Duality and canonicity for Boolean algebra with a relation

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Introduction

Our goal in this talk is to systematically investigate Duality and Canonicity for Boolean Algebra (BA) enriched with a relation satisfying certain axioms.

Let us start with a few examples of (Boolean) algebras with a relation...

Proximity lattices

Proximity lattice [Jung, Sünderhauf (1996)]

A proximity lattice is a pair $(\mathbb{L}, R)$, where L is a lattice and $R \subseteq L \times L$ is a relation satisfying the following axioms:

1. $R \circ R = R$.
2. For any finite set $A \subseteq L$ and $b \in L$, $\bigvee A R b \Leftrightarrow \forall a \in A \; aRb$.
3. For any finite set $B \subseteq L$ and $b \in L$, $aR \bigwedge B \Leftrightarrow \forall b \in B \; aRb$.

Lattice subordination

Lattice subordination [G. Bezhanishvili (2013)]

A lattice subordination is a pair $(A, \prec)$ where A is a BA and $\prec$ is a binary relation on A satisfying:

$(S1)$ $0 \prec 0$ and $1 \prec 1$.

$(S2)$ $a \prec b, c$ implies $a \prec b \wedge c$.

$(S3)$ $a, b \prec c$ implies $a \vee b \prec c$.

$(S4)$ $a \leq b \prec c \leq d$ implies $a \prec d$.

$(S5)$ $a \prec b$ implies that there exists $c \in B$ with $c \prec c$ and $a \leq c \leq b$.

Lattice subordinations are used for an alternative proof of Priestly duality.

Example: Let $X \in$ Stone. For $U, V \in \text{Clop}(X)$ define $U \prec V$ if there exists a clopen up-set $W \subseteq X$ such that $U \subseteq W \subseteq V$

Precontact algebra

Precontact algebra [Düntsch, Vakarelov (2003)]

A precontact algebra is a pair (A, C) where A is a BA and C is a binary relation on A satisfying:
$(C0)$ aCb implies $a, b \neq 0$.
$(C+)$ $aC(b \vee c)$ implies aCb or aCc; $(a \vee b)Cc$ implies aCb or aCc.

Precontact algebra and their subvarieties are used in the algebraic analysis of theory of regions.

Order types

For a poset P, let P^∂ be the dual poset.

A 1-order type ε is an element of the set $\{1, \partial\}$. An n-order type is an element of the set $\{1, \varepsilon\}^n$. So, $\varepsilon = (1, \partial, 1)$ denotes the poset $A \times A^\partial \times A$.

Example

The operation $\rightarrow$: $A \times A \rightarrow A$ is meet-preserving co-ordinate wise with respect to order-type $(\partial, 1)$.

Boolean algebra with a dual operator relation

BA with a dual operator relation

Given a Boolean algebra B, a binary relation R on B is a dual operator relation if it satisfies the following:

(RM1) $(a, \top), (\top, a) \in R$;

(RM2) If $(a, b) \leq (c, d)$ and $(a, b) \in R$, then $(c, d) \in R$;

(RM3) If $(a, c), (b, c) \in R$ then $(a \wedge b, c) \in R$;

(RM4) If $(a, b), (a, c) \in R$ then $(a, b \wedge c) \in R$.

The relation R is an ε-dual operator relation, if $R \subseteq B^{\varepsilon_1} \times B^{\varepsilon_2}$ is a dual operator relation for some $\varepsilon = (\varepsilon_1, \varepsilon_2)$.

Boolean algebra with a dual operator relation

Let BADOR be the category whose objects are BA with an ε-dual operator relation.

Given $(B_1, R_1), (B_2, R_2) \in$ BADOR. The morphisms in the category are Boolean homomorphisms $h : B_1 \to B_2$ which satisfy:

$$(a, b) \in R_1 \quad \text{implies} \quad (h(a), h(b)) \in R_2.$$

Boolean algebra with an operator relation

BA with an operator relation

Given a Boolean algebra B, a binary relation R on B is an operator relation if it satisfies the following:

$(RA1)$ $(a, \bot), (\bot, a) \notin R$;
$(RA2)$ If $(a, b) \leq (c, d)$ and $(a, b) \in R$, then $(c, d) \in R$;
$(RA3)$ If $(a \vee b, c) \in R$, then $(a, c) \in R$ or $(b, c) \in R$;
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The relation R is an ε-operator relation, if $R \subseteq B^{\varepsilon_1} \times B^{\varepsilon_2}$ is an operator relation for some $\varepsilon = (\varepsilon_1, \varepsilon_2)$.

Let BAOR be the category whose objects are BA with an ε-dual operator relation. The morphisms are are Boolean homomorphisms $h : B_1 \to B_2$ which satisfy: $(a, b) \in R_1$ implies $(h(a), h(b)) \in R_2$.

Examples

Lattice subordination

$(S1)$ $0 \prec 0$ and $1 \prec 1$.

$(S2)$ $a \prec b, c$ implies $a \prec b \wedge c$.

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BADOR

(RM1) $(a, \top), (\top, a) \in R$;

(RM4) If $(a, b), (a, c) \in R$ then $(a, b \wedge c) \in R$.

(RM3) If $(a, c), (b, c) \in R$ then $(a \wedge b, c) \in R$;

(RM2) If $(a, b) \leq (c, d)$ and $(a, b) \in R$, then $(c, d) \in R$;

Lemma $(S1) + (S4) \Rightarrow 0 \prec a$ and $1 \prec a$.

Examples

Proposition The category of lattice subordinations is a full subcategory of BADOR.

(Boolean) proximity lattices, de Vries algebras are examples of objects in BADOR.

Precontact algebras and its subvarieties are objects in the category BAOR

Idea: Use the characteristic function of the relation for investigating duality and canocity for BADOR and BAOR

Let $(B, R) \in$ BADOR such that the order-type of R is $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Define $f_R : B^{\varepsilon_1} \times B^{\varepsilon_2} \to \mathbf{2}$ as

$$f_R(x, y) := \begin{cases} 1 & \text{if } (x, y) \in R \\ 0 & \text{otherwise.} \end{cases}$$

Then, f is an ε-dual operator.

Characteristic function of the relation

- Let BACDO be the category whose objects are BA B with ε-dual operator maps $f : B^{\varepsilon_1} \times B^{\varepsilon_2} \to \mathbf{2}$. The morphisms are Boolean homomorphisms $h : B_1 \to B_2$ which satisfy:

  $f_1(a, b) = 1$ implies $f_2(h(a), h(b)) = 1$.

- Given $(B, f : B^{\varepsilon_1} \times B^{\varepsilon_2} \to \mathbf{2}) \in$ BACDO, define a relation $R_f \subseteq B^{\varepsilon_1} \times B^{\varepsilon_2}$ as

  $$(a, b) \in R_f \text{ if } f(a, b) = 1$$

  Then, R_f is an ε-dual operator relation.

Theorem

1. BADOR $\cong$ BACDO
2. BADOR $\cong$ BACO

Discrete duality

- A map $g : (W_1, R_1) \to (W_2, R_2)$ between Kripke frames is a weak p-morphism if it satisfies:

  If $(g(w_1), v_2) \in R_2$ then there exists $v_1 \in W_1$ such that $(w_1, v_1) \in R_1$ and $g(v_1) = v_2$.

  Let FinKrF be the category of Kripke frames and weak p-morphisms.

From FinBADOR to FinKrF

- Given $(B, R) \in$ FinBADOR, define $Q \subseteq coAt(B)^2$ as

$$\boxed{(r,s) \in Q, \text{ if } \quad (r,s) \in R \quad \text{iff} \quad f_R(r,s) = 1}$$

Then, $(coAt(B), Q) \in$ FinKFr.

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Remark From Jónsson-Tarski duality, for a complete dual operator $f : A \times B \to C$, we define $Q' \subseteq coAt(C) \times coAt(A) \times coAt(B)$ as

$$(q, r, s) \in Q' \quad \text{iff} \quad q \leq f(r, s)$$

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- Given $(B, R) \in \text{FinBADOR}$, define $Q \subseteq coAt(B)^2$ as

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$$
\begin{aligned}
&(q, r, s) \in Q' \quad \text{iff} \quad q \leq f(r, s) \\
\Leftrightarrow \quad &(1, r, s) \in Q' \quad \text{iff} \quad 1 \leq f(r, s)
\end{aligned}
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From FinKrF to BADOR

- Given $(W, Q) \in \mathrm{FinKrF}$, define $[Q] : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathbf{2}$

  $$[Q](U, V) := \begin{cases} 1 & \text{if } \forall u \in (W \setminus U), \forall v \in (W \setminus V),\ (u, v) \in Q \\ 0 & \text{otherwise .} \end{cases}$$

  Define $R_{[Q]} \subseteq \mathcal{P}(W)^2$ as

  $$(U, V) \in R_{[Q]} \quad \text{if} \quad [Q](U, V) = 1.$$

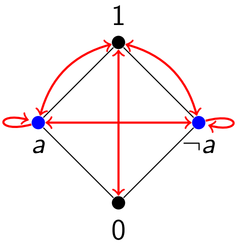
  Then, $(\mathcal{P}(W), R_{[Q]}) \in \mathrm{FinBADOR}$.

- The functors on morphisms in both categories are defined in the usual way as in Jónsson-Tarski duality.

Theorem

The category FinBADOR *is dual to the category of finite Kripke frames with weak p-morphisms.*

Example



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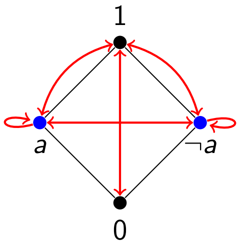

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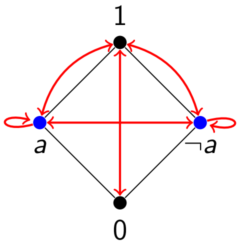

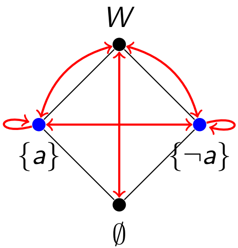

Topological duality

- Let StR be the category of Stone spaces with a closed relation and continuous weak p-morphisms.

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- Given $(B, R) \in \text{BADOR}$, its dual space is $(PrI(B), Q \subseteq PrI(B)^2)$ where Q is defined as

$$\boxed{(i, j) \in Q, \quad \text{if } i \times j \subseteq R \text{ iff } \forall a \in i, \forall b \in j,\ f_R(a, b) = 1}$$

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- From Jónsson-Tarski duality, for a dual operator $f : A \times B \to C$, the dual relation $Q' \subseteq PrI(C) \times PrI(A) \times PrI(B)$ is point closed, i.e.

$$Q'[x] \text{ is closed for each } x \in PrI(C)$$

$$\Rightarrow \quad Q'[*] = Q \text{ is closed.}$$

From StR to BADOR

- Given a Stone space X with a closed binary relation Q, define $[Q] : Clop(X)^2 \to \mathbf{2}$ as

$$[Q](U, V) := \begin{cases} 1 & \text{if } \forall u \in U^\partial, \forall v \in V^\partial,\ (u, v) \in Q \\ 0 & \text{otherwise.} \end{cases}$$

  Then, $(Clop(X), R_{[Q]} \subseteq Clop(X)^2)$ is a BADOR.

- The functors on morphisms in both categories are defined in the usual way as in Jónsson-Tarski duality.

Theorem

The category BADOR *is dual to the category of Stone spaces with a closed relation and continuous weak p-morphisms.*

Canonical extensions

Canonical extension of a BA provides an algebraic characterization of its double dual.

Canonical extension of a BA

The *canonical extension* of a BA A is a complete BA A^δ containing A as a subalgebra, such that

- (*denseness*) Every element of A^δ can be expressed both as a join of meets and as a meet of joins of elements from A;
- (*compactness*) For all $S, T \subseteq A$ with $\bigwedge S \leq \bigvee T$ in A^δ, there exist finite sets $F \subseteq S$ and $G \subseteq T$ such that $\bigwedge F \leq \bigvee G$.

Theorem [Jónsson, Tarksi (1951)] The canonical extension of a BA exists and is unique.

Extension for maps

- An element $x \in A^\delta$ is closed (resp. open) if it is the meet (resp. join) of some subset of A.

- A monotone map $f : A \to B$ can be extended to a map $: A^\delta \to B^\delta$ in two canonical ways. For all $u \in \mathbb{A}^\delta$, define

$$f^\sigma(u) = \bigvee\{\bigwedge\{f(a) : x \leq a \in A\} : u \geq x \in K(A^\delta)\}$$

$$f^\pi(u) = \bigwedge\{\bigvee\{f(a) : y \geq a \in A\} : u \leq y \in O(A^\delta)\}$$

  The map f is smooth if $f^\sigma = f^\pi$.

- Lemma [Gehrke, Jónsson (1994)]
  1. The σ-extension of an operator is a complete operator.
  2. The π-extension of a dual operator is a complete dual operator.

Canonical extension for BADOR

$$
\begin{array}{ccc}
(B^\delta, R_{f_R^\pi} \subseteq (B^\delta)^\varepsilon) & \overset{\cong}{\longleftrightarrow} & (B^\delta, f_R^\pi : (B^\delta)^\varepsilon \to 2) \\
 & & \Big\uparrow (.)^\delta \\
(B, R \subseteq B^\varepsilon) & \overset{\cong}{\longleftrightarrow} & (B, f_R : B^\varepsilon \to 2)
\end{array}
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Canonical extension for BADOR

$$(B^\delta, R_{f_R^\pi} \subseteq (B^\delta)^\varepsilon) \xleftrightarrow{\cong} (B^\delta, f_R^\pi : (B^\delta)^\varepsilon \to 2)$$

$$(.)^\delta \uparrow \qquad\qquad \uparrow (.)^\delta$$

$$(B, R \subseteq B^\varepsilon) \xleftrightarrow{\cong} (B, f_R : B^\varepsilon \to 2)$$

Theorem

The canonical extension of a BADOR exists and is unique.

Using $(B^\delta, R_{f_R^\pi}) \cong (\mathcal{P}(PrI(B)), R_{[Q]})$.

- Recall, that a lattice subordination is a BADOR satisfying:

  (5) $a \prec b$ implies that there exists $c \in B$ with $c \prec c$ and $a \leq c \leq b$.

  Proposition The axiom (5) is preserved under canonical extension of a BADOR.

  This defines the canonical extension of a lattice subordination.

- A pre-contact algebra is a BAOR. Hence, the existence and uniqueness of the canonical extension for pre-contact algebra follows using using the σ-extension of the characteristic map.

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- A pre-contact algebra is a BAOR. Hence, the existence and uniqueness of the canonical extension for pre-contact algebra follows using using the σ-extension of the characteristic map.

  Further, axioms characterizing the pre-contact relation to be reflexive, symmetric are canonical.

Jónsson-style canonicity [Jónsson (1994), Gehrke, Nagahashi, Venema (2005)]

Canonicity $A \models \phi \leq \psi \Rightarrow A^\delta \models \varphi \leq \psi$

$$\mathbb{A} \models \varphi \leq \psi$$

$$\Updownarrow$$

$$\varphi^{\mathbb{A}} \leq \psi^{\mathbb{A}}$$

$$\Downarrow$$

$$\varphi^{\mathbb{A}^\delta} \leq (\varphi^{\mathbb{A}})^\sigma \leq (\psi^{\mathbb{A}})^\sigma \leq \psi^{\mathbb{A}^\delta}$$

σ-expanding σ-contracting

$$\mathbb{A}^\delta \models \varphi \leq \psi$$

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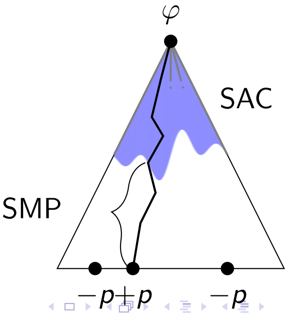
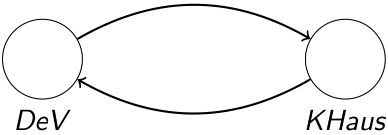
add. coord.				mult. prod.	
$+$	$\vee$	$\wedge$	g	$+$	$\wedge$
$-$	$\wedge$	$\vee$	f	$-$	$\vee$



Future work

- Characterize the classes of Kripke frames dual to lattice subordinations, de Vries algebras (Correspondence theory).
- Generalize this approach to (distributive) lattice setting and compare it to the notion of canonical extension for stably compact spaces in [van Gool 2012].
- Topological characterization of a KHaus as a subspace of a Stone space with a closed relation.



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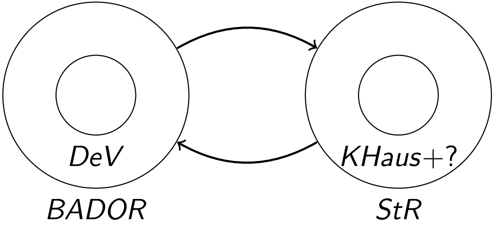

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- A quasi-order on a Stone space X is a Priestly quasi-order if $x \not\leq y$ implies that there exists a clopen up-set U of X with $x \in U$ and $y \notin U$.

  A pair $(X, \leq)$ is a quasi-ordered Priestly space (QPS) if X is a Stone space and $\leq$ is a Priestly quasi-order on X.

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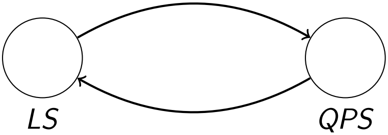

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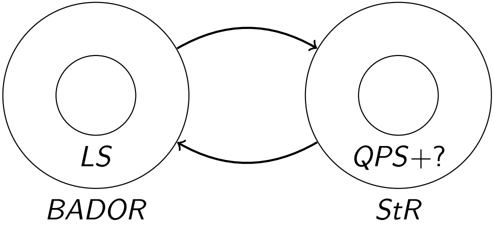

Topological characterization of a Quasi-ordered Priestly space as a subspace of a Stone space with a closed relation.

Thank you!