On skew Heyting algebras

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Workshop on Topological Methods in Logic V
June 13–17, 2016
Tbilisi, Georgia

Motivation

Skew lattices: noncommutative lattices.

"Nice" distributive skew lattices are dual to sheaves over Priestley spaces:

- Andrej Bauer, KCV, Mai Gehrke, Sam Van Gool and Ganna Kudryavtseva: *A non-commutative Priestley duality*, preprint.

Is there a notion of a skew Heyting algebra?

Skew lattices

Pascual Jordan 1940's and 1960's (classified report)
Jonathan Leech 1980's
A *skew lattice* is an algebra $(S; \wedge, \vee)$ such that $\wedge$ and $\vee$ are both idempotent and associative, and they dualize each other in that

$$x \wedge y = x \text{ iff } x \vee y = y \text{ and}$$
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A *skew lattice with zero* is an algebra $(S; \wedge, \vee, 0)$ such that $(S; \wedge, \vee)$ is a skew lattice and $x \wedge 0 = 0 = 0 \wedge x$ for all $x \in S$.

Rectangular algebras

- A *rectangular band* $(A, \wedge)$:
  - $\wedge$ is idempotent and associative
  - $x \wedge y \wedge z = x \wedge z$
- It becomes a skew lattice if we define $x \vee y = y \wedge x$.

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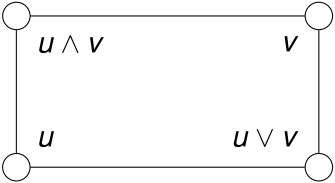
- It becomes a skew lattice if we define $x \vee y = y \wedge x$.
- For sets X and Y define $\wedge$ on $X \times Y$:

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1, y_2)$$



  $(X \times Y, \wedge)$ is a rectangular algebra.
- Every rectangular algebra is isomorphic to one such.

The order and Green's relation $\mathcal{D}$

S a skew lattice.

Natural preorder: $x \preceq y$ iff $x \wedge y \wedge x = x$ (and dually $y \vee x \vee y = y$).

Natural partial order: $x \leq y$ iff $x \wedge y = x = y \wedge x$ (and dually $y \vee x = y = x \vee y$).

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Green's relation $\mathcal{D}$ is defined by

$$x \mathcal{D} y \Leftrightarrow (x \preceq y \text{ and } y \preceq x).$$

Leech's decomposition theorems

The first decomposition theorem

Theorem (Leech, 1989)

- *$\mathcal{D}$ is a congruence;*
- *$S/\mathcal{D}$ is a lattice: the maximal lattice image of S, and*
- *each $\mathcal{D}$-class is a rectangular band.*

So: a skew lattice is a lattice of rectangular bands.

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So: a skew lattice is a lattice of rectangular bands.

Fact: $x \preceq y$ in S iff $\mathcal{D}_x \leq \mathcal{D}_y$ in $S/\mathcal{D}$.

A SL S is:

- *left handed* if it satisfies $x \wedge y \wedge x = x \wedge y$ and $x \vee y \vee x = y \vee x$.
- *right handed* if it satisfies $x \wedge y \wedge x = y \wedge x$ and $x \vee y \vee x = x \vee y$.

Leech's decomposition theorems

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Most natural examples of SLs are either left or right handed.

Leech, 1989: Any SL factors as a fiber product of a left handed SL by a right handed SL over their common maximal lattice image. (Pullback.)

Strongly distributive lattices

A *strongly distributive lattice* is a skew lattice S which satisfies the identities

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$
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S a skew distributive lattice $\Rightarrow S/\mathcal{D}$ is a distributive lattice.

Given any $x \in S$: $x\!\downarrow = \{y \in S \mid y \leq x\}$ is a distributive lattice.

If S has a top $\mathcal{D}$-class T, $t \in T$. Then: $t\!\downarrow \cong S/\mathcal{D}$.

Skew Boolean algebras

A *skew Boolean algebra* is an algebra $(S; \wedge, \vee, \setminus, 0)$ of type $(2, 2, 2, 0)$ where

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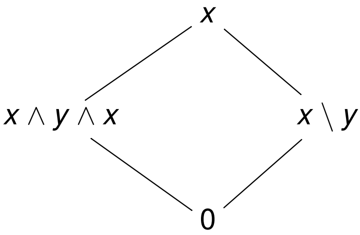
- $(S; \wedge, \vee, 0)$ is a skew distributive lattice with 0,
- $x{\downarrow}$ is a Boolean lattice for all x, and
- $x \setminus y$ is the complement of $x \wedge y \wedge x$ in $x{\downarrow}$.



Example of a skew Boolean algebra: $X \rightharpoonup Y$

- For partial maps $f, g : X \rightharpoonup Y$ define:

$$\begin{aligned} 0 &= \emptyset \\ f \wedge g &= f\restriction_{\mathrm{dom} f \cap \mathrm{dom} g} \\ f \vee g &= f\restriction_{\mathrm{dom} f \setminus \mathrm{dom} g} \cup g \\ f \setminus g &= f\restriction_{\mathrm{dom} f \setminus \mathrm{dom} g} \end{aligned}$$

  With these operations $X \rightharpoonup Y$ is a skew Boolean algebra.
- The maximal lattice image: $(X \rightharpoonup Y)/\mathcal{D} = \mathcal{P}(X)$.
- $X \rightharpoonup Y$ is left-handed: $f \wedge g \wedge f = f \wedge g$.

Duality for skew Boolean algebras

Bauer, CV, Kudryavtseva:

- A *Boolean space* is a locally compact zero-dimensional Hausdorff space.
- A *Boolean sheaf* is a local homeomorphism $p : E \to B$ where B is a Boolean space.

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- We only consider sheaves for which $p : E \to B$ is *surjective*.
- **Theorem:**
  Boolean sheaves are dual to left-handed skew Boolean algebras.

Duality for strongly distributive skew lattices

A skew lattice is *strongly distributive* if it satisfies:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z).$$

Bauer, CV, Gehrke, Van Gool, Kudryavtseva: *Left-handed strongly distributive skew lattices are dual to sheaves over Priestley spaces.*

Heyting algebras

A *Heyting algebra* is an algebra $\mathbf{H} = (H; \wedge, \vee, \rightarrow, 1, 0)$ such that $(H, \wedge, \vee, 1, 0)$ is a bounded distributive lattice that satisfies the condition:

(HA) $x \wedge y \leq z$ iff $x \leq y \rightarrow z$.

Equivalently, (HA) can be replaced by the following set of identities:

(H1) $(x \rightarrow x) = 1$,

(H2) $x \wedge (x \rightarrow y) = x \wedge y$,

(H3) $y \wedge (x \rightarrow y) = y$,

(H4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

Skew Heyting algebras - motivation

We want a skew notion of $\to$ s. t.:

- $\to$ satisfies a "skew version" of (H1)–(H4)
- $\to$ compatible with $\mathcal{D}$, $S/\mathcal{D}$ Heyting algebra
- $\to$ is the Heyting implication when S commutative

Skew Heyting algebras - problem

$(S, \wedge, \vee, 0)$ a strongly distributive SL with 0
If S has a top element then it is commutative.

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$(S, \wedge, \vee, 0)$ a strongly distributive SL with 0
If S has a top element then it is commutative.

Moreover:

- $(S, \wedge, \vee, 0)$ a strongly distributive SL with 0
- S has a top $\mathcal{D}$-class T, $t \in T$ fixed

Operation $\to$ defined on S s.t.:

(A1) $x \to x = x \vee t \vee x$

(A2) $x \wedge (x \to y) \wedge x = x \wedge y \wedge x$

(A3) $y \wedge (x \to y) = y$ and $(x \to y) \wedge y = y$

(A4) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

Then: S commutative.

Skew Heyting algebras - definition

A skew lattice is *co-strongly distributive* if it satisfies:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$
$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

A *skew Heyting lattice* is an algebra $(S; \wedge, \vee, 1)$ such that:

- $(S; \wedge, \vee, 1)$ is a co-strongly distributive skew lattice with top 1. Thus: Each upset $u\uparrow = \{x \in S \mid x \geq u\}$ is a bounded distributive lattice.
- $\rightarrow_u$ can be defined on $u\uparrow$ such that $(u\uparrow; \wedge, \vee, \rightarrow_u, 1, u)$ is a Heyting algebra with top 1 and bottom u.

Define $\rightarrow$ on a skew Heyting lattice S by setting:

$$x \rightarrow y = (y \vee x \vee y) \rightarrow_y y.$$

Skew Heyting algebras - axiomatization

Theorem

$(S; \wedge, \vee, \to, 1)$ is a skew Heyting algebra if and only if $\to$ satisfies the following axioms:

(SH0) $x \to y = (y \vee x \vee y) \to y$.

(SH1) $x \to x = 1$,

(SH2) $x \wedge (x \to y) \wedge x = x \wedge y \wedge x$,

(SH3) $y \wedge (x \to y) = y$ *and* $(x \to y) \wedge y = y$,

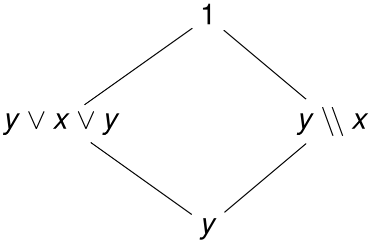
(SH4) $x \to (u \vee (y \wedge z) \vee u) = (x \to (u \vee y \vee u)) \wedge (x \to (u \vee z \vee u))$.

Skew Heyting algebras - examples

- All finite co-strongly distributive skew lattices with a top 1.
- Co-strongly distributive skew chins with 1 ($S/\mathcal{D}$ chain). We set:

$$x \to y = \begin{cases} 1; & \text{if } x \preceq y. \\ y; & \text{otherwise.} \end{cases}$$

- $(S; \wedge, \vee, \backslash\!\backslash, 1)$ is a *dual skew Boolean algebra* if $(S; \wedge, \vee, 1)$ and



$x \to y = y \,\backslash\!\backslash\, x = y \,\backslash\!\backslash\, (y \vee x \vee y)$ in $y\!\uparrow$.

The partial functions example

$X \rightharpoonup Y$ all partial functions from X to Y

skew Heyting operation	description	skew Boolean op.
$f \wedge g$	$g \cup (f\|_{\mathrm{dom}f - \mathrm{dom}g})$	$f \vee g$
$f \vee g$	$f\|_{\mathrm{dom}g \cap \mathrm{dom}f}$	$f \wedge g$
$f \to g$	$g\|_{\mathrm{dom}g - \mathrm{dom}f}$	$g \setminus f$
1	$\emptyset$	0