Admissibility of Multiple-Conclusion Rules in Logics with the Disjunction Property

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Introduction.

The disjunction property (DP) was observed by Gödel for intuitionistic logic (**Int**). Specifically, for any (propositional) formulas A, B if $A \vee B \in \mathbf{Int}$, then $A \in \mathbf{Int}$ or $B \in \mathbf{Int}$. It turned out [Wroński, 1973] that there are continuum many intermediate logics enjoying the DP. For such logics, if we have a basis R of admissible rules, the set

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Moreover, if R consists of rules of form $A \vee q/B \vee q$, where q is not occurring in A, B, and R is an independent basis of admissible rules, (1) is an independent basis for admissible m-rules.

For instance, m-rules $p \vee q/p, q$ and $p \wedge \neg p/\emptyset$ form an independent basis for m-rules admissible in the Medvedev logic.

The converse also holds: if an intermediate logic L enjoys the DP and $\Gamma_i/\Delta_i, i \in I$ is a basis of admissible m-rules,

$$\bigwedge \Gamma_i \vee q / \bigvee \Delta_i \vee q, i \in I, \tag{2}$$

where q does not occur in formulas from Γ_i, Δ_i, forms a basis of rules admissible for L (comp. [Rybakov, 1985]).
The same holds for Gabbay - de Jongh logics and modal logics with the DP
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The goal: to show how these relations between bases of admissible rules and m-rules can be extended to a very broad class of logics.

Outline.

- Basic Definitions
- Derivations in Multiple-Conclusion Logics
- Disjunction in Multiple-Conclusion Logics
- From Basis to m-Basis
- q-Extensions: from Basis to m-Basis preserving Independence
- From m-Basis to Basis

Logics.

We consider a countable set $\mathcal{P}$ of (propositional) variables and finite set of connectives $\mathcal{C}$. The (propositional) formulas are constructed in a usual way. Let Fm be a set of all propositional formulas.

A set of formulas $\mathsf{L} \subseteq \mathsf{Fm}$ closed under substitutions is a (propositional) *logic*.

An ordered pair Γ/Δ of finite sets of formulas $\Gamma, \Delta \subseteq \mathsf{Fm}$ is called an *m-rule*. Formulas from Γ are *premises*. Formulas from Δ are *alternatives*.

We use $\top$ and $\bot$ to denote empty sets of premises and alternatives. That is, we write $\top/\Delta$ and $\Gamma/\bot$ instead of $\emptyset/\Delta$ and $\Gamma/\emptyset$.

m-Rule System.

A *(finitary structural) m-rule system* is a set R of m-rules (writing $\Gamma \vdash_{\mathsf{R}} \Delta$ instead of $\Gamma/\Delta \in \mathsf{R}$) satisfying for all finite sets $\Gamma, \Gamma', \Delta, \Delta' \subseteq \mathsf{Fm}$ and formulas $A \in \mathsf{Fm}$:

1. $A \vdash_{\mathsf{R}} A$, (R)
2. if $\Gamma \vdash_{\mathsf{R}} \Delta$, then$\Gamma, \Gamma' \vdash_{\mathsf{R}} \Delta, \Delta'$, (M)
3. if $\Gamma, A \vdash_{\mathsf{R}} \Delta$ and $\Gamma \vdash_{\mathsf{R}} A, \Delta$, then $\Gamma \vdash_{\mathsf{R}} \Delta$, (T)
4. if $\Gamma \vdash_{\mathsf{R}} \Delta$, then $\sigma(\Gamma) \vdash_{\mathsf{R}} \sigma(\Delta)$ for each substitution σ. (S)

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(M) and (T) entail: for any finite sets $\Gamma_0, \Gamma_1, \Delta$ and any formula A

$$\text{if } \Gamma_0, A \vdash_{R} \Delta \text{ and } \Gamma_1 \vdash_{R} A, \Delta, \text{then } \Gamma_0, \Gamma_1 \vdash_{R} \Delta, \qquad (\mathrm{T}^*)$$

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Given an m-rule system R, the set $Th(\mathsf{R}) := \{A \in \mathsf{Fm} : \vdash_{\mathsf{R}} A\}$ is called a set of *theorems* of R. Clearly, $Th(\mathsf{R})$ is a logic. If L is a logic and $\mathsf{L} = Th(\mathsf{R})$, we say that m-rule system R *defines logic* L.

m-Rules.

A logic L is *consistent* if there is a formula A, such that $A \notin$ L. We call a logic *substitutionally consistent* (s-consistent for short) if there is a finite set of formulas $A_i, i < k$, such that for neither substitution σ, $\sigma(A_i) \in$ L for all $i < k$ (*not* L*-unifiable set of formulas*).

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If R is a set of m-rules, by [R] we denote the least m-rule system containing all m-rules from R. If R is an m-rule system and $\mathrm{R} = [\mathrm{R}_0]$ for some set of m-rules R_0, we say that R_0 is an *m-basis* of R.

If R is m-rule system and R_0 is a set of m-rules, we let

$$\mathrm{R} + \mathrm{R}_0 := [\mathrm{R} \cup \mathrm{R}_0].$$

The m-rules having a single conclusion are called the *rules*. By R° or by $\vdash^\circ_{\mathsf{R}}$ we denote a set of all rules from R, that is,

$$\mathsf{R}^\circ := \{\Gamma/A : \Gamma/A \in \mathsf{R}\}.$$

A set of rules satisfying the following conditions is called a *rule system*:

1. $A \vdash_{\mathsf{L}} A,$ (R)
2. if $\Gamma \vdash_{\mathsf{L}} A,$ then $\Gamma, \Gamma' \vdash_{\mathsf{L}} A,$ (M')
3. if $\Gamma, A \vdash_{\mathsf{L}} B$ and $\Gamma \vdash_{\mathsf{L}} A,$ then $\Gamma \vdash_{\mathsf{L}} B,$ (T')
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Since every theorem A corresponds to m-rule $\top/A$ and the latter is a rule, we have

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Since every theorem A corresponds to m-rule $\top/A$ and the latter is a rule, we have

$$Th(\mathsf{R}) = Th(\mathsf{R}^\circ).$$

If R is a rule system, we let $[\mathsf{R}]^\circ$ to be the least rule system containing all rules from R, and $\mathsf{R}_0 +^\circ \mathsf{R}_1 := [\mathsf{R}_0 \cup \mathsf{R}_1]^\circ$.

Admissible vis-à-vis Conservative m-Rules.

Let L be a logic.

An m-rule Γ/Δ is *admissible for* L if for every substitution σ, $\sigma(\Gamma) \subseteq \mathsf{L}$ entails $\sigma(\Delta) \cap \mathsf{L} \neq \emptyset$ (strict - [Iemhoff, 2015]).

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For the rules all three classes of rules systems coincide, while for m-rule systems all three classes may be distinct. For instance, for classical logic m-rule $p \vee q/p, q$ is strongly conservative, but not admissible. For intermediate logic of seven-element cyclic Heyting algebra this m-rule is conservative, but not strongly conservative.

m-Inference.

Let R be a set of m-rules and Γ be a finite (maybe empty) set of formulas. An *inference from* Γ *by* R ((R, Γ)-inference for short) is a labeled tree, defined by induction:

(-) A tree containing only a root labeled by $\top$ is an (R, Γ)-inference – a *trivial inference*;

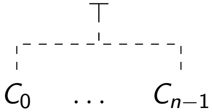
(a) If I is an (R, Γ)-inference, then a tree obtained from I by adjoining to an extendable (that is, not labeled by $\bot$) leaf, an immediate successor labeled by a formula from Γ is an (R, Γ)-inference;

(b) If I is an (R, Γ)-inference and $\mathfrak{n}$ is an extendable leaf, then a tree, obtained from I by adjoining to $\mathfrak{n}$ immediate successors $\mathfrak{n}_0, \ldots, \mathfrak{n}_{m-1}$ labeled by formulas $B_0, \ldots, B_{m-1}$, is (R, Γ)-inference, provided there is an instance Γ/Δ of a rule from R such that

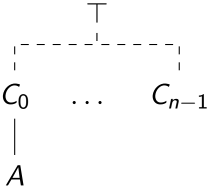
$$\Gamma \subseteq \lambda(\mathfrak{n}\uparrow) \text{ and } \Delta = \{B_0, \ldots, B_{m-1}\},$$

where $\lambda(\mathfrak{n}\uparrow)$ is a set of labels of all predecessors of $\mathfrak{n}$.

Given an (R, Γ)-inference



and an m-rule $\{A_i, i < n\}/\{B_j, j < m\} \in \mathsf{R}$, the trees



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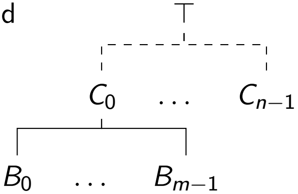

are (R, Γ)-inferences, provided $A \in \Gamma$ and all premises $A_i, i < n$ can be found on the branch between leaf C_0 and the root.

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Let R be a set of m-rules and r $:= \Gamma/\Delta$ be an m-rule. We say that r is *derivable from rules* R (in written R$\vdash$r) if there is a (R, Γ)-inference I such that $\lambda(lv(I)) \subseteq \Delta$, i.e. every leaf formula is an alternative from Δ.

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Theorem (comp. [Iemhoff, 2015])

Suppose R *is a set of rules. Then*

$$[\mathrm{R}] = \{\mathrm{r} : \mathrm{R} \vdash \mathrm{r}\}.$$

A proof can be done by induction on the length of inference.

Remark. The above definition of inference is slightly different from the definitions in [Shoesmith and Smiley, 2008] and [Iemhoff, 2015].

$\vee$-Introduction.

We assume that there is a formula $D(p, q)$ (a finite set of formulas $D_i(p, q), i < k$) on two variables that possesses properties of disjunction. To make the notation more suggestive we write $p \nabla q$.

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First, we need $\vee$-introduction rules:

$$\mathrm{dir} := p/p\nabla q \quad \text{and} \quad \mathrm{dil} := q/p\nabla q.$$

In the case of a set of formulas, the $\vee$-introduction rules are

$$p/D_i(p,q) \quad \text{and} \quad q/D_i(p,q), \text{ where } i < k.$$

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For $\vee$-elimination rules we use a multiple-alternative rule.

In the setting of natural deduction, $\vee$-elimination rule is

$$\dfrac{A \vee B \qquad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}$$

$\vee$-Elimination.

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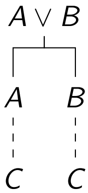
$\vee$-Elimination in the multiple-alternative setting:



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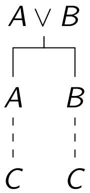

$$\text{de} \;:=\; p \nabla q / p, q.$$

Properties of ∇.

If R is a set of rules and $\mathrm{R}\vdash\mathrm{dil}, \mathrm{R}\vdash\mathrm{dir}, \mathrm{R}\vdash\mathrm{de}$, we say that ∇ is *m-disjunction* for R.

Given a set of m-rules, the m-disjunction for R is unique modulo $\vdash$.

Proposition

Let R *be a set m-rules. If $D(p,q)$ and $D'(p,q)$ are m-disjunctions for* R *then*

$$\mathrm{R}\vdash D(p,q)/D'(p,q) \textit{ and } \mathrm{R}\vdash D'(p,q)/D(p,q).$$

In the multiple-formula setting, if $\nabla = \{D_j(p,q), j < k\}$ and $\nabla' = \{D'_l(p,q), l < s\}$,

$$\mathrm{R}\vdash D_0, \ldots, D_{k-1}/D'_l, \text{ for every } l < s.$$

and

$$\mathrm{R}\vdash D'_0, \ldots, D'_{s-1}/D_j, \text{ for every } j < k.$$

Properties of ∇.

The following rules play special role:

$$\mathrm{dir} := p/p\nabla q$$
$$\mathrm{dil} := p/q\nabla p$$
$$\mathrm{dc} := p\nabla q/q\nabla p$$
$$\mathrm{dar} := (p\nabla q)\nabla r/p\nabla(q\nabla r)$$
$$\mathrm{dal} := p\nabla(q\nabla r)/(p\nabla q)\nabla r$$
$$\mathrm{dsd} := (p\nabla r)\nabla(q\nabla r)/(p\nabla q)\nabla r$$
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\end{aligned}$$

Let $\mathrm{D}^\circ := \{\mathrm{dir}, \mathrm{dil}, \mathrm{dc}, \mathrm{dar}, \mathrm{dal}, \mathrm{dsd}, \mathrm{did}\}$

Example

For **Int**, $D(p,q) = p \vee q$. For **S4**, $D(p,q) = \Box p \vee \Box q$. For **BCK**, $D(p,q) = (p \to q) \to q$ [Kowalski, 2014].

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Proposition

Let R *be a set m-rules and* ∇ *be an m-disjunction for* R*. Then the following holds:*

(a) $\mathsf{R} \vdash \mathsf{r}$ *for every* $\mathsf{r} \in \mathsf{D}^\circ$*;*

(b) *If* $R \vdash A \nabla B$ *and* $R \vdash A/C$ *and* $R \vdash B/C$*, then* $\mathsf{R} \vdash A \nabla B/C$ *for every* $A, B, C \in \mathsf{Fm}$*.*

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(b) *If* $R \vdash A \nabla B$ *and* $R \vdash A/C$ *and* $R \vdash B/C$*, then* $\mathsf{R} \vdash A \nabla B / C$ *for every* $A, B, C \in \mathsf{Fm}$*.*

Rule de is not the same as the disjunction property: for **every** intermediate logic L, $\vee$ is an m-disjunction for a rule system consisting of axiom schemas of L (treated as rules) and modus ponens, regardless whether L enjoys the DP. m-Rule de is "equivalent" to the DP only when it is **admissible** for $Th(\mathsf{R})$.

From basis to m-basis.

Let L be a logic. Then the set of all m-rules admissible in L forms an m-rule system $Adm(\mathsf{L})$, and the set of all admissible in L rules forms a rule system $Adm^{\circ}(\mathsf{L})$.

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If $\Gamma/\Delta \in Adm(\mathsf{L})$ and Γ is not L-unifiable, m-rule Γ/Δ is called *passive*. Clearly, if Γ/Δ is a passive m-rule, then the rule $\Gamma/\bot$ is passive too.

Proposition

Let L *be a logic,* $\Gamma \subseteq \mathsf{L}$ *be a not* L*-unifiable set of formulas, and let* R° *be a basis of admissible rules. Then any passive m-rule can be derived from m-rules*

$$\mathsf{R}^{\circ} \textit{ and } \Gamma/\bot.$$

Let L be a logic. Then the set of all m-rules admissible in L forms an m-rule system $Adm(\mathrm{L})$, and the set of all admissible in L rules forms a rule system $Adm^{\circ}(\mathrm{L})$.

If $\Gamma/\Delta \in Adm(\mathrm{L})$ and Γ is not L-unifiable, m-rule Γ/Δ is called *passive*. Clearly, if Γ/Δ is a passive m-rule, then the rule $\Gamma/\bot$ is passive too.

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For instance, in **S4.3** all passive m-rules can be derived from $\Diamond p \wedge \Diamond \neg p/\bot$ (comp. [Rybakov et al., 2000]) or from $p \wedge \neg p/\bot$.

From basis to m-basis.

Suppose sets $\Gamma := \{A_i, i < k\}$ and $\Delta := \{B_j, j < m\}$ are not L-unifiable. We derive $\Delta/\bot$ from $\Gamma/\bot$, using $\mid\!\sim \Delta/A_i$, $i < k$:

$$\begin{array}{l} B_1 \\ \mid \\ \dots \\ \mid \ \text{premises} \\ B_{m-1} \\ \mid \\ A_1 \\ \mid \\ \dots \\ \mid \ \text{by } \Delta/A_i \\ A_{k-1} \\ \mid \ \text{by } \Gamma/\bot \\ \bot \end{array}$$

Theorem

Let L *be a logic,* ∇ *be m-disjunction for* $Adm(\mathsf{L})$ *and* R° *be a basis of* $Adm^\circ(\mathsf{L})$. *Then,*

(a) *If* L *is s-consistent and* Γ *is any finite not* L*-unifiable set of formulas,*

$$\mathsf{R}^\circ, \qquad \mathsf{de}, \qquad \Gamma/\bot$$

is a basis for $Adm(\mathsf{L})$*;*

(b) *If* L *is not s-consistent,*

$$\mathsf{R}^\circ, \qquad \mathsf{de}$$

is a basis for $Adm(\mathsf{L})$*;*

From basis to m-basis: some applications.

Corollary

Let L *be a logic and* $Adm(\mathsf{L})$ *has an m-disjunction. Then*

(a) $Adm^{\circ}(\mathsf{L})$ *is decidable if and only if* $Adm(\mathsf{L})$ *is decidable;*

(b) *If* $Adm^{\circ}(\mathsf{L})$ *has a finite base, then* $Adm(\mathsf{L})$ *has a finite base;*

(c) *If* $Adm^{\circ}(\mathsf{L})$ *has a finite basis relative to* L, *then* $Adm(\mathsf{L})$ *has a finite basis relative to* L.

From basis to m-basis: some applications.

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(c) *If* $Adm^{\circ}(\mathsf{L})$ *has a finite basis relative to* L*, then* $Adm(\mathsf{L})$ *has a finite basis relative to* L*.*

Due to $Adm^{\circ}(\mathsf{L})$ being decidable for every $\mathsf{L} \in \{\mathbf{K4}, \mathbf{K4}.\mathbf{1}, \mathbf{S4}, \mathbf{S4}.\mathbf{1}, \mathbf{Grz}, \mathbf{Int}, , \mathbf{D_n}, n \geq 1\}$ ([Rybakov, 1997]), we have that for all these logics $Adm(\mathsf{L})$ is decidable too ([Rybakov, 1997]).

Due to $Adm^{\circ}(\mathbf{Int}^{+})$ and $Adm^{\circ}(\mathbf{Jhn})$ being decidable ([Odintsov and Rybakov, 2013]), $Adm(\mathbf{Int}^{+})$ and $Adm(\mathbf{Jhn})$ is decidable too.

From basis to m-basis: some applications.

Visser's rules form a basis of $Adm^{\circ}(\mathbf{Int})$ ([Iemhoff, 2001]), hence Visser's rules together with de and $p \wedge \neg p/\bot$ is a basis of $Adm(\mathbf{Int})$.

m-Rule $p \wedge \neg p/\bot$ is not derivable from Visser's rules. In general, m-rule of form $\Gamma/\bot$ is not derivable from any set of rules that have non-empty set of alternatives.

q-Extension.

Let A be a formula and q be a variable not occurring in A. Then formula $(A \nabla q)$ is a *q-extension* of A. We let $\top^q := \top$ and $\bot^q := q$.

Definition

q-Extension of an m-rule $\mathrm{r} := \Gamma/\Delta$ is a rule r^q obtained from r by replacing every premise from Γ by its q-extension, and by replacing alternatives Δ by q-extension of the formula obtained by connecting all formulas from Δ by ∇.

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For instance, q-extension of m-rule $A_0, A_1/B_0, B_1$ is rule

$$A_0 \nabla q, A_1 \nabla q / B_0 \nabla B_1 \nabla q,$$

where q has no occurrences in A_0, A_1, B_0, B_1. And q-extensions of rules $A/\bot$ and $\top/B$ are respectfully rules

$$A \nabla q/q \text{ and } \top/B \nabla q.$$

q-Extension.

Proposition

Let R *be a set of m-rules and* ∇ *be an m-disjunction for* R*. For any rule* r*,*

$$\mathsf{R}, \mathsf{r}^q \vdash \mathsf{r}.$$

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Let $\mathsf{r} := A/B$.

$$\begin{array}{c} \top \\ \big| \; \text{premise} \\ A \\ \big| \; \text{dir} \\ A \nabla B \\ \big| \; \mathsf{r}^q \\ B \nabla B \\ \big| \; \text{de} \\ B \end{array}$$

q-Extension.

A set of m-rules R (rules R°) is *independent* if neither m-rule $\mathsf{r} \in \mathsf{R}$ (neither rule $\mathsf{r} \in \mathsf{R}^\circ$) can be derived from $\mathsf{R} \setminus \{\mathsf{r}\}$ ($\circ$-derived from $\mathsf{R}^\circ \setminus \{\mathsf{r}\}$).

Theorem

Let L *be a logic,* ∇ *be m-disjunction for* $Adm(\mathsf{L})$ *and* R° *be an* **independent basis** *of* $Adm^\circ(\mathsf{L})$ *consisting of q-extended rules. Then,*

(a) *If* L *is s-consistent and* Γ *is any finite not* L*-unifiable set of formulas,*

$$\mathsf{R}^\circ, \qquad \mathsf{de}, \qquad \Gamma/\bot$$

is an **independent basis** *for* $Adm(\mathsf{L})$*;*

(b) *If* L *is not s-consistent,*

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q-Extension.

For instance [Jeřábek, 2008], the following is an independent basis of admissible rules for **Int**:

$$\pi_n := ((\bigvee_{i<n} p_i \to p) \to \bigvee_{i<n} p_i) \vee q / \bigvee_{i<n} (p \wedge \bigwedge_{j \neq i} p_j \to p_i) \vee q.$$

Thus, $\{\pi_n; n > 1, \text{de}, p \wedge \neg p/\bot\}$ is an independent m-basis for **Int**. In [Jeřábek, 2008] Jeřábek proves that the following is independent m-basis ($n \neq 1$)

$$\Pi_n := ((\bigvee_{i<n} p_i \to p) \to \bigvee_{i<n} p_i)/\{p \wedge \bigwedge_{j \neq i} p_j \to p_i; i < n\}.$$

As we see, in the Jeřábek's basis m-rule de is not needed, because the conclusions of rules π_n are already "decomposed".

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As we see, in the Jeřábek's basis m-rule de is not needed, because the conclusions of rules π_n are already "decomposed".

Problem

Is there an intermediate logic with the DP *that has an independent basis of admissible rules, but does not have an independent m-basis?*

From m-basis to basis.

Theorem

Let L *be a logic and* R *be a m-basis of* $Adm(\mathsf{L})$. *Then* $\mathsf{D}^\circ \cup \mathsf{R}^\nabla$ *is basis of* $Adm^\circ(\mathsf{L})$.

From m-basis to basis.

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From the above theorem and absence of finite basis for admissible in **Int** rules [Rybakov, 1984], we have [Rybakov, 1985]

Corollary

There is no finite m-basis for **Int**.

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For example, we can take the m-basis $\{\mathrm{r}_n, \mathrm{de}, p \wedge \neg p/\bot\}$ of Gabbay-de Jongh logic BB_n constructed in [Goudsmit, 2015, Theorem 5.36] and convert it into a basis $\{\mathrm{r}_n^q\}$.

$$\mathrm{r}_n := \quad (\vee_{i=1}^n (p_i \to p) \to \vee_{j=1}^n p_j) / \vee_{j=1}^n ((\vee_{i=1}^n p_i \to p) \to p_j),$$
$$\mathrm{r}_n^q := q \vee (\vee_{i=1}^n (p_i \to p) \to \vee_{j=1}^n p_j) / \vee_{j=1}^n ((\vee_{i=1}^n p_i \to p) \to p_j) \vee q.$$

The proof is based on the following observation:

$\mathrm{r} := A_0, \dots, A_{n-1}/B_0, B_1$	$\mathrm{r}' := A_0, \dots, A_{n-1}/B_0 \nabla B_1$
$\mathrm{r}_i := B_i/C; i < 2$	$\mathrm{r}_i^q := B_i \vee q/C \vee q; i < 2$

$$\begin{tikzcd}
& A_0,\dots,A_{n-1} \arrow[dl, no head, "\mathrm{r}"'] \arrow[dr, no head] & \\
B_0 \arrow[d, no head, "\mathrm{r}_1"'] & & B_1 \arrow[d, no head, "\mathrm{r}_2"] \\
C & & C
\end{tikzcd}$$

$$\begin{tikzcd}
A_0,\dots,A_{n-1} \arrow[d, no head, "\mathrm{r}'"] \\
B_0 \nabla B_1 \arrow[d, no head, "\mathrm{r}_0^q"] \\
C \nabla B_1 \arrow[d, no head, "\mathrm{r}_1^q"] \\
C \nabla C \arrow[d, no head] \\
C
\end{tikzcd}$$

Final Remarks.

So far we were considering transitions from a given m-basis to a basis, and vise versa. But we can try to use a set of conservative m-rules in order to define all admissible rules. For instance, despite the fact [Rybakov, 1995] that there are tabular intermediate and modal logics whose admissible rules have no finite basis, the following holds.

Proposition

Let L *be a tabular extension of* **Int** *(or* **K4***) logic. Then there is a finite set of m-rules* R *such that* R *is conservative relative to* L *and every admissible in* L *rule is derivable from* R*.*

Final Remarks.

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Problem

Is there a finite set R *of m-rules that is conservative relative to* **Int** *and such that every admissible in* **Int** *rule is derivable from* R*?*

Thank You

Thank You

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