Duality for sheaf representations of distributive-lattice-ordered algebras

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ToLo 4
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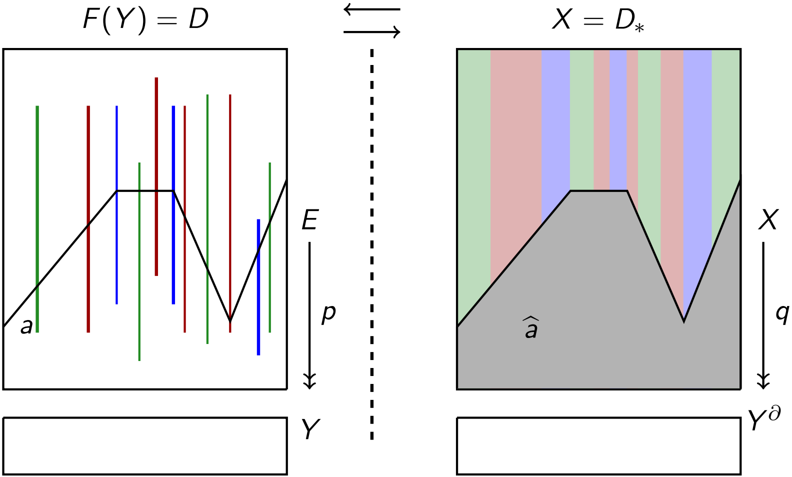
This talk in a picture



A different kind of picture

Co-authors: Mai Gehrke and Vincenzo Marra

Definition of étale space

- Let $\mathcal{V}$ be a variety of algebras, (Y, ρ) a topological space.
- Let $(A_y)_{y \in Y}$ be a Y-indexed family of $\mathcal{V}$-algebras.
- Let $E := \bigsqcup_{y \in Y} A_y$, with $p : E \twoheadrightarrow Y$ the natural surjection.
- Suppose τ is a topology on E such that
  - $p : (E, \tau) \twoheadrightarrow (Y, \rho)$ is a local homeomorphism, and
  - for each n-ary f the partial map $f^E : E^n \nrightarrow E$ is continuous.
- $p : (E, \tau) \twoheadrightarrow (Y, \rho)$ is called an étale space of $\mathcal{V}$-algebras.

Sheaf from an étale space

- Let $p : (E, \tau) \twoheadrightarrow (Y, \rho)$ be an étale space of $\mathcal{V}$-algebras.
- For any $U \in \rho$, write $F(U)$ for the set of local sections over U:

$$F(U) := \{s : U \to E \text{ continuous s.t. } p \circ s = \mathbf{id}_U\}.$$

- Note: $F(U)$ is a $\mathcal{V}$-algebra (it is a subalgebra of $\prod_{y \in U} A_y$).
- If $U \subseteq V$, there is a natural restriction map $F(V) \to F(U)$.
- F is called the sheaf associated with p.

Definition of sheaf

- In general, a sheaf F on Y consists of the data:
  - For each open U, a $\mathcal{V}$-algebra $F(U)$ ("local sections");
  - For each open $U \subseteq V$, a $\mathcal{V}$-homomorphism $()|_U : F(V) \to F(U)$ ("restriction maps");

  such that the appropriate diagrams commute, and such that it satisfies the following patching property:
  - For any open cover $(U_i)_{i \in I}$ of an open set U, if $(s_i)_{i \in I}$ is a "compatible family" of local sections, i.e., $s_i \in F(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then $\exists! s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.
- $F(Y)$ is called the algebra of global sections of the sheaf F.

Sheaves vs. étale spaces

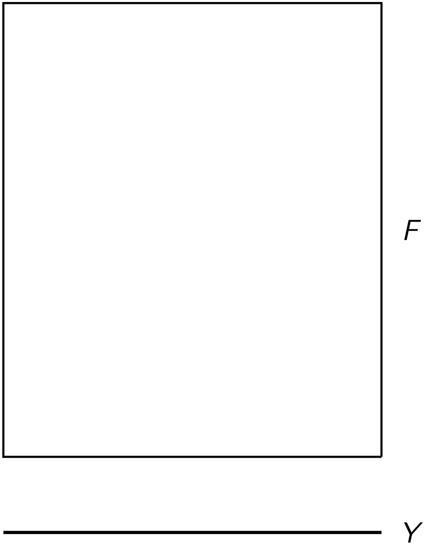
Fact

Any sheaf arises from an étale space, and vice versa.

Boolean product representation

- A Boolean space is a compact Hausdorff space which has a basis of clopen sets.
- A Boolean product representation of an algebra A is a sheaf F on a Boolean space Y such that A is isomorphic to the algebra of global sections of F.
- Equivalent: a subdirect embedding $A \rightarrowtail \prod_{y \in Y} A_y$ satisfying:
  - (Open equalizers) For any $a, b \in A$, the equalizer $\|a = b\| := \{y \in Y \mid a_y = b_y\}$ is open;
  - (Patch) For K clopen in Y, $a, b \in A$, there exists $c \in A$ such that $a|_K = c|_K$ and $b|_{K^c} = c|_{K^c}$.

Boolean product, pictorially



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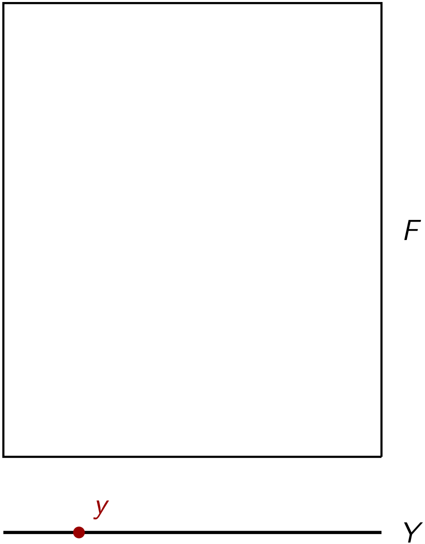

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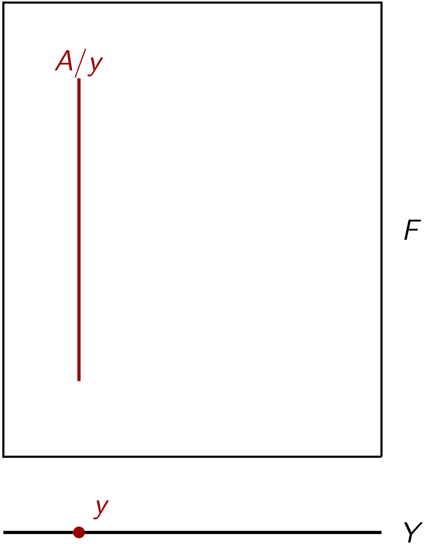

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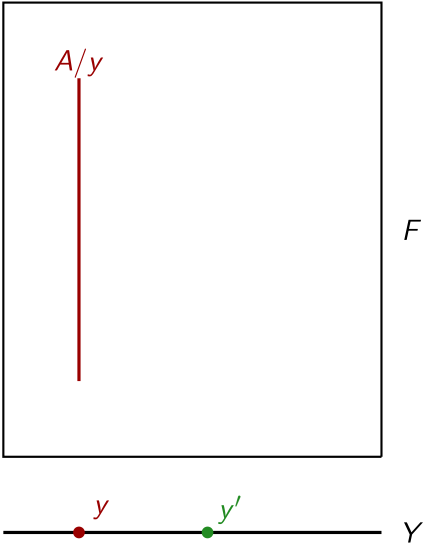

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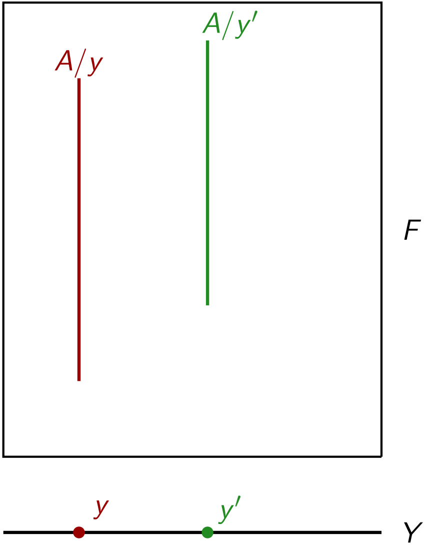

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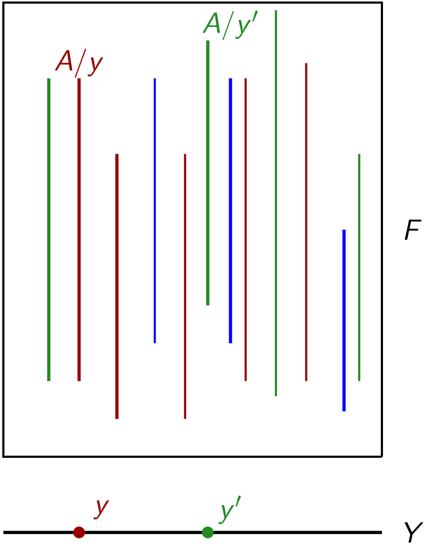

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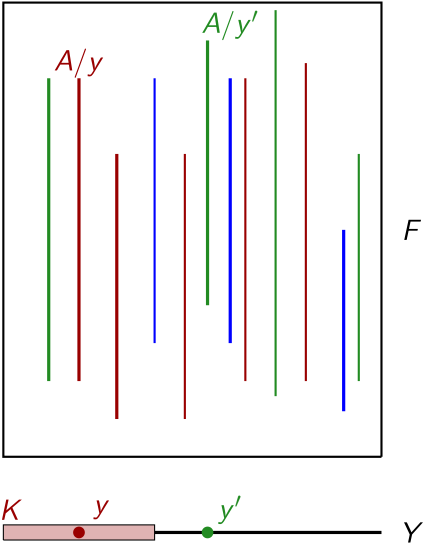

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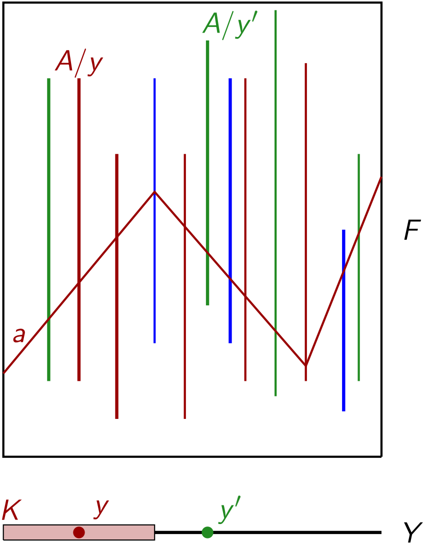

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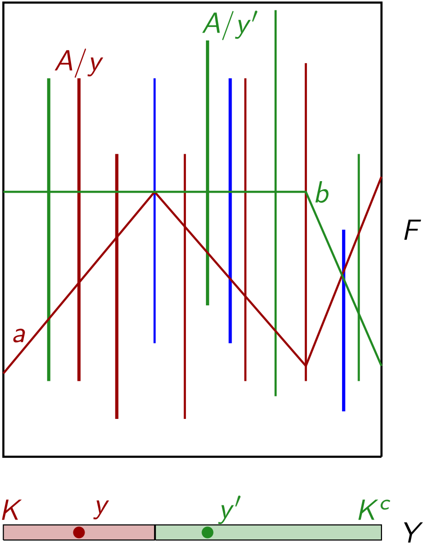

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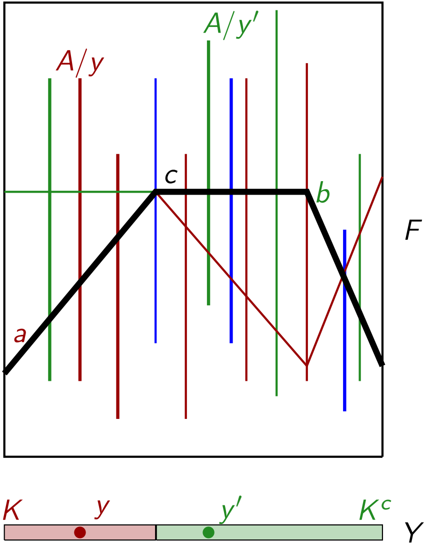


Lattices of congruences

Theorem (Comer 1971, Burris & Werner 1980)

The Boolean product representations of A are in a natural one-to-one correspondence with relatively complemented distributive lattices of permuting congruences on A.

Duals of Boolean products

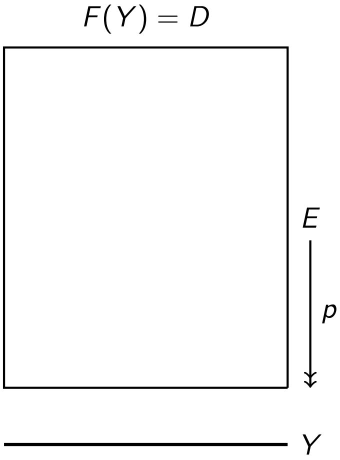
- Let D be a distributive lattice.

Theorem (Gehrke 1991)

Boolean product representations $D \rightarrowtail \prod_{y \in Y} D_y$ are in a natural one-to-one correspondence with Boolean sum decompositions of the Stone dual space X of D into the Stone dual spaces $(X_y)_{y \in Y}$ of the lattices $(D_y)_{y \in Y}$.

- Also see [Hansoul & Vrancken-Mawet 1984] for a version for the Priestley dual spaces.

Dual characterization, pictorially



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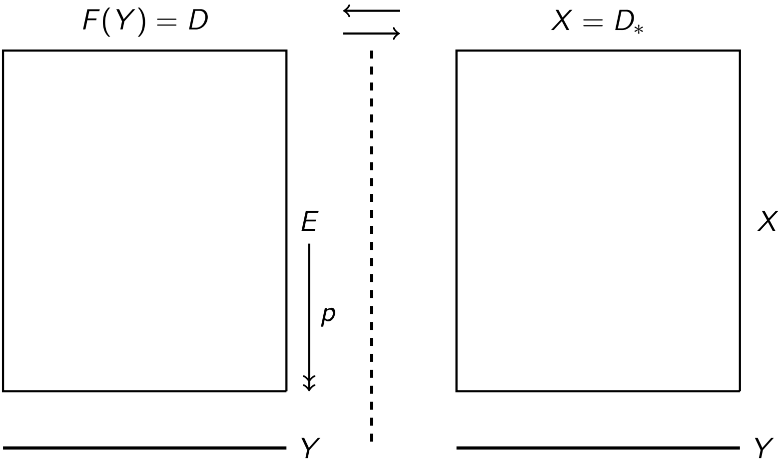

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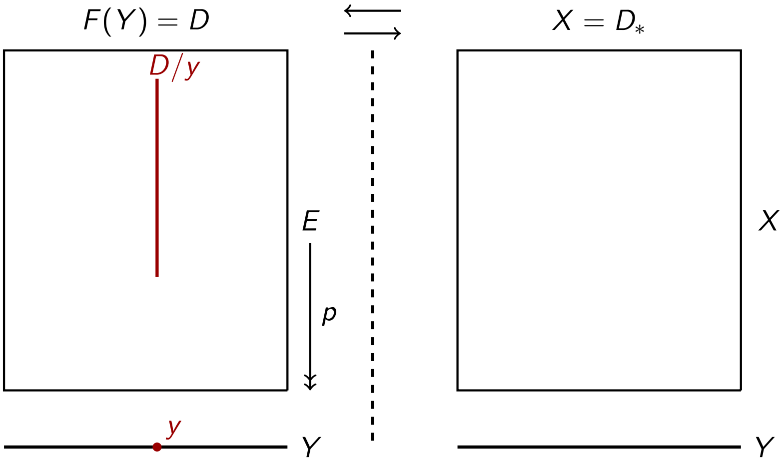

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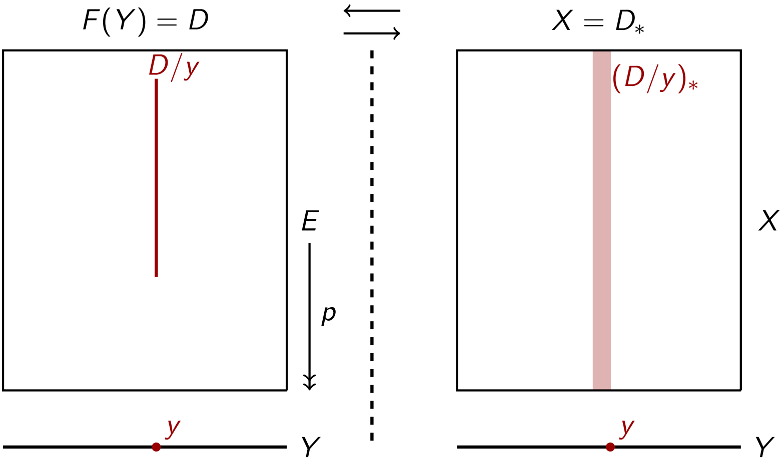

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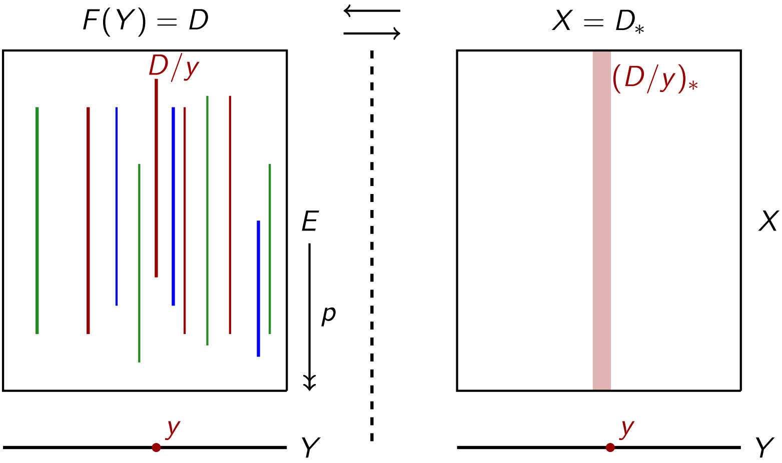

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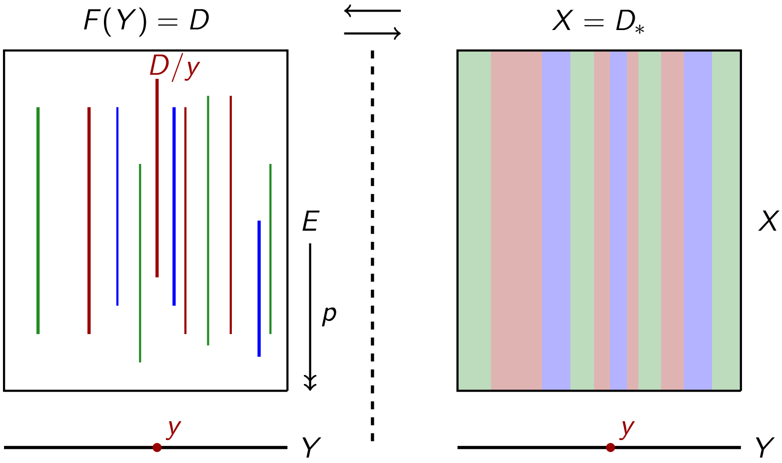

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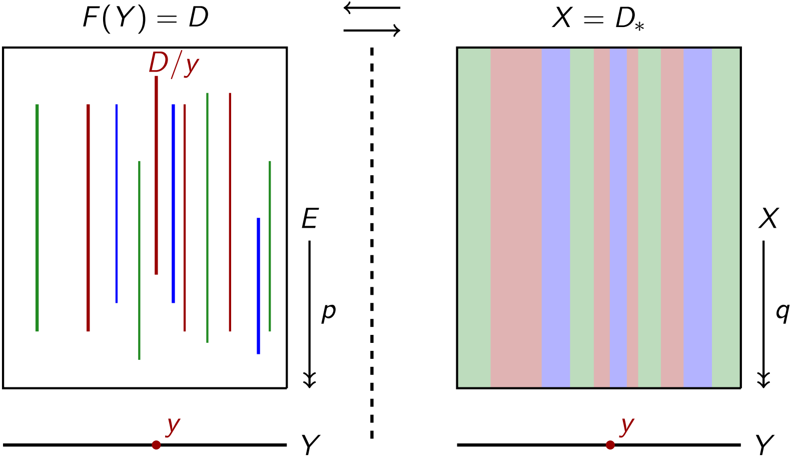

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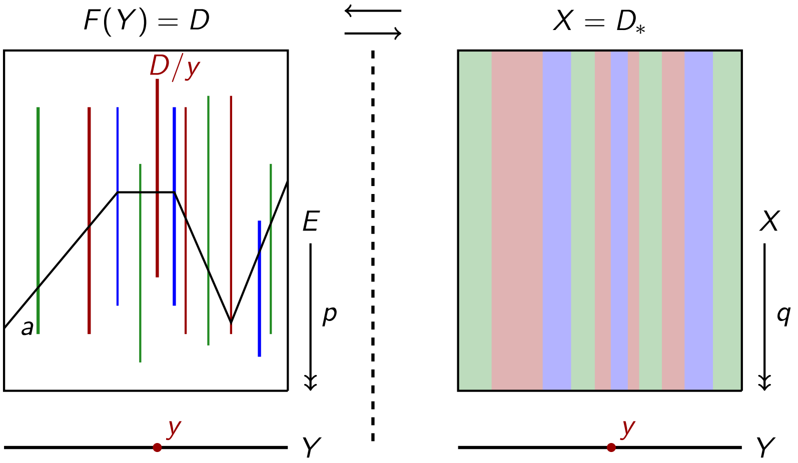

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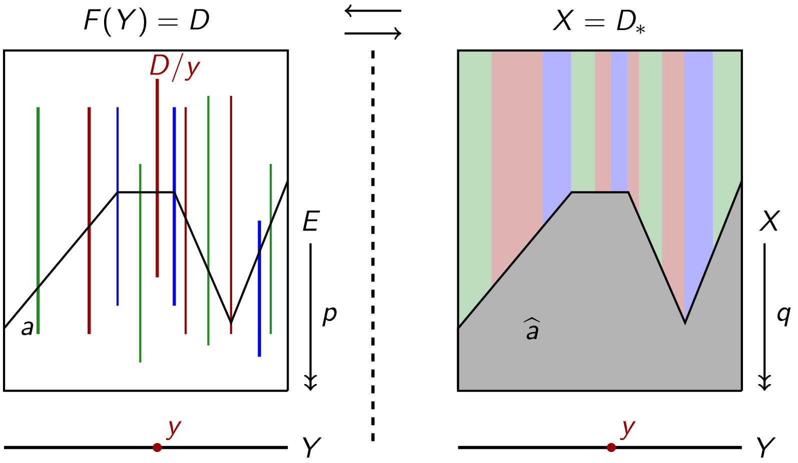

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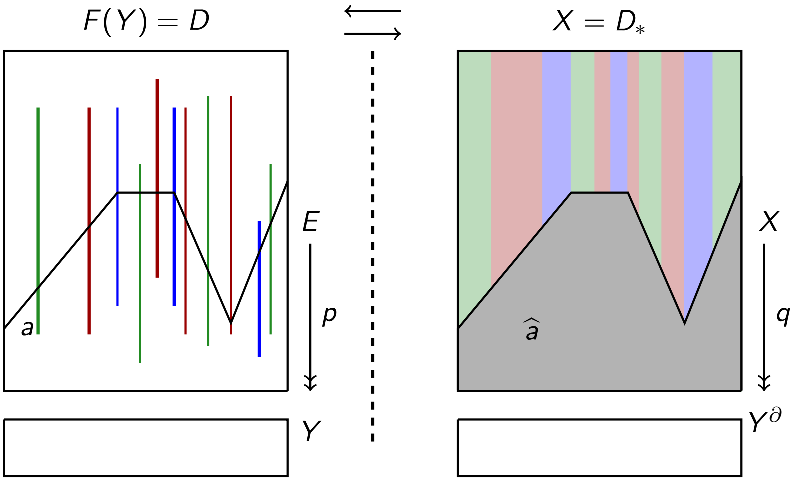

Question

- What if Y is no longer a Boolean space?

Stably compact spaces

- "Generalisation of compact Hausdorff to T_0-setting"

Definition

Stably compact space =

- T_0,
- Sober,
- Locally compact,
- Intersection of compact-saturated is compact.

Co-compact dual and patch topology

- For any topological space (Y, ρ), define its co-compact dual

$$\rho^{\partial} := \langle U \subseteq Y \mid Y \setminus U \text{ is compact-saturated in } \rho \rangle_{\mathrm{top}}$$

Co-compact dual and patch topology

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- Fact: (Y, ρ^p) is a compact Hausdorff space.

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- Fact: $\leq$ is a closed subspace of $(Y \times Y, \rho^p \times \rho^p)$.

Co-compact dual and patch topology

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- Fact: (Y, ρ^p) is a compact Hausdorff space.
- Let $y \leq y' \iff y' \in \overline{\{y\}}$, the specialization order of ρ.
- Fact: $\leq$ is a closed subspace of $(Y \times Y, \rho^p \times \rho^p)$.
- So $(Y, \rho^p, \leq)$ is a compact ordered space (Nachbin 1965).

Compact ordered spaces

- Conversely, given a compact ordered space $(Y, \pi, \leq)$, denote by $\pi^{\downarrow}$ the topology of open down-sets.
- Then $(Y, \pi^{\downarrow})$ is a stably compact space, and $(\pi^{\downarrow})^{\partial} = \pi^{\uparrow}$.

Compact ordered spaces

- Conversely, given a compact ordered space $(Y, \pi, \leq)$, denote by $\pi^{\downarrow}$ the topology of open down-sets.
- Then $(Y, \pi^{\downarrow})$ is a stably compact space, and $(\pi^{\downarrow})^{\partial} = \pi^{\uparrow}$.

Fact

The categories of stably compact spaces and compact ordered spaces are isomorphic.

Representing stably compact spaces

- Let (Y, ρ) be a stably compact space
  and let B be a lattice basis of ρ-open sets.
- For U, V in B, define

$$U \prec V \iff \exists K \text{ compact such that } U \subseteq K \subseteq V.$$

- The pair $(B, \prec)$ forms a join-strong proximity lattice
  which determines the space (Y, ρ).

Basis and co-compact dual

Lemma (Dual basis)

Let (Y, ρ) be a stably compact space and B a lattice basis for ρ. For any open set W in ρ^∂, we have

$$W = \bigcup \{V \in \rho^\partial \mid \exists U \in B : V \subseteq U^c \subseteq W\}.$$

Flasque sheaves

- Let F be a sheaf on a space Y with $F(Y) \neq \emptyset$.
- If $K \subseteq Y$ is a clopen set, then the restriction map $F(Y) \to F(K)$ is surjective.
- Thus, if Y is a Boolean space, then there is a basis B for which all restriction maps are surjective.
- If (Y, ρ) is a stably compact space with lattice basis of opens B, then F is called B-flasque if $F(Y) \to F(U)$ is surjective for all $U \in B$.

Flasque sheaves and congruences

- Let B be a lattice basis for a stably compact space (Y, ρ).
- For a B-flasque sheaf F, define

$$\theta_F : B^{\mathsf{op}} \to \mathrm{Con}(F(Y))$$

by

$$U \in B \mapsto \ker(F(Y) \twoheadrightarrow F(U)).$$

- The map θ_F is a homomorphism and maps into the permuting congruences on $F(Y)$.
- In case B is a Boolean algebra, we get a Boolean subalgebra of factor congruences (cf. Comer's result).

Lifting to frames

- The homomorphism $\theta_F : B^{\mathrm{op}} \to \mathrm{Con}(F(Y))$ can be lifted:
- Define $\widetilde{\theta_F} : \mathcal{O}(Y^\partial) \to F$ by

$$\widetilde{\theta_F}(W) := \bigvee \{\theta_F(U) \mid U \in B,\ U^c \subseteq W\}.$$

- Then $\widetilde{\theta_F}$ is a frame homomorphism.

  (Here we use the dual basis lemma of stably compact spaces.)

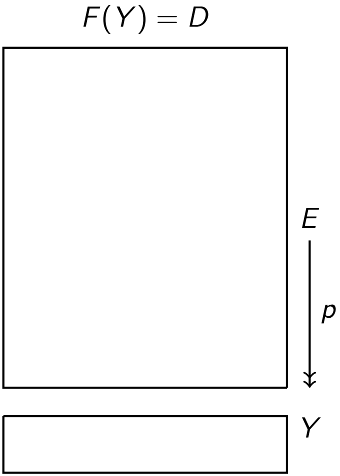
Frame homomorphism gives continuous map

- Since $F(Y)$ is a distributive lattice, there is an isomorphism
$$\psi : \mathrm{Con}(F(Y)) \to \mathcal{O}(X),$$
where X is the Priestley dual space of $F(Y)$.
- Therefore, there is a frame homomorphism
$$\psi \circ \widetilde{\theta_F} : \mathcal{O}(Y^\partial) \to \mathcal{O}(X),$$
which corresponds to a continuous map $q_F : X \to Y^\partial$.

Proposition

- *The dual of the stalk of F at $y \in Y$ is $q_F^{-1}(\downarrow y)$.*
- *For $U \in B$, the dual of the lattice $F(U)$ is $\overline{q_F^{-1}(U)}$.*

Sheaves and decompositions in a picture



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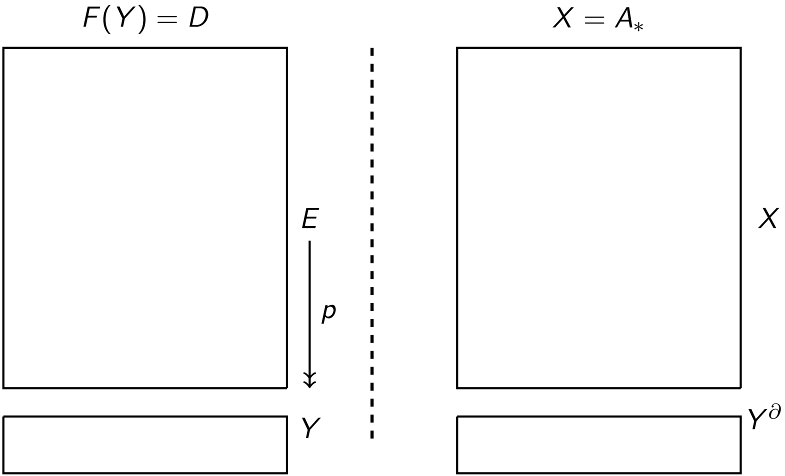

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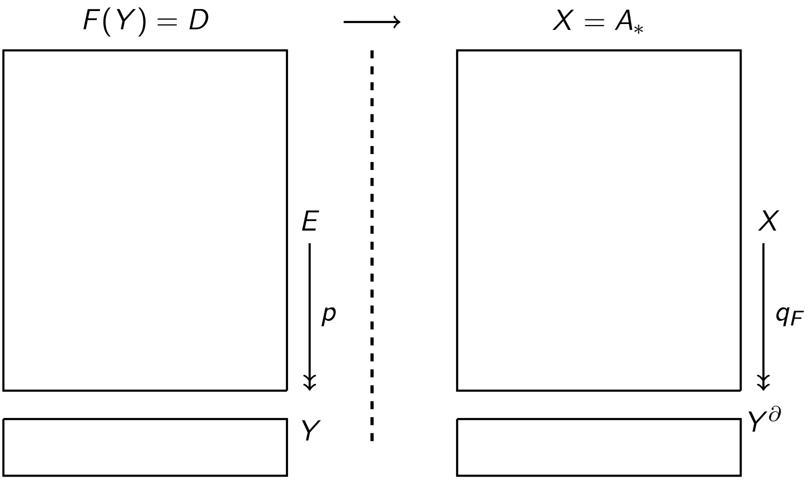

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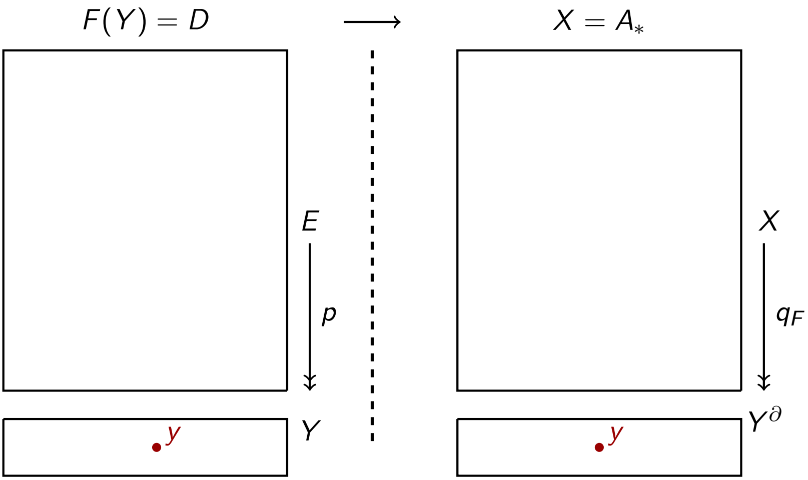

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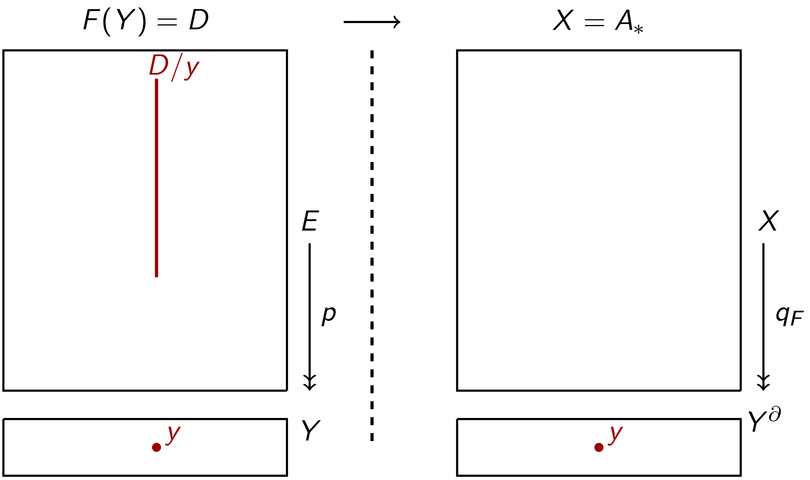
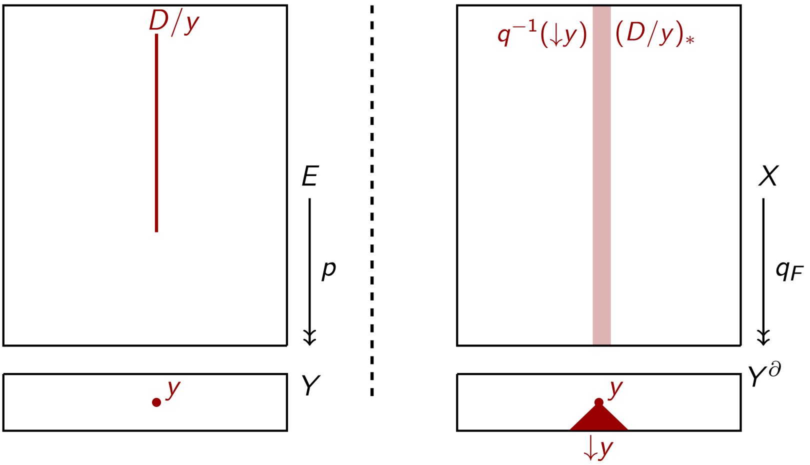


Sheaves and decompositions in a picture

$F(Y) = D \longrightarrow X = A_*$



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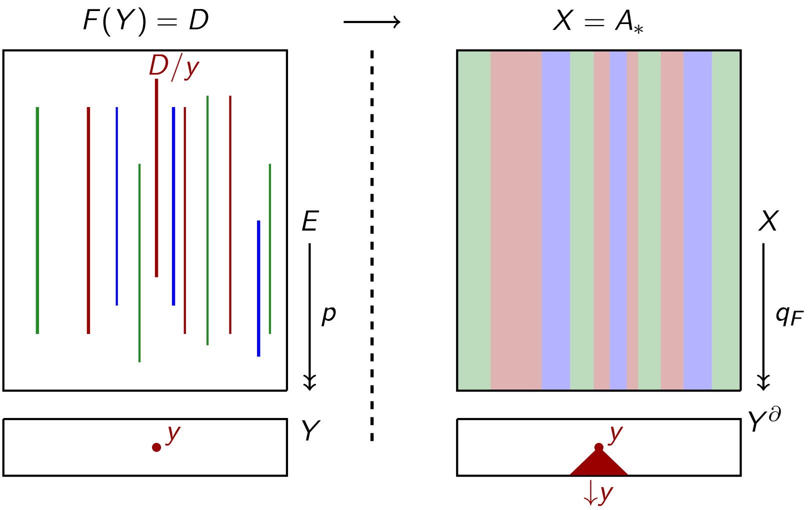


From decomposition to étale space

- Let D be a distributive lattice with dual space X.
- For $q : X \to Y^\partial$ continuous to a stably compact space Y, we may define an étale space $p : E \to Y$ such that $p^{-1}(y)$ is the lattice dual to the closed subspace $q^{-1}(\downarrow y) \subseteq X$.
- Write F for the sheaf associated to $p : E \to Y$.
- There is a natural embedding

$$\begin{aligned} \eta : D &\to F(Y) \\ a &\mapsto (y \in Y \mapsto \widehat{a} \cap q^{-1}(\downarrow y)). \end{aligned}$$

Characterizing sheaf representations dually

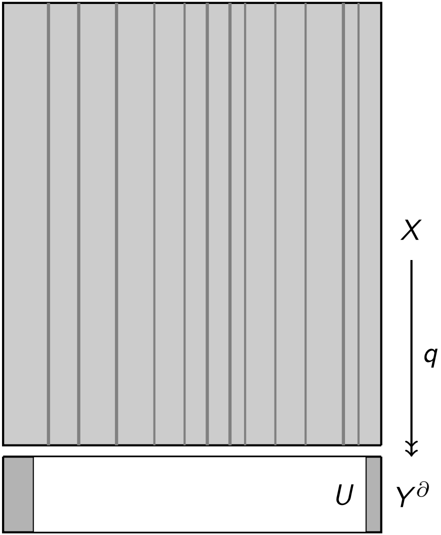
- Thus, any continuous map $q : X \to Y^\partial$ yields a sheaf F such that D embeds into the lattice of global sections $F(Y)$ of Y.
- Question: When is the embedding η an isomorphism?

Lemma (Dual characterization)

For any open set $U \subseteq Y$, the following are equivalent:

1. *Each local section $s \in F(U)$ is equal to $\eta(a)|_U$ for some $a \in D$;*
2. *If $U = \bigcup_{i \in I} U_i$ and $(K_i)_{i \in I}$ is a family of clopen downsets in X such that $K_i \cap q^{-1}(U_i \cap U_j) = K_j \cap q^{-1}(U_i \cap U_j)$, $(i, j \in I)$ then there is a clopen downset K in X such that $K \cap q^{-1}(U) = \bigcup_{i \in I}(K_i \cap q^{-1}(U_i))$. ("Property (P_U)")*

Property (P_U) in a picture



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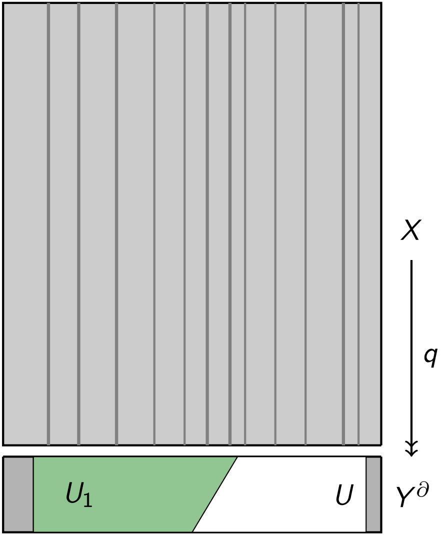

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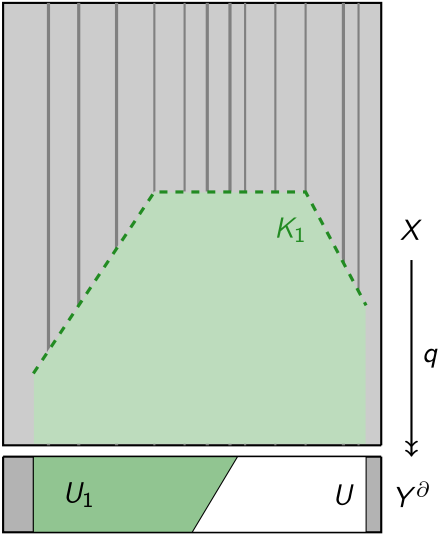

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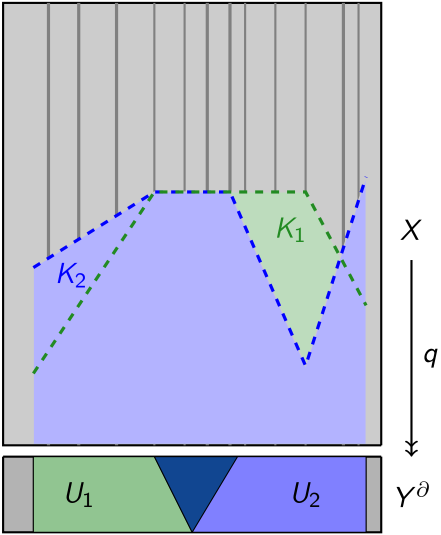

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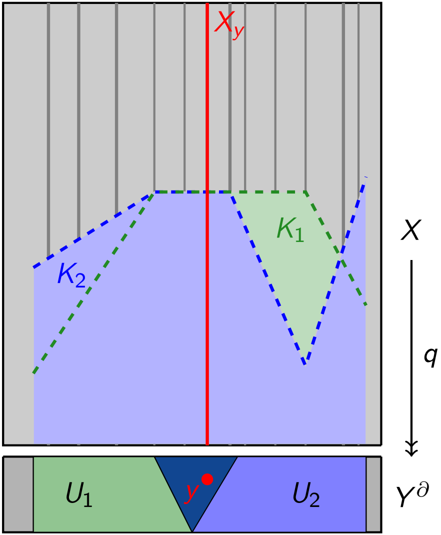

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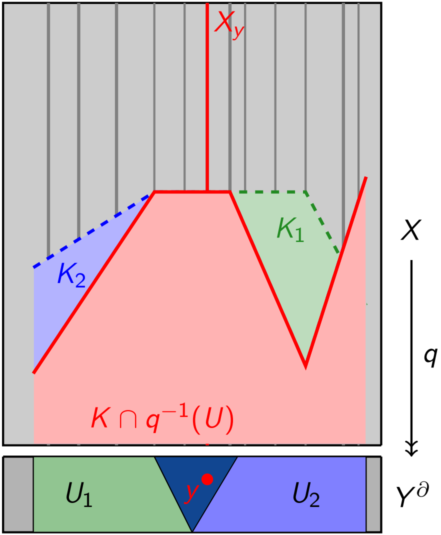


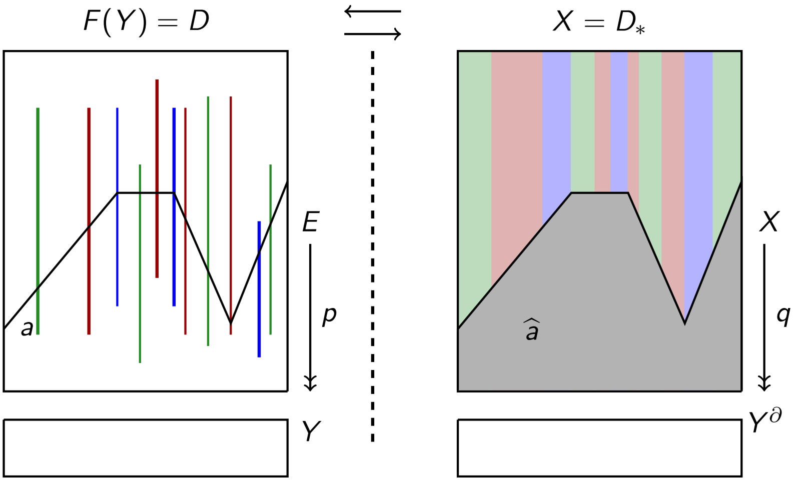
Flasque sheaves and patching decompositions

- If Y is a stably compact space with lattice basis B, we say a map $q : X \to Y^\partial$ is B-patching if (P_U) holds for all $U \in B$.

Theorem

Let D be a distributive lattice with dual Priestley space X.
The B-flasque sheaf representations of D over Y are in one-to-one correspondence with the B-patching decompositions of X over Y^∂.

This talk in a picture



The dual space of a Cartesian product

- Suppose that $D = \prod_{i \in I} D_i$ is a Cartesian product of distributive lattices.
- Note that $D = F(Y)$, where $Y = \beta I$, the Stone-Čech compactification of I as a discrete space, and F is the sheaf whose stalk at $y \in \beta I$ is the ultraproduct $D_y := (\prod_{i \in I} D_i)/y$. (cf. Jónsson's Lemma)
- Therefore, the Priestley space X dual to D decomposes as the disjoint union of closed subspaces X_y, where X_y is the Priestley dual space of D_y.

MV-algebras

- Infinite-valued logic (Łukasiewicz, 1917): truth values in $[0, 1]$.
  - a formula (= polynomial), e.g., $\varphi = (p \oplus q) \wedge r$
    is interpreted as $[\varphi] : [0, 1]^3 \to [0, 1]$,
  - collection of all formulas is a Multi-Valued (MV) algebra.

Definition

An MV-algebra is a tuple $(A, \vee, \wedge, 0, 1, \oplus, \ominus)$ such that

- $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice,
- $(A, \oplus, 0)$ is a commutative monoid and $\ominus$ is the residual of $\oplus$:

$$a \ominus b \leq c \iff a \leq b \oplus c,$$

- $x \vee y = (x \ominus y) \oplus y$.

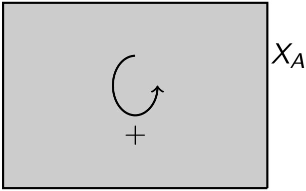
Dual spaces of MV-algebras

- Since an MV-algebra A is in particular a distributive lattice, let $(X_A, \pi, \leq)$ be the Priestley dual space of the lattice reduct.
- There is a subspace Y_A of X_A consisting of prime MV-ideals, i.e., those prime ideals I which satisfy $I \oplus I \subseteq I$.
- The Zariski topology on Y_A is the subspace topology of $\pi^{\downarrow}$, a lattice basis for this topology is $B := \{\widehat{a} \cap Y_A \mid a \in A\}$.
- There is a subspace Z_A of Y_A consisting of maximal MV-ideals.

Dual spaces of MV-algebras

Theorem

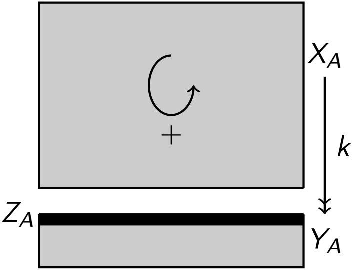
The dual space X_A of any MV-algebra A is a topological partial commutative semigroup,



Dual spaces of MV-algebras

Theorem

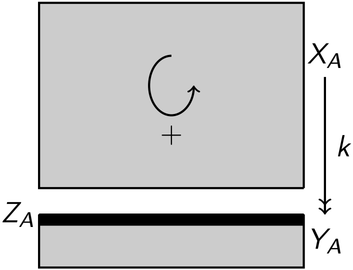
The dual space X_A of any MV-algebra A is a topological partial commutative semigroup, which admits a B-patching decomposition $k : (X_A, \pi) \to (Y_A, \pi^{\downarrow})$ over the prime MV-spectrum Y_A with the Zariski topology,



Dual spaces of MV-algebras

Theorem

The dual space X_A of any MV-algebra A is a topological partial commutative semigroup, which admits a B-patching decomposition $k : (X_A, \pi) \to (Y_A, \pi^{\downarrow})$ over the prime MV-spectrum Y_A with the Zariski topology, and there is a retraction $m : (Y_A, \pi^{\downarrow}) \to (Z_A, \pi^{\downarrow})$.



Sheaf representations of MV-algebras

Corollary (Keimel, Filipoiu-Georgescu, Yang, Dubuc-Poveda, ...)

Any MV-algebra A can be represented as the global sections of:

1. *a sheaf F_{pr} of totally ordered MV-algebras over the space Y_A with the co-Zariski topology;*
2. *a sheaf $F_{\max}$ of local MV-algebras over the space Z_A.*

Further work

- Can we go beyond flasque sheaves?
- What do these results say about canonical extensions?
- Applications to other classes of DL-ordered algebras?
- Relation to Jipsen's Priestley & Esakia products for n-potent GBL-algebras?

Duality for sheaf representations of distributive-lattice-ordered algebras

Sam van Gool

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