On skew Heyting algebras

Karin Cvetko-Vah

University of Ljubljana

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Skew lattices: noncommutative lattices.

“Nice” distributive skew lattices are dual to sheaves over Priestley spaces:


Is there a notion of a skew Heyting algebra?
A skew lattice is an algebra \((S; \wedge, \vee)\) such that \(\wedge\) and \(\vee\) are both idempotent and associative, and they dualize each other in that

\[
x \wedge y = x \text{ iff } x \vee y = y \text{ and } \\
x \wedge y = y \text{ iff } x \vee y = x.
\]
Skew lattices

Pascual Jordan 1940’s and 1960’s (classified report)
Jonathan Leech 1980’s

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A *skew lattice with zero* is an algebra \((S; \wedge, \vee, 0)\) such that 
\((S; \wedge, \vee)\) is a skew lattice and \(x \wedge 0 = 0 = 0 \wedge x\) for all \(x \in S\).
Rectangular algebras

- A rectangular band \((A, \wedge)\):
  - \(\wedge\) is idempotent and associative
  - \(x \wedge y \wedge z = x \wedge z\)
- It becomes a skew lattice if we define \(x \vee y = y \wedge x\).
Rectangular algebras

- A *rectangular band* \((A, \wedge)\):
  - \(\wedge\) is idempotent and associative
  - \(x \wedge y \wedge z = x \wedge z\)
- It becomes a skew lattice if we define \(x \lor y = y \wedge x\).
- For sets \(X\) and \(Y\) define \(\wedge\) on \(X \times Y\):
  \[
  (x_1, y_1) \wedge (x_2, y_2) = (x_1, y_2)
  \]

\[(X \times Y, \wedge)\] is a rectangular algebra.
- Every rectangular algebra is isomorphic to one such.
The order and Green’s relation $\mathcal{D}$

$S$ a skew lattice.

Natural preorder: $x \preceq y$ iff $x \land y \land x = x$ (and dually $y \lor x \lor y = y$).

Natural partial order: $x \leq y$ iff $x \land y = x = y \land x$ (and dually $y \lor x = y = x \lor y$).
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*Green’s relation* $\mathcal{D}$ is defined by

$$xDy \iff (x \preceq y \text{ and } y \preceq x).$$
Leech’s decomposition theorems
The first decomposition theorem

Theorem (Leech, 1989)

- $\mathcal{D}$ is a congruence;
- $S/\mathcal{D}$ is a lattice: the maximal lattice image of $S$, and
- each $\mathcal{D}$-class is a rectangular band.

So: a skew lattice is a lattice of rectangular bands.
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So: a skew lattice is a lattice of rectangular bands.

Fact: $x \preceq y$ in $S$ iff $\mathcal{D}_x \leq \mathcal{D}_y$ in $S/\mathcal{D}$. 
Leech’s decomposition theorems

The second decomposition theorem

A SL $S$ is:

- **left handed** if it satisfies $x \land y \land x = x \land y$ and $x \lor y \lor x = y \lor x$.

- **right handed** if it satisfies $x \land y \land x = y \land x$ and $x \lor y \lor x = x \lor y$. 

Most natural examples of SLs are either left or right handed.

Leech, 1989: Any SL factors as a fiber product of a left handed SL by a right handed SL over their common maximal lattice image. (Pullback.)
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A *strongly distributive lattice* is a skew lattice $S$ which satisfies the identities

\[ x \land (y \lor z) = (x \land y) \lor (x \land z), \]
\[ (x \lor y) \land z = (x \land z) \lor (y \land z). \]
Strongly distributive lattices

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\end{align*}
\]

$S$ a skew distributive lattice $\Rightarrow S/\mathcal{D}$ is a distributive lattice.

Given any $x \in S$: $x\downarrow = \{y \in S \mid y \leq x\}$ is a distributive lattice.

If $S$ has a top $\mathcal{D}$-class $T$, $t \in T$. Then: $t\downarrow \cong S/\mathcal{D}$. 
Skew Boolean algebras

A *skew Boolean algebra* is an algebra \((S; \wedge, \vee, \setminus, 0)\) of type \((2, 2, 2, 0)\) where

- \((S; \wedge, \vee, 0)\) is a skew distributive lattice with 0,
Skew Boolean algebras

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- \((S; \land, \lor, 0)\) is a skew distributive lattice with 0,
- \(x\downarrow\) is a Boolean lattice for all \(x\), and
- \(x \setminus y\) is the complement of \(x \land y \land x\) in \(x\downarrow\).
Example of a skew Boolean algebra: $X \rightarrow Y$

- For partial maps $f, g : X \rightarrow Y$ define:

  $$
  0 = \emptyset
  $$

  $$
  f \land g = f\upharpoonright_{\text{dom}f \cap \text{dom}g}
  $$

  $$
  f \lor g = f\upharpoonright_{\text{dom}f \setminus \text{dom}g} \cup g
  $$

  $$
  f \setminus g = f\upharpoonright_{\text{dom}f \setminus \text{dom}g}
  $$

  With these operations $X \rightarrow Y$ is a skew Boolean algebra.

- The maximal lattice image: $(X \rightarrow Y)/\mathcal{D} = \mathcal{P}(X)$.

- $X \rightarrow Y$ is left-handed: $f \land g \land f = f \land g$. 
Duality for skew Boolean algebras

Bauer, CV, Kudryavtseva:

- A *Boolean space* is a locally compact zero-dimensional Hausdorff space.
- A *Boolean sheaf* is a local homeomorphism $p : E \to B$ where $B$ is a Boolean space.
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- We only consider sheaves for which $p : E \to B$ is surjective.
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- A *Boolean sheaf* is a local homeomorphism $p : E \to B$ where $B$ is a Boolean space.
- We only consider sheaves for which $p : E \to B$ is *surjective*.

**Theorem:**

*Boolean sheaves are dual to left-handed skew Boolean algebras.*
Duality for strongly distributive skew lattices

A skew lattice is \textit{strongly distributive} if it satisfies:

\[ x \land (y \lor z) = (x \land y) \lor (x \land z), \]
\[ (x \lor y) \land z = (x \land z) \lor (y \land z). \]

Bauer, CV, Gehrke, Van Gool, Kudryavtseva: \textit{Left-handed strongly distributive skew lattices are dual to sheaves over Priestley spaces}. 
A *Heyting algebra* is an algebra $\mathbf{H} = (H; \land, \lor, \to, 1, 0)$ such that $(H, \land, \lor, 1, 0)$ is a bounded distributive lattice that satisfies the condition:

\[(HA) \quad x \land y \leq z \text{ iff } x \leq y \to z.\]

Equivalently, $(HA)$ can be replaced by the following set of identities:

\[(H1) \quad (x \to x) = 1,\]
\[(H2) \quad x \land (x \to y) = x \land y,\]
\[(H3) \quad y \land (x \to y) = y,\]
\[(H4) \quad x \to (y \land z) = (x \to y) \land (x \to z).\]
We want a skew notion of $\rightarrow$ s. t.:
- $\rightarrow$ satisfies a "skew version" of (H1)–(H4)
- $\rightarrow$ compatible with $\mathcal{D}$, $S/\mathcal{D}$ Heyting algebra
- $\rightarrow$ is the Heyting implication when $S$ commutative
Skew Heyting algebras - problem

$(S, \land, \lor, 0)$ a strongly distributive SL with 0
If $S$ has a top element then it is commutative.
Skew Heyting algebras - problem

\((S, \land, \lor, 0)\) a strongly distributive SL with 0
If \(S\) has a top element then it is commutative.

Moreover:
- \((S, \land, \lor, 0)\) a strongly distributive SL with 0
- \(S\) has a top \(D\)-class \(T, t \in T\) fixed

Operation \(\rightarrow\) defined on \(S\) s.t.:

(A1) \(x \rightarrow x = x \lor t \lor x\)

(A2) \(x \land (x \rightarrow y) \land x = x \land y \land x\)

(A3) \(y \land (x \rightarrow y) = y\) and \((x \rightarrow y) \land y = y\)

(A4) \(x \rightarrow (y \land z) = (x \rightarrow y) \land (x \rightarrow z)\)

Then: \(S\) commutative.
Skew Heyting algebras - definition

A skew lattice is *co-strongly distributive* if it satisfies:

\[
x \lor (y \land z) = (x \lor y) \land (x \lor z),
\]
\[
(x \land y) \lor z = (x \lor z) \land (y \lor z).
\]

A *skew Heyting lattice* is an algebra \((S; \land, \lor, 1)\) such that:

1. \((S; \land, \lor, 1)\) is a co-strongly distributive skew lattice with top 1. Thus: Each upset \(u\uparrow = \{x \in S \mid x \geq u\}\) is a bounded distributive lattice.
2. \(\rightarrow_u\) can be defined on \(u\uparrow\) such that \((u\uparrow; \land, \lor, \rightarrow_u, 1, u)\) is a Heyting algebra with top 1 and bottom \(u\).

Define \(\rightarrow\) on a skew Heyting lattice \(S\) by setting:

\[
x \rightarrow y = (y \lor x \lor y) \rightarrow_y y.
\]
Skew Heyting algebras - axiomatization

Theorem

\((S; \land, \lor, \rightarrow, 1)\) is a skew Heyting algebra if and only if \(\rightarrow\) satisfies the following axioms:

- \((\text{SH0})\)  \(x \rightarrow y = (y \lor x \lor y) \rightarrow y\).
- \((\text{SH1})\)  \(x \rightarrow x = 1\),
- \((\text{SH2})\)  \(x \land (x \rightarrow y) \land x = x \land y \land x\),
- \((\text{SH3})\)  \(y \land (x \rightarrow y) = y\) and \((x \rightarrow y) \land y = y\),
- \((\text{SH4})\)  \(x \rightarrow (u \lor (y \land z) \lor u) = (x \rightarrow (u \lor y \lor u)) \land (x \rightarrow (u \lor z \lor u))\).
Skew Heyting algebras - examples

- All finite co-strongly distributive skew lattices with a top 1.
- Co-strongly distributive skew chins with 1 ($S/D$ chain). We set:
  \[
x \rightarrow y = \begin{cases} 
  1; & \text{if } x \leq y. \\
  y; & \text{otherwise.} 
\end{cases}
\]

- $(S; \wedge, \vee, \parallel, 1)$ is a dual skew Boolean algebra if $(S; \wedge, \vee, 1)$ and

\[
x \rightarrow y = y \parallel x = y \parallel (y \vee x \vee y) \text{ in } y\uparrow.
\]
The partial functions example

$X \rightarrow Y$ all partial functions from $X$ to $Y$

<table>
<thead>
<tr>
<th>skew Heyting operation</th>
<th>description</th>
<th>skew Boolean op.</th>
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<tbody>
<tr>
<td>$f \land g$</td>
<td>$g \cup (f</td>
<td>_{\text{dom}f-\text{dom}g})$</td>
</tr>
<tr>
<td>$f \lor g$</td>
<td>$f</td>
<td>_{\text{dom}g \cap \text{dom}f}$</td>
</tr>
<tr>
<td>$f \rightarrow g$</td>
<td>$g</td>
<td>_{\text{dom}g-\text{dom}f}$</td>
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<tr>
<td>1</td>
<td>$\emptyset$</td>
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