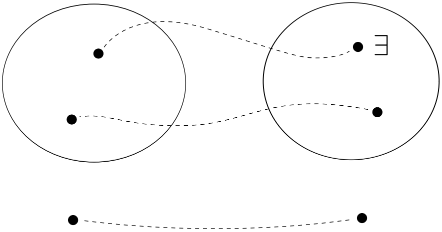
Instantial neighborhood bisimulations



Instantial neighborhood bisimulations



Instantial neighborhood bisimulations



Coalgebra

- Neighborhood frames = coalgebras for $\mathcal{I} = \mathcal{P} \circ \mathcal{P}$, a w.p.b. preserving functor!
- Behavioural equivalence = instantial neighborhood bisimilarity
- $\overline{\mathcal{I}}$-bisimulations = instantial neighborhood bisimulations
- Instantial neighborhood modalities are predicate liftings!

Instantial neighborhood modality on topological spaces

Proposition

Over topological spaces, INL has the same expressive power as standard neighborhood + global modality.

Proof.

$$\begin{array}{lcl} \mathsf{E}\varphi & \mapsto & \Box(\varphi, \top) \\ \Box(\psi_1, ..., \psi_n; \varphi) & \mapsto & \Box\varphi \wedge \mathsf{E}(\psi_1 \wedge \Box\varphi) ... \wedge \mathsf{E}(\psi_n \wedge \Box\varphi) \end{array}$$

∎

Basic results:

- Restriction to *n*-existential fragment decreases expressive power for all $n \in \omega$!

***n*-existential fragment:**

Formulas $\Box(\psi_1, ..., \psi_k; \varphi)$ restricted so that $k < n$.

- Bisimulation invariance + Hennessy-Milner theorem for finite models
- Satisfiability preserving translations into normal (bi-)modal logic
- Complexity = PSPACE-complete

Axioms

(NW) $\Box(\gamma_1, ..., \gamma_j; \psi) \to \Box(\gamma_1, ..., \gamma_j; \psi \vee \chi)$,

(SW) $\Box(\gamma_1, ..., \gamma_j, \alpha; \psi) \to \Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta; \psi)$,

(SR) $\Box(\gamma_1, ..., \gamma_j, \varphi; \psi) \to \Box(\gamma_1, ..., \gamma_j, \varphi \wedge \psi; \psi)$,

(SC) $\neg\Box(\bot; \psi)$,

(NT) $\Box(\gamma_1, ..., \gamma_j; \psi) \to \Box(\gamma_1, ..., \gamma_j, \delta; \psi) \vee \Box(\gamma_1, ..., \gamma_j; \psi \wedge \neg\delta)$,

(AD) $\Box(\gamma_1, ..., \gamma_j, \varphi, \delta_1, ..., \delta_n; \psi) \to \Box(\gamma_1, ..., \gamma_j, \delta_1, ..., \delta_n; \psi)$,

(AI) $\Box(\gamma_1, ..., \gamma_j, \delta_1, ..., \delta_n; \psi) \to \Box(\gamma_1, ..., \gamma_j, \gamma_j, \delta_1, ..., \delta_n; \psi)$

The canonical model

Theorem

The axioms for INL are sound and strongly complete.

Proof is by a canonical model construction:

Definition

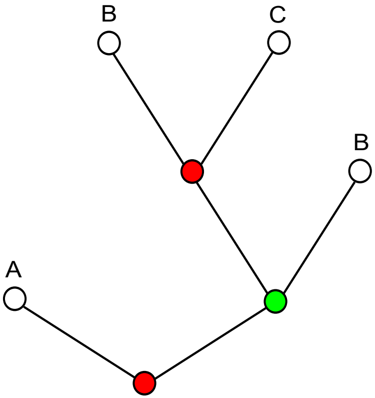
Let Γ be an MC set and Z a family of MC sets. Set $(\Gamma, Z) \in R_C$ iff: for all $\psi_1, \ldots, \psi_n, \varphi$, if

- $\varphi \in \bigcap Z$ and
- for each i, $\psi_i \in \bigcup Z$,

then $\Box(\psi_1, \ldots, \psi_n; \varphi) \in \Gamma$.

Part II: Game Logic

Games



Powers

Definition

Let G be a game with outcomes in O. Then $P \subseteq O$ is a *power* of Player I in G if:

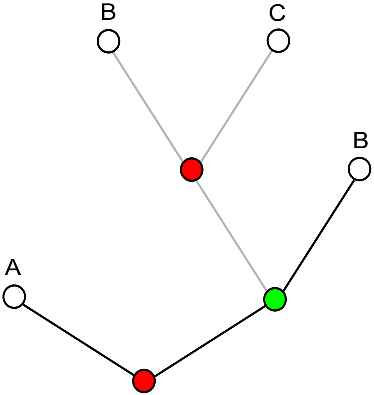
$$\exists \sigma \in \mathrm{Strat}(\mathrm{I}) \forall \sigma' \in \mathrm{strat}(\mathrm{II}) : \ \mathrm{Out}(\sigma, \sigma') \in P$$

Same for Player II.

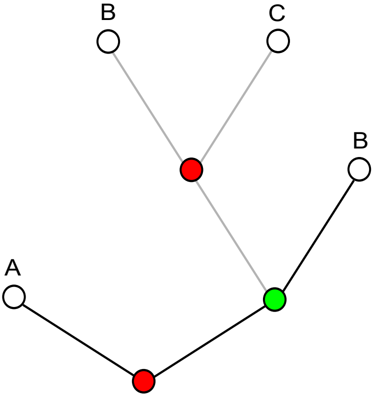
Definition

If $N_I(G_1) = N_I(G_2)$ and $N_{II}(G_1) = N_{II}(G_2)$, we say G_1 and G_2 are *power equivalent*.

$\{A, B\} \in N_I(G)$



$\Box(A \vee B)$



Game logic

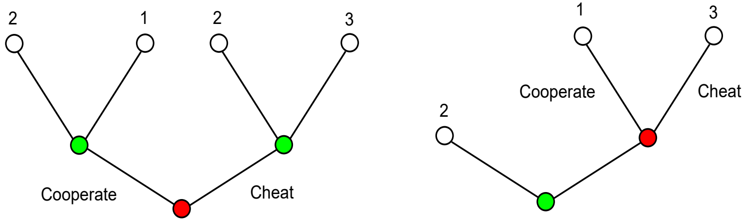
Language (minus unrestricted dual):

$$\varphi := p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid (G)\varphi$$

$$G := g \mid G \cup G \mid G \cap G \mid G \circ G \mid G^\star$$

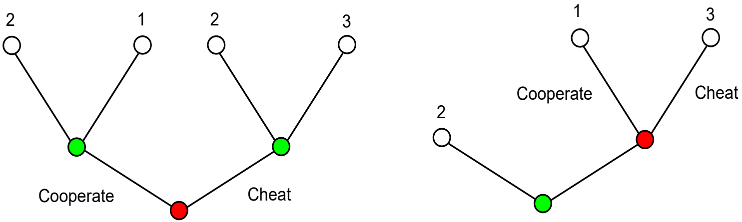
Game logic is suitable for reasoning about *powers*, but does not describe the *individual strategies* available in the game. Power equivalent games can still have strategies that behave differently in terms of possible outcomes of the game!

Example



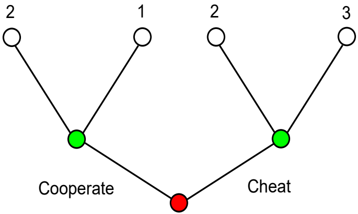
Set $2 \prec_I 3 \prec_I 1$, and $2 \prec_{II} 1 \prec_{II} 3$.

$\Box(1 \vee 2)$ $\Box(1 \vee 2)$

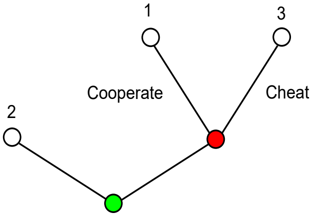


$\{1, 2\}$ is a power in both games...

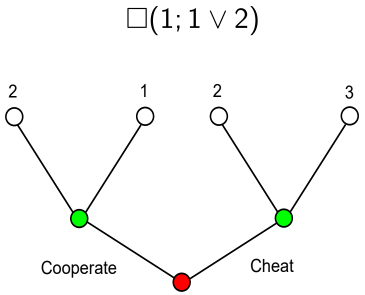
$\Box(1; 1 \vee 2)$

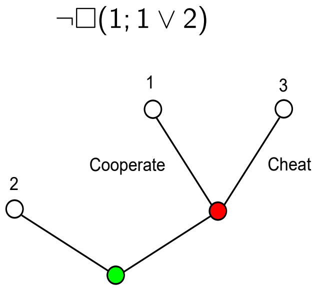


$\neg\Box(1; 1 \vee 2)$



...but can only be forced by a *strictly dominated strategy* in the right game!





By contrast, the left game has a Nash equilibrium in which Player I plays a strategy forcing $\{1, 2\}$.

Strategy equivalence

Definition

A set $P \subseteq O$ is said to be an *exact power* of Player I in G if:

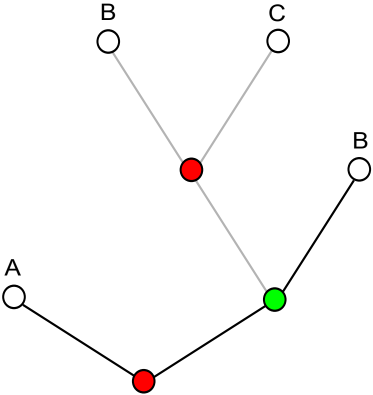
$$\exists \sigma \in \mathrm{Strat}(\mathrm{I}): \; P = \{o \in O \mid \exists \sigma' \in \mathrm{Strat}(\mathrm{II}): \; o = \mathrm{Out}(\sigma, \sigma')\}$$

Same for Player II. We say that G_1 and G_2 are *strategy equivalent* if the exact powers of each player are the same in both games: $E_I(G_1) = E_I(G_2)$ and $E_{II}(G_1) = E_{II}(G_2)$.

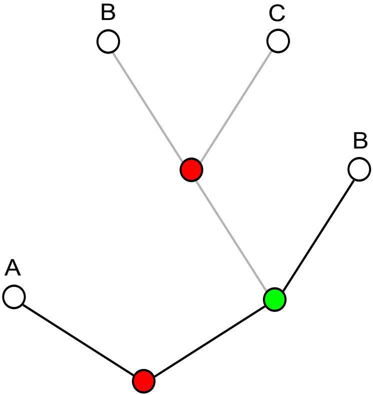
Make strategies first-class citizens in game logic:

Exact power = set of possible outcomes of playing one of the available strategies. Strategy equivalence = every strategy in G_1 has the same possible outcomes as some strategy in G_2, and vice versa.

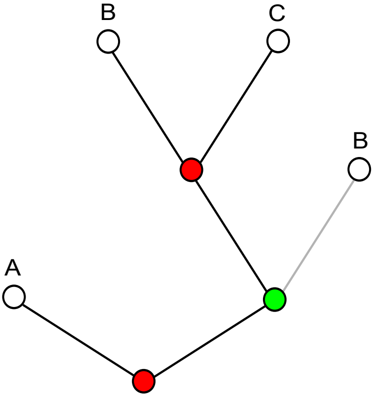
$\{A, B\} \in E(G)$



$\Box(A, B; A \vee B)$



$\square(A, B, C; A \vee B \vee C)$



Strategic normal form:

	LL	LR	RL	RR
L	A	A	B	C
R	A	A	B	B

Exact powers = rows in SNF.

Exact powers as basis:

Proposition

For any game G:

$$N(G) = \{P \in O \mid P' \subseteq P \text{ for some } P' \in E(G)\}$$

A step towards equilibria

Definition

Say that $p \in O$ is a *stable outcome* of a strategy σ for Player I if p is an outcome of some σ-guided match, and there is no σ-guided outcome which is better for Player II.

p is a stable outcome:

$$\Box(p; \bigwedge_{p \prec_{II} q} \neg q)$$

Proposition

Strategy equivalent games have the same stable outcomes for both players.

Proposition

Every equilibrium has a stable outcome for each player. If p has maximal payoff for either player, then it is a stable outcome in G iff there is an equilibrium with outcome p.

Instantial game logic

$$\varphi := p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid (G)(\psi_1, \ldots, \psi_n; \varphi)$$

$$G := g \mid G \cup G \mid G \cap G \mid G \circ G \mid G^\star$$

Semantics

Definition

A *game frame* is a pair (S, R) where R associates with every atomic game g a relation $R_g \subseteq S \times \mathcal{P}^+(S)$.

$$u \Vdash (G)(\psi_1, \ldots, \psi_n; \varphi) \;\Leftrightarrow\; \exists Z \in R_G[u] : Z \subseteq [\![\varphi]\!] \;\&\; Z \cap [\![\psi_i]\!] \neq \emptyset$$

Angelic choice

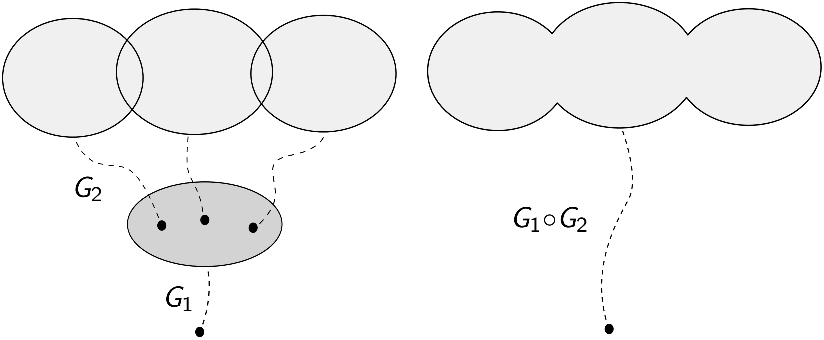
$$R_{G_1 \cup G_2} = R_{G_1} \cup R_{G_2}$$

Demonic choice

Only slightly more complicated:

$$R_{G_1 \cap G_2} = \{(u, Z_1 \cup Z_2) \mid (u, Z_1) \in R_{G_1} \;\&\; (u, Z_2) \in R_{G_2}\}$$

Composition



Kleene star

- $(u, Z) \in R_{G^0}$ iff $Z = \{u\}$
- $G^{n+1} = G^n \cup G \circ G^n$
- $(u, Z) \in R_{G^\star}$ iff $(u, Z) \in R_{G^n}$ for some $n \in \omega$

Basic properties

- Dual-free game logic as a fragment
- Bisimulation invariance

Proof.

Game operations are safe for instantial neighborhood bisimulation. ■

- Complexity $\in$ 2EXPTIME

Proof.

Satisfiability preserving translation into modal mu-calculus (exponential growth in formula size). ■

- Admits a variant of filtration.

Axioms

Angelic choice

$$(G_1 \cup G_2)(\vec{\psi}; \varphi) \leftrightarrow (G_1)(\vec{\psi}; \varphi) \vee (G_2)(\vec{\psi}; \varphi)$$

Demonic choice

Definition

If $\vec{\psi} = \psi_1, ..., \psi_n$, then let $\mathsf{Split}(G_1, G_2, \vec{\psi}, \varphi)$ be the disjunction of all formulas

$$(G_1)(\theta_1, ..., \theta_k; \varphi) \wedge (G_2)(\theta_{k+1}, ..., \theta_m; \varphi)$$

such that $\{\psi_1, ..., \psi_n\} = \{\theta_1, ..., \theta_m\}$.

Demonic choice

$$(G_1 \cap G_2)(\vec{\psi}; \varphi) \leftrightarrow \mathsf{Split}(G_1, G_2, \vec{\psi}, \varphi)$$

The Composition Law

Definition

Given game terms G_1, G_2, and finite tuple of formulas $\vec{\psi}, \varphi$: let $\delta(G_1, G_2, \vec{\psi}, \varphi)$ be the disjunction of all formulas

$$(G_1)((G_2)(\vec{\theta}_1; \varphi), \ldots, (G_2)(\vec{\theta}_n; \varphi); (G_2)\varphi)$$

where:

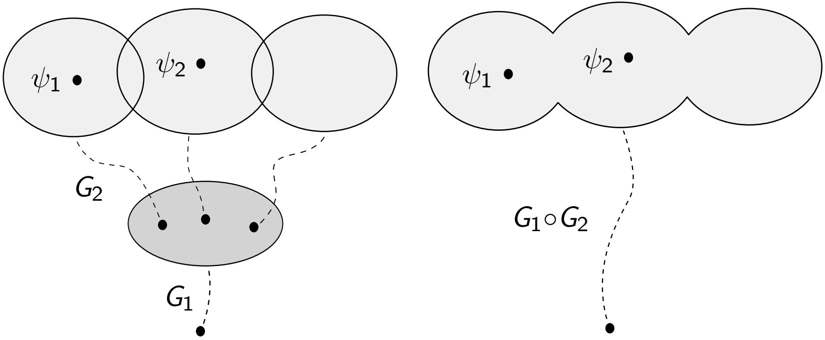
1. $\vec{\theta}_1 \cdot \ldots \cdot \vec{\theta}_n = \vec{\psi}$ and
2. $|\vec{\theta}_i| < |\vec{\psi}|$ for each i.

Note that if $\vec{\psi}$ is a singleton or empty, $\delta(G_1, G_2, \vec{\psi}, \varphi) = \perp$.

Composition law

$$(G_1 \circ G_2)(\vec{\psi}; \varphi) \leftrightarrow \delta(G_1, G_2, \vec{\psi}, \varphi) \vee (G_1)((G_2)(\vec{\psi}; \varphi); (G_2)\varphi)$$

Example:



$(G_1)((G_2)(\psi_1;\varphi),(G_2)(\psi_2;\varphi);(G_2)\varphi)$ $\qquad$ $(G_1 \circ G_2)(\psi_1,\psi_2;\varphi)$

Dealing with the Kleene star

- The Kleene star is a least fixpoint construction.
- Axiomatizing least fixpoints: fixpoint axiom + induction rule. (cf. Kozen's axioms for the μ-calculus)
- Fixpoint axiom: $(G^\star)(\psi_1, ..., \psi_n; \varphi) \leftrightarrow (\psi_1 \wedge ... \wedge \psi_n \wedge \varphi) \vee (G \circ G^\star)(\psi_1, ..., \psi_n; \varphi)$
- This leaves the problem of finding the right induction rules!

First induction principle

$$\frac{\varphi \rightarrow \gamma \qquad (G)\gamma \rightarrow \gamma}{(G^\star)\varphi \rightarrow \gamma}$$

By the composition law:

If $\gamma = (G^\star)(\vec{\psi}; \varphi)$ then:

$$
\begin{aligned}
\gamma &\equiv \\
&(\bigwedge \vec{\psi} \wedge \varphi) \vee (G \circ G^\star)(\vec{\psi}; \varphi) \equiv \\
&(\bigwedge \vec{\psi} \wedge \varphi) \vee \delta(G, G^\star, \vec{\psi}, \varphi) \vee (G)((G^\star)(\vec{\psi}; \varphi); (G^\star)\varphi) = \\
&(\bigwedge \vec{\psi} \wedge \varphi) \vee \delta(G, G^\star, \vec{\psi}, \varphi) \vee (G)(\gamma; (G^\star)\varphi)
\end{aligned}
$$

Second induction principle

$$\frac{\bigwedge \vec{\psi} \wedge \varphi \to \gamma \qquad \delta(G, G^\star, \vec{\psi}, \varphi) \to \gamma \qquad (G)(\gamma; (G^\star)\varphi) \to \gamma}{(G^\star)(\vec{\psi}; \varphi) \to \gamma}$$

Special case for a single instantial formula:

$$\frac{\psi \wedge \varphi \to \gamma \qquad (G)(\gamma; (G^\star)\varphi) \to \gamma}{(G^\star)(\psi; \varphi) \to \gamma}$$

An axiom system for IGL

1. All axioms and rules for INL
2. Angelic and demonic choice axioms
3. Composition law
4. Fixpoint axiom for Kleene star
5. Both induction rules

Completeness

Theorem

The axiom system for IGL is sound and weakly complete for validity on game frames.

Ongoing and future work

Unrestricted dual

Definition

Write $G_1 \simeq_I G_2$ if Player I has the same exact powers in G_1, G_2.

Problem:

The equivalence $\simeq_I$ is not a congruence for game dual, even with determinacy!

Example

G_1 and G_1:

A	B

A	B	B
A	B	A

G_1^∂ and G_2^∂:

A
B

A	A
B	B
B	A

Representation theorem for exact powers

Let $F_I, F_{II} \subseteq \mathcal{P}(O)$. Consider the following conditions:

- (Non-emptiness) $F_I \neq \emptyset$ and $F_{II} \neq \emptyset$.
- (Forth) Given $P \in F_I$ ($P \in F_{II}$): for any $x \in P$, there is some $P' \in F_{II}$ ($P' \in F_I$) with $x \in P'$.
- (Back) For any $P \in F_I$ and $P' \in F_{II}$ we have $P \cap P' \neq \emptyset$.

Theorem

Suppose $F_I, F_{II} \subseteq \mathcal{P}(O)$. Then the pair (F_I, F_{II}) satisfies the Non-emptiness, Back and Forth conditions if, and only if, there exists a game G such that $F_I = E_I(G)$ and $F_{II} = E_{II}(G)$.

Game algebra

- Let $\mathcal{G}$ the set of all games with outcomes in O, with operations $\cup$ and dual $(-)^{\partial}$.
- Strong power equivalence $\sim$ is a congruence for dual and choice.

Definition

The *strong algebra of games* is the quotient $\mathcal{G}/\sim$.

Failure of idempotent laws

Proposition

The equation $x \cap x = x$ is not valid on the strong game algebra.

Proof.

$$G = \begin{array}{|c|} \hline A \\ \hline B \\ \hline \end{array} \qquad G \cap G = \begin{array}{|c|c|} \hline A & A \\ \hline A & B \\ \hline B & A \\ \hline B & B \\ \hline \end{array}$$

$\blacksquare$

Modularity... sort of

Proposition

The following quasi-equation is valid on the strong game algebra:

$$x \cap z = x \quad x \cup z = z \quad \Rightarrow \quad x \cup (y \cap z) = (x \cup y) \cap z$$

Because of the failure of idempotent laws, this does not seem to reduce to an equation.

More problems

- What game operations are safe for instantial neighborhood bisimulations?
- Precise complexity?
- Axiomatize strong game algebra!
- Instantial semantics for ATL?
- Applications!

Thank you!