Generalized Heyting Algebras
with a Unary Operator

Almudena Colacito

This is a Master Thesis project under the supervision of Marta Bílková and Dick de Jongh

INSTITUTE FOR LOGIC,
LANGUAGE AND COMPUTATION

June 16, 2016

Master Thesis

Kripke Semantics
(Completeness, Finite Model Property, Applications)

Master Thesis

Kripke Semantics
(Completeness, Finite Model Property, Applications)

Algebraic Semantics (Algebraic Completeness, Duality)

Master Thesis

Kripke Semantics
(Completeness, Finite Model Property, Applications)

Algebraic Semantics (Algebraic Completeness, Duality)

Sequent Calculi
(Cut Elimination, Craig's Interpolation, Translation)

Master Thesis

Kripke Semantics
(Completeness, Finite Model Property, Applications)

Algebraic Semantics (Algebraic Completeness, Duality)

Sequent Calculi
(Cut Elimination, Craig's Interpolation, Translation)

Outline

Outline

Introduction

Language: $\mathcal{L}^{\neg} = \mathcal{L}^{+} \cup \{\neg\}$, where $\mathcal{L}^{+} = \{\wedge, \vee, \rightarrow\}$

Introduction

Language: $\mathcal{L}^{\neg} = \mathcal{L}^{+} \cup \{\neg\}$, where $\mathcal{L}^{+} = \{\wedge, \vee, \to\}$

Setting: Positive logic

Introduction

$$\bot \to \varphi$$

Introduction

$$\cancel{\bot \longrightarrow \varphi}$$

Basic System of a Unary Operator

Basic System of a Unary Operator

Basic Logic of Negation (N) $\Rightarrow$ $(p \leftrightarrow q) \rightarrow (\neg p \leftrightarrow \neg q)$

Examples of Extensions of N

Examples of Extensions of N

Negative ex Falso Logic (NeF) $\Rightarrow$ $\mathsf{N} + p \to (\neg p \to \neg q)$

Examples of Extensions of N

Negative ex Falso Logic (NeF) $\Rightarrow$ $\mathrm{N} + p \to (\neg p \to \neg q)$

Contraposition Logic (CoPC) $\Rightarrow$ $(p \to q) \to (\neg q \to \neg p)$

Introduction

Examples of Extensions of N

Negative ex Falso Logic (NeF) $\Rightarrow$ $\mathrm{N} + p \to (\neg p \to \neg q)$

Contraposition Logic (CoPC) $\Rightarrow$ $(p \to q) \to (\neg q \to \neg p)$

Minimal Logic (MPC) $\Rightarrow$ $\big((p \to q) \wedge (p \to \neg q)\big) \to \neg p$

Outline

Semantics for the Basic Logic of a Unary Operator

- Intuitionistic frame
- A function N between upsets
- A persistent propositional valuation V

Semantics for the Basic Logic of a Unary Operator

- Intuitionistic frame
- A function N between upsets
- A persistent propositional valuation V

Semantics for the Basic Logic of a Unary Operator

- Intuitionistic frame
- A function N between upsets
- A persistent propositional valuation V

Semantics for the Basic Logic of a Unary Operator

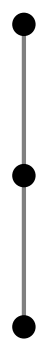
- Intuitionistic frame
- A function N between upsets
- A persistent propositional valuation V

Semantics for the Basic Logic of a Unary Operator

$$[\![\neg\varphi]\!] := N([\![\varphi]\!])$$

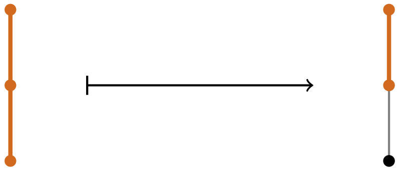
Kripke Semantics

Semantics for the Basic Logic of a Unary Operator $\langle W, R \rangle$



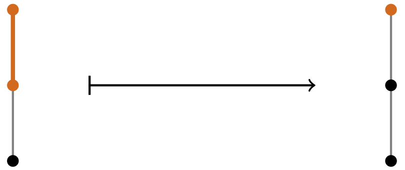
Kripke Semantics

Semantics for the Basic Logic of a Unary Operator $\mathfrak{F} = \langle W, R, N \rangle$

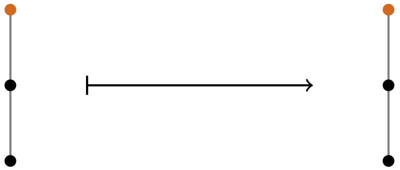


Kripke Semantics

Semantics for the Basic Logic of a Unary Operator $\mathfrak{F} = \langle W, R, N \rangle$

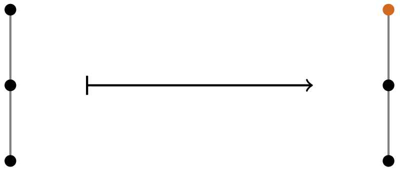


Semantics for the Basic Logic of a Unary Operator $\mathfrak{F} = \langle W, R, N \rangle$



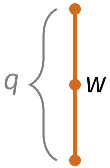
Kripke Semantics

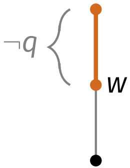
Semantics for the Basic Logic of a Unary Operator $\mathfrak{F} = \langle W, R, N \rangle$



Kripke Semantics

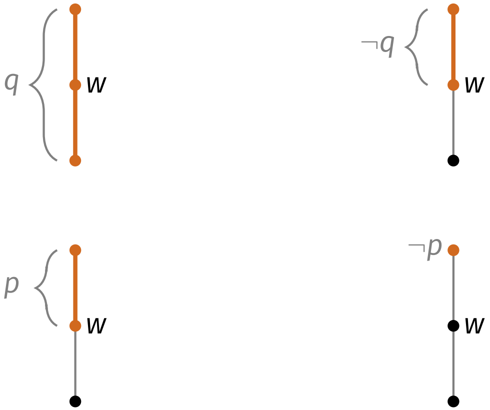
Semantics for the Basic Logic of a Unary Operator $\langle W, R, N, V \rangle$





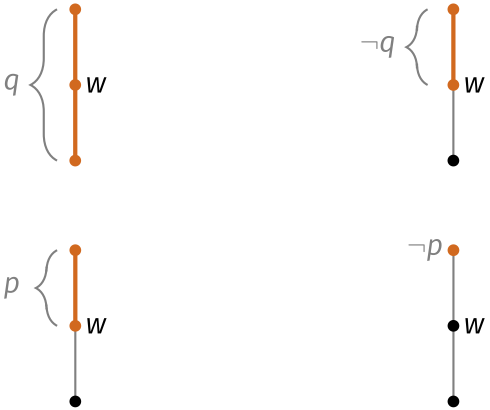
Kripke Semantics

Semantics for the Basic Logic of a Unary Operator $\langle W, R, N, V \rangle$



Kripke Semantics

Semantics for the Basic Logic of a Unary Operator $\langle W, R, N, V\rangle$



$$\mathfrak{F}, w \vDash_V p \leftrightarrow q \text{ and } \mathfrak{M}, w \nvDash_V \neg q \rightarrow \neg p$$

Semantics for the Basic Logic of a Unary Operator

?

Kripke Semantics

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

Kripke Semantics

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

$$\mathfrak{F}, w \vDash_V p \leftrightarrow q$$

Kripke Semantics

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

$$\mathfrak{F}, w \vDash_V p \leftrightarrow q$$
$$V(p) \cap R(w) = V(q) \cap R(w)$$

Kripke Semantics

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

$$\mathfrak{F}, w \models_V p \leftrightarrow q$$
$$V(p) \cap R(w) = V(q) \cap R(w)$$
$$w \in N(V(p)) \Leftrightarrow w \in N(V(p) \cap R(w))$$

Kripke Semantics

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

$$\mathfrak{F}, w \vDash_V p \leftrightarrow q$$
$$V(p) \cap R(w) = V(q) \cap R(w)$$
$$w \in N(V(p)) \Leftrightarrow w \in N(V(p) \cap R(w))$$
$$w \in N(V(p) \cap R(w)) \Leftrightarrow w \in N(V(q) \cap R(w))$$

Kripke Semantics

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

$$\mathfrak{F}, w \vDash_V p \leftrightarrow q$$

$$V(p) \cap R(w) = V(q) \cap R(w)$$

$$w \in N(V(p)) \Leftrightarrow w \in N(V(p) \cap R(w))$$

$$w \in N(V(p) \cap R(w)) \Leftrightarrow w \in N(V(q) \cap R(w))$$

$$w \in N(V(q) \cap R(w)) \Leftrightarrow w \in N(V(q))$$

Kripke Semantics

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

$$\mathfrak{F}, w \vDash_V p \leftrightarrow q$$

$$V(p) \cap R(w) = V(q) \cap R(w)$$

$$w \in N(V(p)) \Leftrightarrow w \in N(V(p) \cap R(w))$$

$$w \in N(V(p) \cap R(w)) \Leftrightarrow w \in N(V(q) \cap R(w))$$

$$w \in N(V(q) \cap R(w)) \Leftrightarrow w \in N(V(q))$$

Kripke Semantics

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

$$\mathfrak{F}, w \vDash_V p \leftrightarrow q$$
$$V(p) \cap R(w) = V(q) \cap R(w)$$
$$w \in N(V(p)) \Leftrightarrow w \in N(V(p) \cap R(w))$$
$$w \in N(V(p) \cap R(w)) \Leftrightarrow w \in N(V(q) \cap R(w))$$
$$w \in N(V(q) \cap R(w)) \Leftrightarrow w \in N(V(q))$$
$$\mathfrak{F}, w \vDash_V \neg p \leftrightarrow \neg q$$

Kripke Semantics

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$ ✓

$$\mathfrak{F}, w \models_V p \leftrightarrow q$$
$$V(p) \cap R(w) = V(q) \cap R(w)$$
$$w \in N(V(p)) \Leftrightarrow w \in N(V(p) \cap R(w))$$
$$w \in N(V(p) \cap R(w)) \Leftrightarrow w \in N(V(q) \cap R(w))$$
$$w \in N(V(q) \cap R(w)) \Leftrightarrow w \in N(V(q))$$
$$\mathfrak{F}, w \models_V \neg p \leftrightarrow \neg q$$

Kripke Semantics

Locality: $N(U) \cap V = N(U \cap V) \cap V$ ✓

$$\mathfrak{F}, w \vDash_V p \leftrightarrow q$$
$$V(p) \cap R(w) = V(q) \cap R(w)$$
$$w \in N(V(p)) \Leftrightarrow w \in N(V(p) \cap R(w))$$
$$w \in N(V(p) \cap R(w)) \Leftrightarrow w \in N(V(q) \cap R(w))$$
$$w \in N(V(q) \cap R(w)) \Leftrightarrow w \in N(V(q))$$
$$\mathfrak{F}, w \vDash_V \neg p \leftrightarrow \neg q$$

Example

$$(p \to q) \to (\neg q \to \neg p)$$

Example

$$(p \to q) \to (\neg q \to \neg p)$$

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

Anti-monotonicity: $U \subseteq V$ implies $N(V) \subseteq N(U)$

Example

$$(p \to q) \to (\neg q \to \neg p)$$

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

Anti-monotonicity: $V(p) \subseteq V(q)$ implies $N(V) \subseteq N(U)$

Example

$$(p \to q) \to (\neg q \to \neg p)$$

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

Anti-monotonicity: $V(p) \subseteq V(q)$ implies $N(V(q)) \subseteq N(V(p))$

Example

$$(p \to q) \to (\neg q \to \neg p)$$

Locality: $w \in N(U)$ if and only if $w \in N(U \cap R(w))$

Anti-monotonicity: $U \subseteq V$ implies $N(V) \subseteq N(U)$ ✓

Outline

Algebraic Semantics

gH-algebras A **generalized Heyting algebra** (*gH-algebra* for short) $\langle A, \wedge, \vee, \rightarrow, 1\rangle$ is a lattice $\langle A, \wedge, \vee\rangle$ such that for every pair of elements $a, b \in A$, the element $a \rightarrow b$ defining the supremum of the set $\{c \in A | a \wedge c \leq b\}$ exists.

Algebraic Semantics

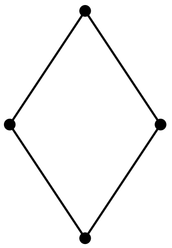
gH-algebras A **generalized Heyting algebra** (*gH-algebra* for short) $\langle A, \wedge, \vee, \rightarrow, 1\rangle$ is a lattice $\langle A, \wedge, \vee\rangle$ such that for every pair of elements $a, b \in A$, the element $a \rightarrow b$ defining the supremum of the set $\{c \in A | a \wedge c \leq b\}$ exists.

gH-algebras & Heyting algebras

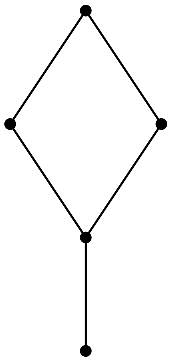
gH-algebra $\mathfrak{A}$ | Heyting algebra $\mathfrak{A}_\perp$

gH-algebras & Heyting algebras

gH-algebra $\mathfrak{A}$



Heyting algebra $\mathfrak{A}_\perp$



Algebraic Semantics

From gH-algebras to N-algebras An **N-algebra** $\langle A, \wedge, \vee, \rightarrow, \neg, 1\rangle$ is given by a gH-algebra $\langle A, \wedge, \vee, \rightarrow, 1\rangle$ equipped with a unary operator $\neg$ such that

$$(x \leftrightarrow y) \rightarrow (\neg x \leftrightarrow \neg y) = 1$$

Algebraic Semantics

From gH-algebras to N-algebras An **N-algebra** $\langle A, \wedge, \vee, \rightarrow, \neg, 1\rangle$ is given by a gH-algebra $\langle A, \wedge, \vee, \rightarrow, 1\rangle$ equipped with a unary operator $\neg$ such that

$$(x \leftrightarrow y) \rightarrow (\neg x \leftrightarrow \neg y) = 1$$

Algebraic Semantics

From gH-algebras to N-algebras An **N-algebra** $\langle A, \wedge, \vee, \rightarrow, \neg, 1\rangle$ is given by a gH-algebra $\langle A, \wedge, \vee, \rightarrow, 1\rangle$ equipped with a unary operator $\neg$ such that

$$(x \leftrightarrow y) \rightarrow (\neg x \leftrightarrow \neg y) = 1$$

Algebraic Semantics

From gH-algebras to N-algebras An **N-algebra** $\langle A, \wedge, \vee, \rightarrow, \neg, 1\rangle$ is given by a gH-algebra $\langle A, \wedge, \vee, \rightarrow, 1\rangle$ equipped with a unary operator $\neg$ such that

$$\neg x \wedge y = \neg(x \wedge y) \wedge y$$

Algebraic Semantics

Algebraic Completeness ✓

Outline

Duality

General frames An **N-general frame** is a quadruple $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$, where $\langle W, R \rangle$ is an intuitionistic frame, $\mathcal{P}$ is a set of upsets of W, containing W and which is closed under $\cup$, (finite) $\cap$, $\to$, where $\to$ is defined by

$$U \to V := \{w \in W | \forall v(wRv \wedge v \in U \to v \in V)\},$$

and $N : \mathcal{P} \to \mathcal{P}$ which satisfies locality.

Duality

General frames An **N-general frame** is a quadruple $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$, where $\langle W, R \rangle$ is an intuitionistic frame, $\mathcal{P}$ is a set of upsets of W, containing W and which is closed under $\cup$, (finite) $\cap$, $\to$, where $\to$ is defined by

$$U \to V := \{w \in W | \forall v(wRv \wedge v \in U \to v \in V)\},$$

and $N : \mathcal{P} \to \mathcal{P}$ which satisfies locality.

Duality

General frames An **N-general frame** is a quadruple $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$, where $\langle W, R \rangle$ is an intuitionistic frame, $\mathcal{P}$ is a set of upsets of W, containing W and which is closed under $\cup$, (finite) $\cap$, $\to$, where $\to$ is defined by

$$U \to V := \{w \in W | \forall v (wRv \wedge v \in U \to v \in V)\},$$

and $N : \mathcal{P} \to \mathcal{P}$ which satisfies locality.

Duality

General frames An **N-general frame** is a quadruple $\mathfrak{F} = \langle W, R, N, \mathcal{P}\rangle$, where $\langle W, R\rangle$ is an intuitionistic frame, $\mathcal{P}$ is a set of upsets of W, containing W and which is closed under $\cup$, (finite) $\cap$, $\to$, where $\to$ is defined by

$$U \to V := \{w \in W | \forall v(wRv \wedge v \in U \to v \in V)\},$$

and $N : \mathcal{P} \to \mathcal{P}$ which satisfies locality.

Duality

General frames An **N-general frame** is a quadruple $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$, where $\langle W, R \rangle$ is an intuitionistic frame, $\mathcal{P}$ is a set of upsets of W, containing W and which is closed under $\cup$, (finite) $\cap$, $\to$, where $\to$ is defined by

$$U \to V := \{w \in W | \forall v(wRv \wedge v \in U \to v \in V)\},$$

and $N : \mathcal{P} \to \mathcal{P}$ which satisfies locality.

General frames An **N-general frame** is a quadruple $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$, where $\langle W, R \rangle$ is an intuitionistic frame, $\mathcal{P}$ is a set of upsets of W, containing W and which is closed under $\cup$, (finite) $\cap$, $\rightarrow$, where $\rightarrow$ is defined by

$$U \rightarrow V := \{w \in W | \forall v(wRv \wedge v \in U \rightarrow v \in V)\},$$

and $N : \mathcal{P} \rightarrow \mathcal{P}$ which satisfies locality.

We do not require $\emptyset$ to be in $\mathcal{P}$!

Duality

Descriptive frames Let $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$ be an N-general frame.

- We say that the frame $\mathfrak{F}$ is *refined* if, for every $w, v \in W$: $\neg(wRv)$ implies the existence of an upset $U \in \mathcal{P}$ which contains w and does not contain v, i.e., $w \in U$ and $v \notin U$.
- We say that the frame $\mathfrak{F}$ is *compact* if for every $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \{W \setminus U | U \in \mathcal{P}\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the *finite intersection property*, then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

An **N-descriptive frame** is a refined and compact N-general frame $\mathfrak{F}$.

Descriptive frames Let $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$ be an N-general frame.

- We say that the frame $\mathfrak{F}$ is *refined* if, for every $w, v \in W$: $\neg(wRv)$ implies the existence of an upset $U \in \mathcal{P}$ which contains w and does not contain v, i.e., $w \in U$ and $v \notin U$.
- We say that the frame $\mathfrak{F}$ is *compact* if for every $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \{W \setminus U | U \in \mathcal{P}\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the *finite intersection property*, then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

An **N-descriptive frame** is a refined and compact N-general frame $\mathfrak{F}$.

Descriptive frames Let $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$ be an N-general frame.

- We say that the frame $\mathfrak{F}$ is *refined* if, for every $w, v \in W$: $\neg(wRv)$ implies the existence of an upset $U \in \mathcal{P}$ which contains w and does not contain v, i.e., $w \in U$ and $v \notin U$.
- We say that the frame $\mathfrak{F}$ is *compact* if for every $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \{W \setminus U | U \in \mathcal{P}\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the *finite intersection property*, then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

An **N-descriptive frame** is a refined and compact N-general frame $\mathfrak{F}$.

Duality

Descriptive frames Let $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$ be an N-general frame.

- We say that the frame $\mathfrak{F}$ is *refined* if, for every $w, v \in W$: $\neg(wRv)$ implies the existence of an upset $U \in \mathcal{P}$ which contains w and does not contain v, i.e., $w \in U$ and $v \notin U$.
- We say that the frame $\mathfrak{F}$ is *compact* if for every $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \{W \setminus U | U \in \mathcal{P}\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the *finite intersection property*, then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

An **N-descriptive frame** is a refined and compact N-general frame $\mathfrak{F}$.

Duality

Top Descriptive frames A **top descriptive frame** is a descriptive frame such that $\langle W, R\rangle$ has a greatest element t such that, for every upset $U \in \mathcal{P}$, we have $t \in U$.

Duality

Top Descriptive frames A **top descriptive frame** is a descriptive frame such that $\langle W, R \rangle$ has a greatest element t such that, for every upset $U \in \mathcal{P}$, we have $t \in U$.

Duality

Top Descriptive frames A **top descriptive frame** is a descriptive frame such that $\langle W, R\rangle$ has a greatest element t such that, for every upset $U \in \mathcal{P}$, we have $t \in U$.

$\emptyset$ cannot be in $\mathcal{P}$!

Duality

From frames to N-algebras Let $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$ be a top descriptive frame for N. Then, the structure

$$\langle \mathcal{P}, \cap, \cup, \rightarrow, N, W \rangle$$

is an N-algebra.

Duality

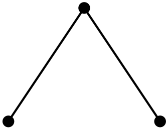
From frames to N-algebras Let $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$ be a top descriptive frame for N. Then, the structure

$$\langle \mathcal{P}, \cap, \cup, \to, N, W \rangle$$

is an N-algebra.

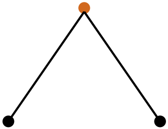
Duality

From frames to N-algebras



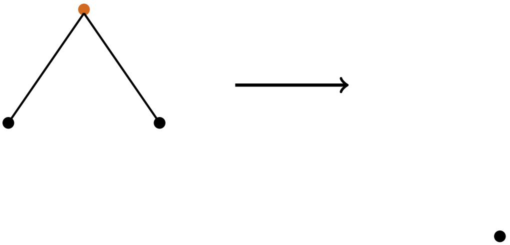
Duality

From frames to N-algebras



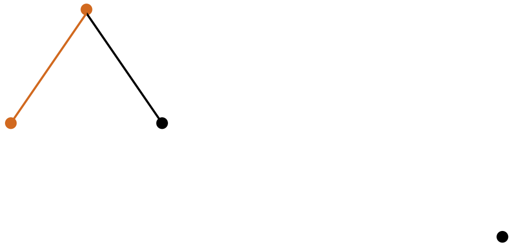
Duality

From frames to N-algebras



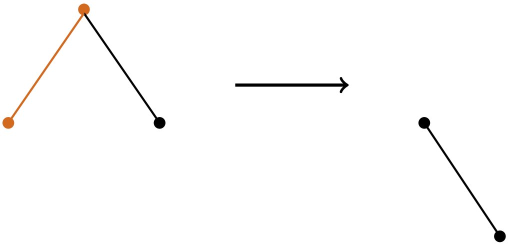
Duality

From frames to N-algebras



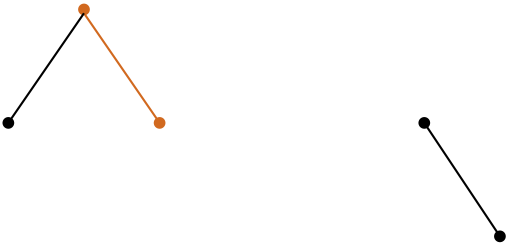
Duality

From frames to N-algebras



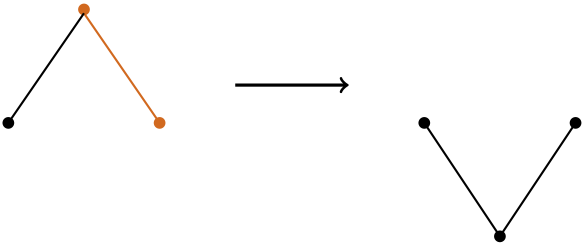
Duality

From frames to N-algebras



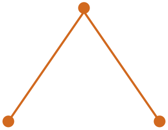
Duality

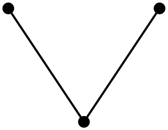
From frames to N-algebras



Duality

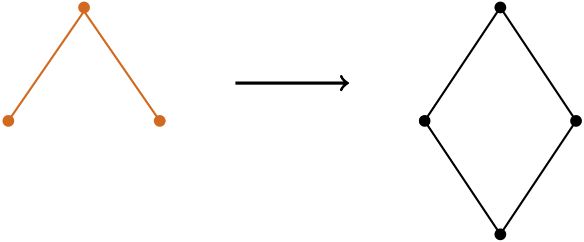
From frames to N-algebras





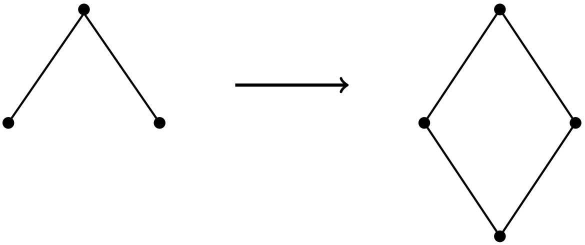
Duality

From frames to N-algebras



Duality

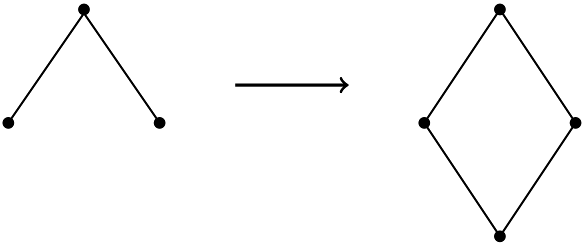
From frames to N-algebras



$$N(U) \cap V = N(U \cap V) \cap V$$

Duality

From frames to N-algebras



$$N(U) \cap V = N(U \cap V) \cap V \quad \checkmark$$

Duality

From N-algebras to frames Let $\mathfrak{A} = \langle A, \wedge, \vee, \to, \neg, 1\rangle$ be an N-algebra. A non-empty subset of A is called a **filter** if

- $a, b \in F$ implies $a \wedge b \in F$,
- $a \in F$ and $a \leq b$ imply $b \in F$.

Moreover, a filter F is called a *prime* filter if

- $a \vee b \in F$ implies $a \in F$ or $b \in F$.

Duality

From N-algebras to frames Let $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \neg, 1 \rangle$ be an N-algebra. A non-empty subset of A is called a **filter** if

- $a, b \in F$ implies $a \wedge b \in F$,
- $a \in F$ and $a \leq b$ imply $b \in F$.

Moreover, a filter F is called a *prime* filter if

- $a \vee b \in F$ implies $a \in F$ or $b \in F$.

Duality

From N-algebras to frames Let $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \neg, 1 \rangle$ be an N-algebra. A non-empty subset of A is called a **filter** if

- $a, b \in F$ implies $a \wedge b \in F$,
- $a \in F$ and $a \leq b$ imply $b \in F$.

Moreover, a filter F is called a *prime* filter if

- $a \vee b \in F$ implies $a \in F$ or $b \in F$.

The improper filter $F = A$ is a prime filter!

Duality

From N-algebras to frames

$W_{\mathfrak{A}} := \{F | F \text{ is a prime filter of } \mathfrak{A}\}$,

$F R_{\mathfrak{A}} F'$ if and only if $F \subseteq F'$,

$\mathcal{P} := \{\hat{a} | a \in A\}$, where $\hat{a} := \{F \in W_{\mathfrak{A}} | a \in F\}$,

$N_{\mathfrak{A}} : \mathcal{P} \to \mathcal{P}$ defined as $N_{\mathfrak{A}}(\hat{a}) := \widehat{(\neg a)}$

Duality

From N-algebras to frames

$W_{\mathfrak{A}} := \{F | F \text{ is a prime filter of } \mathfrak{A}\}$,

$FR_{\mathfrak{A}}F'$ if and only if $F \subseteq F'$,

$\mathcal{P} := \{\hat{a} | a \in A\}$, where $\hat{a} := \{F \in W_{\mathfrak{A}} | a \in F\}$,

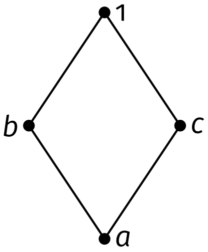
$N_{\mathfrak{A}} : \mathcal{P} \to \mathcal{P}$ defined as $N_{\mathfrak{A}}(\hat{a}) := \widehat{(\neg a)}$

Duality

From N-algebras to frames

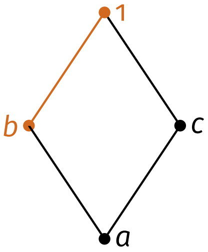
$$W_{\mathfrak{A}} := \{F \mid F \text{ is a prime filter of } \mathfrak{A}\},$$
$$F R_{\mathfrak{A}} F' \text{ if and only if } F \subseteq F',$$
$$\mathcal{P} := \{\hat{a} \mid a \in A\}, \text{ where } \hat{a} := \{F \in W_{\mathfrak{A}} \mid a \in F\},$$
$$N_{\mathfrak{A}} : \mathcal{P} \to \mathcal{P} \text{ defined as } N_{\mathfrak{A}}(\hat{a}) := \widehat{(\neg a)}$$

From N-algebras to frames

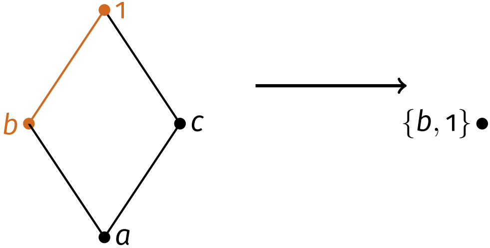


Duality

From N-algebras to frames

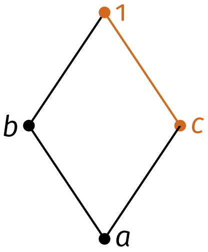


From N-algebras to frames



Duality

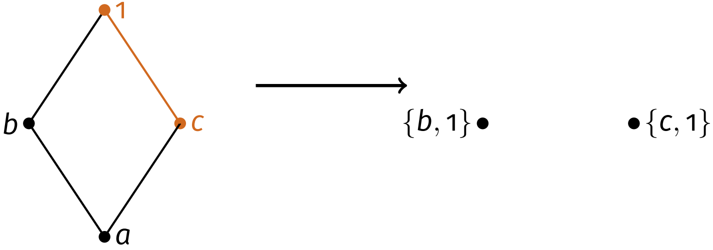
From N-algebras to frames



$\{b, 1\}$ •

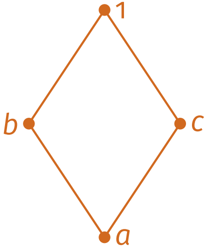
Duality

From N-algebras to frames

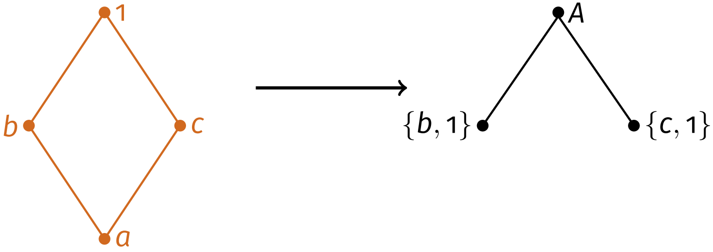


Duality

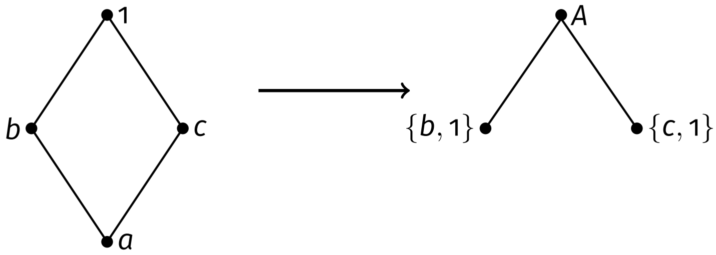
From N-algebras to frames



From N-algebras to frames

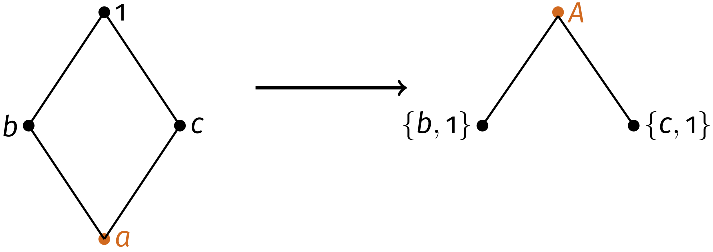


From N-algebras to frames



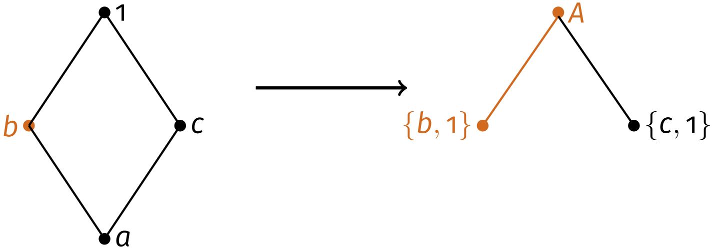
Duality

From N-algebras to frames



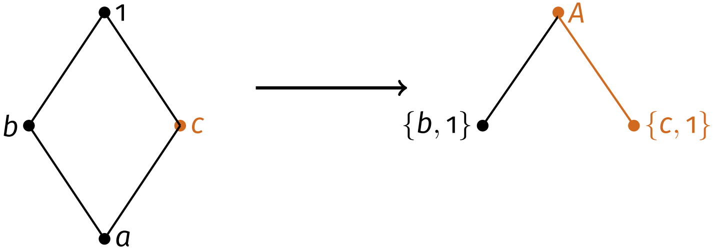
Duality

From N-algebras to frames



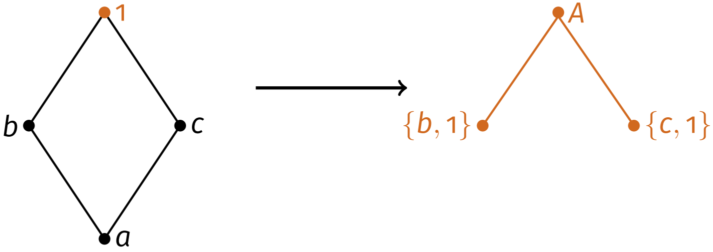
Duality

From N-algebras to frames



Duality

From N-algebras to frames



Duality

From N-algebras to frames

Does locality hold for $N_{\mathfrak{A}}$ defined as $N_{\mathfrak{A}}(\hat{a}) = \widehat{(\neg a)}$?

Duality

From N-algebras to frames

Does locality hold for $N_{\mathfrak{A}}$ defined as $N_{\mathfrak{A}}(\hat{a}) = \widehat{(\neg a)}$?

Duality

From N-algebras to frames

Does locality hold for $N_{\mathfrak{A}}$ defined as $N_{\mathfrak{A}}(\hat{a}) = \widehat{(\neg a)}$?

$N_{\mathfrak{A}}(\hat{a}) \cap \hat{b} = \{F | \neg a \in F \text{ and } b \in F\}$

From N-algebras to frames

Does locality hold for $N_{\mathfrak{A}}$ defined as $N_{\mathfrak{A}}(\hat{a}) = \widehat{(\neg a)}$?

$$\begin{aligned} N_{\mathfrak{A}}(\hat{a}) \cap \hat{b} &= \{F | \neg a \in F \text{ and } b \in F\} \\ &= \{F | \neg a \wedge b \in F\} \end{aligned}$$

Duality

From N-algebras to frames

Does locality hold for $N_{\mathfrak{A}}$ defined as $N_{\mathfrak{A}}(\hat{a}) = \widehat{(\neg a)}$?

$$\begin{aligned} N_{\mathfrak{A}}(\hat{a}) \cap \hat{b} &= \{F | \neg a \in F \text{ and } b \in F\} \\ &= \{F | \neg a \wedge b \in F\} \\ &= \{F | \neg(a \wedge b) \wedge b \in F\}, \text{ since } \neg a \wedge b = \neg(a \wedge b) \wedge b \end{aligned}$$

Duality

From N-algebras to frames

Does locality hold for $N_{\mathfrak{A}}$ defined as $N_{\mathfrak{A}}(\hat{a}) = \widehat{(\neg a)}$?

$$\begin{aligned} N_{\mathfrak{A}}(\hat{a}) \cap \hat{b} &= \{F | \neg a \in F \text{ and } b \in F\} \\ &= \{F | \neg a \wedge b \in F\} \\ &= \{F | \neg(a \wedge b) \wedge b \in F\}, \text{ since } \neg a \wedge b = \neg(a \wedge b) \wedge b \\ &= \{F | \neg(a \wedge b) \in F \text{ and } b \in F\} \end{aligned}$$

Duality

From N-algebras to frames

Does locality hold for $N_{\mathfrak{A}}$ defined as $N_{\mathfrak{A}}(\hat{a}) = \widehat{(\neg a)}$?

$$
\begin{aligned}
N_{\mathfrak{A}}(\hat{a}) \cap \hat{b} &= \{F | \neg a \in F \text{ and } b \in F\} \\
&= \{F | \neg a \wedge b \in F\} \\
&= \{F | \neg(a \wedge b) \wedge b \in F\}, \text{ since } \neg a \wedge b = \neg(a \wedge b) \wedge b \\
&= \{F | \neg(a \wedge b) \in F \text{ and } b \in F\} \\
&= N_{\mathfrak{A}}(\widehat{a \wedge b}) \cap \hat{b}
\end{aligned}
$$

Duality

From N-algebras to frames

Does locality hold for $N_{\mathfrak{A}}$ defined as $N_{\mathfrak{A}}(\hat{a}) = \widehat{(\neg a)}$?

$$
\begin{aligned}
N_{\mathfrak{A}}(\hat{a}) \cap \hat{b} &= \{F | \neg a \in F \text{ and } b \in F\} \\
&= \{F | \neg a \wedge b \in F\} \\
&= \{F | \neg(a \wedge b) \wedge b \in F\}, \text{ since } \neg a \wedge b = \neg(a \wedge b) \wedge b \\
&= \{F | \neg(a \wedge b) \in F \text{ and } b \in F\} \\
&= N_{\mathfrak{A}}(\widehat{a \wedge b}) \cap \hat{b} \\
&= N_{\mathfrak{A}}(\hat{a} \cap \hat{b}) \cap \hat{b}
\end{aligned}
$$

Duality

From N-algebras to frames

Does locality hold for $N_{\mathfrak{A}}$ defined as $N_{\mathfrak{A}}(\hat{a}) = \widehat{(\neg a)}$?

$$
\begin{aligned}
N_{\mathfrak{A}}(\hat{a}) \cap \hat{b} &= \{F | \neg a \in F \text{ and } b \in F\} \\
&= \{F | \neg a \wedge b \in F\} \\
&= \{F | \neg(a \wedge b) \wedge b \in F\}, \text{ since } \neg a \wedge b = \neg(a \wedge b) \wedge b \\
&= \{F | \neg(a \wedge b) \in F \text{ and } b \in F\} \\
&= N_{\mathfrak{A}}(\widehat{a \wedge b}) \cap \hat{b} \\
&= N_{\mathfrak{A}}(\hat{a} \cap \hat{b}) \cap \hat{b}
\end{aligned}
$$

Frame-based completeness

Let $\mathfrak{A}$ be an N-algebra. Then,

$$\langle \mathfrak{A}, v \rangle \models \varphi \Leftrightarrow \langle \mathfrak{A}_*, V \rangle \models \varphi,$$

where $V(p) = \widehat{v(p)}$.

Duality

Frame-based completeness

Let $\mathfrak{A}$ be an N-algebra. Then,

$$\langle \mathfrak{A}, v \rangle \vDash \varphi \Leftrightarrow \langle \mathfrak{A}_*, V \rangle \vDash \varphi,$$

where $V(p) = \widehat{v(p)}$.

Duality

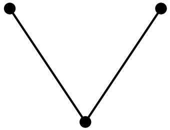
Frame-based completeness

Every extension $\mathbf{L}$ of the basic logic of a unary operator N is sound and complete with respect to the class of top descriptive frames $(\mathbf{V_L})_*$

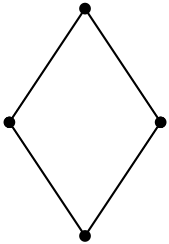
Duality

Top descriptive frames & Descriptive frames

Descriptive Frame $\mathfrak{F}$



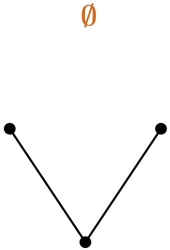
Top Descriptive Frame $\mathfrak{F}_T$



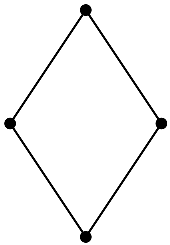
Duality

Top descriptive frames & Descriptive frames

Descriptive Frame $\mathfrak{F}$



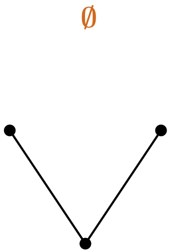
Top Descriptive Frame $\mathfrak{F}_T$



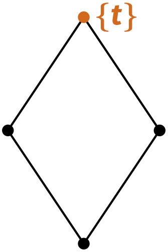
Duality

Top descriptive frames & Descriptive frames

Descriptive Frame $\mathfrak{F}$



Top Descriptive Frame $\mathfrak{F}_T$



Frame-based completeness

The basic logic of a unary operator N is sound and complete with respect to the class of descriptive frames.

Outline

Conclusions

What we have done

- Kripke-style Semantics
- N-algebras
- Algebraic Completeness
- (Top) Descriptive Frames
- Frame-based Completeness

Conclusions

What we have done

- Kripke-style Semantics
- N-algebras
- Algebraic Completeness
- (Top) Descriptive Frames
- Frame-based Completeness

Conclusions

What we have done

- Kripke-style Semantics
- N-algebras
- Algebraic Completeness
- (Top) Descriptive Frames
- Frame-based Completeness

Conclusions

What we have done

- Kripke-style Semantics
- N-algebras
- Algebraic Completeness
- (Top) Descriptive Frames
- Frame-based Completeness

Conclusions

What we have done

- Kripke-style Semantics
- N-algebras
- Algebraic Completeness
- (Top) Descriptive Frames
- Frame-based Completeness

Conclusions

What we have done

- Kripke-style Semantics
- N-algebras
- Algebraic Completeness
- (Top) Descriptive Frames
- Frame-based Completeness

Conclusions

What we want to do

Conclusions

What we want to do

Everything else!

Conclusions

What we want to do

- Order-topological Duality
- Universal Models
- Jankov-de Jongh Formulas