Squares of modal logics and relation algebras

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Introduction

In the early days of modal logic (before 1980s) there was interest in studying multiple particular systems.

Contemporary modal logic also investigates classes of logics and general constructions for combining different systems.

Products were introduced in the 1970s; their intensive study started in the 1990s.

Products are a natural type of combined modal logics
Connections:

first-order classical logic

first-order modal logics

relation algebras

description logics

Modal logics

We consider normal modal logics with unary modalities.

An *n-modal logic* has basic modalities $\Box_1, \ldots, \Box_n$

It is an extension of the *minimal logic* $\mathbf{K}_n$.

Frames $F=(W,R_1,\ldots,R_n)$

$F \vDash A$ means that a formula A is valid in a frame F

(= true at all worlds in all Kripke models over F).

$\mathbf{L}(F) := \{ A \mid F \vDash A \}$ (the *logic of* F).

$\mathbf{L}(\mathcal{C}) := \bigcap \{\mathbf{L}(F) \mid F \in \mathcal{C}\}$ (the *logic of a class* $\mathcal{C}$).

Logics of this form are called *Kripke complete.*

Def. If F is finite, $\mathbf{L}(F)$ is called *tabular.*

A modal logic has the *finite model property (FMP)* if it is an intersection of tabular logics.

Some particular complete logics

$\mathbf{K}_n$ is the minimal n-modal logic, $\mathbf{K}=\mathbf{K}_1$.

$\mathbf{K.t}_n$ is the minimal n-temporal logic, $\mathbf{K.t}=\mathbf{K.t}_1$.

Modalities: $\Box_1, \ldots, \Box_1, \Box_{-1}, \ldots, \Box_{-n}$

Axioms: $\Diamond_i \Box_{-i}\, p \rightarrow p$

$\mathbf{K.t}_n$-frames are $(W, R_1, (R_1)^{-1}, \ldots, R_n, (R_n)^{-1})$.

$\mathbf{T} = \mathbf{K} + \Box p \rightarrow p = \mathbf{L}(\text{all reflexive frames})$

$\mathbf{K4} = \mathbf{K} + \Box p \rightarrow \Box\Box p = \mathbf{L}(\text{all transitive frames})$

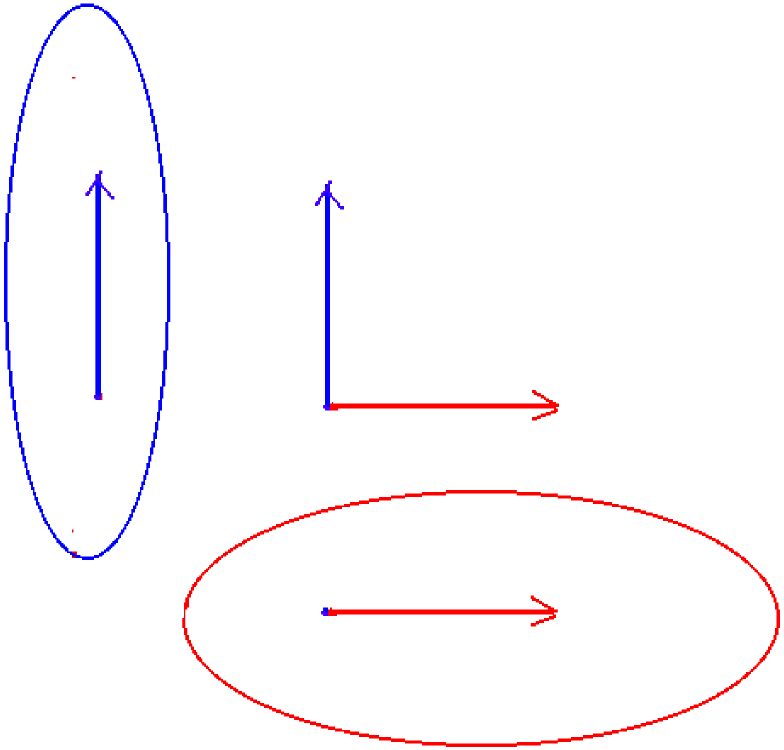
$\mathbf{S4} = \mathbf{K4} + \Box p \rightarrow p = \mathbf{L}(\text{all transitive reflexive frames})$

$\mathbf{KB} = \mathbf{K} + \Diamond\Box p \rightarrow p = \mathbf{L}(\text{all symmetric frames})$

$\mathbf{S5} = \mathbf{S4} + \Diamond\Box p \rightarrow p = \mathbf{L}(\text{all equivalence frames})$

$= \mathbf{L}(\text{all universal frames})$

Products of frames and logics



Def. $(W,R_1,\dots,R_n) \times (V,S_1,\dots,S_m) :=$
$(W\times V, R_{11},\dots,R_{n1},S_{12},\dots,S_{m2})$,

where

$(x,y)\ R_{i1}\ (x', y')$ iff $xR_ix'\ \&\ y = y'$

$(x,y)\ R_{j2}\ (x', y')$ iff $yS_jy'\ \&\ x = x'$

Def. The product of two modal logics

$$L_1\times L_2 := \mathbf{L}(\{F_1\times F_2 \mid F_1\vDash L_1,\ F_2\vDash L_2\})$$

If L_1 is n-modal with $\Box_1,\dots,\Box_n$,

L_2 is m-modal with $\blacksquare_1,\dots,\blacksquare_m$,

then $L_1\times L_2$ is (n+m)-modal. In a Kripke model over $F_1\times F_2$

$(x,y)\vDash \Box_i A$ iff $\forall x'(xR_ix' \Rightarrow (x',y)\vDash A)$

$(x,y)\vDash \blacksquare_j A$ iff $\forall x'(yS_jy' \Rightarrow (x,y')\vDash A)$

Remark on squares

Prop. The square of a logic is the logic of squares of its frames

$$L_1 \times L_1 := \mathbf{L}(\{F_1 \times F_1 \mid F_1 \vDash L_1\})$$

Proof:

$(F_1 \sqcup F_2) \times (F_1 \sqcup F_2) \cong$

$(F_1 \times F_1) \sqcup (F_1 \times F_2) \sqcup (F_2 \times F_1) \sqcup (F_2 \times F_2)$.

Hence

$$\mathbf{L}(F_1 \times F_2) \subseteq \mathbf{L}((F_1 \sqcup F_2)^2)$$

So the logic of all L_1-squares contains the logic of all L_1-products.

Axiomatization problem:

Given axioms for L_1, L_2, *find axioms for* $L_1 \times L_2$.

This problem is solved for Horn axiomatizable logics

Def. The fusion $L_1 * L_2$ of two modal logics

is the smallest logic containing them both. (If the modalities are not disjoint, rename them. In this talk we use $\blacksquare$)

Def. The commutative join (commutator)

$$[L_1, L_2] := \quad L_1 * L_2 +$$

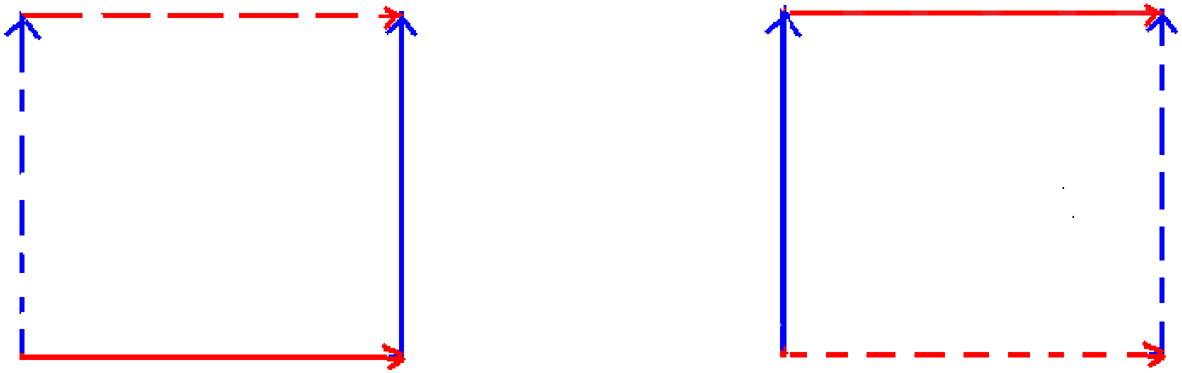
$\blacksquare_j \Box_i p \leftrightarrow \Box_i \blacksquare_j p$ (commutation axioms)

$\blacklozenge_j \Box_i p \rightarrow \blacksquare_j \Diamond_i p$ (Church-Rosser axioms)

The corresponding conditions on frames

$$R_i^{-1} \circ R_j \subseteq R_j \circ R_i^{-1} \text{ (Church – Rosser)}$$

$R_i \circ R_j = R_i \circ R_j$ (commutation)



Both conditions hold for products frames, so

$$[L_1, L_2] \subseteq L_1 \times L_2$$

Product-matching logics

Def. L_1, L_2 are product-matching if

$$L_1 \times L_2 = [L_1, L_2]$$

Def. A Horn sentence is a universal first order sentence of the form

$$\forall x \forall y \ldots (\phi(x,y,\ldots) \rightarrow R(x,y)),$$

where ϕ is positive quantifier-free, $R(x,y)$ is atomic.

A modal formula A is Horn if it corresponds to a Horn sentence (i.e., the class of its frames is

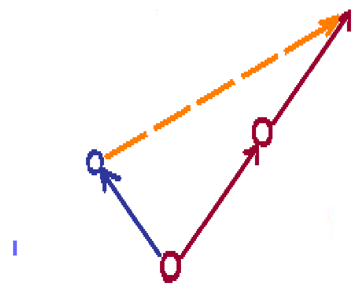
definable by a Horn sentence).

Example Modal formulas of the form

$$(\Diamond\ldots\Diamond)\Box p \to (\Box\ldots\Box)p$$

correspond to Horn sentences.

The corresponding property of frames



Logics with such axioms are always Kripke complete.

Def. A modal logic is Horn axiomatizable if it is axiomatizable by formulas that are either variable-free or Horn.

Theorem (Gabbay, Sh 1998<<BOOK03) If L_1, L_2 are Kripke complete and Horn axiomatizable, then they are product matching.

Product FMP

Def. $L_1 \times L_2$ has the product FMP if

$$L_1 \times L_2 := \mathbf{L}(\{F_1 \times F_2 \mid F_1 \models L_1,\ F_2 \models L_2\ ;\ F_1, F_2 \text{ are finite}\})$$

(Equivalently: $L_1 \times L_2$ is the logic of a class of products of finite frames)

FMP ⇒ product FMP

Def. A QTC-logic is axiomatizable by variable-free formulas and formulas or axioms of the form

$$\Diamond_i \Box_j p \rightarrow p,\ \Box_i p \rightarrow (\Box_i)^k p.$$

Theorem [Sh 2005] If L_2 is a QTC-logic, then

$$\mathbf{K.t}_n \times L_2 = [\mathbf{K.t}_n, L_2] \text{ has the fmp.}$$

Remark. These logics may not have the product FMP, e.g.

K4× **S5** (Wolter)

Theorem [Sh 2011] $(\mathbf{K.t}_n)^2 = [\mathbf{K.t}_n, \mathbf{K.t}_n]$ has the product fmp.

Corollary $\mathbf{KB}^2 = [\mathbf{KB}, \mathbf{KB}]$ has the product fmp.

Proof: **KB** is embeddable in **K.t**: translate $\Box A$ as $(\Box_1 A \wedge \Box_{-1} A)$ (the omnitemporal modality).

Conjecture The same holds with the reflexivity axioms:

$(\mathbf{T.t}_n)^2$, $\mathbf{TB}^2$ have the product fmp.

Segerberg squares

These are square frames with additional functions.

Krister Segerberg (1973) studied a special type - squares of frames with the universal relation. He considered the following functions on squares.

$f_{\bigcirc}$: $(x,y) \mapsto (y,x)$ (the diagonal symmetry)

f_1: $(x,y) \mapsto (y,y)$ (the first diagonal projection)

f_2: $(x,y) \mapsto (x,x)$ (the second diagonal projection)

These functions can be associated with extra modal operators $\bigcirc$, $\bigcirc_1$, $\bigcirc_2$. So in square frames they are interpreted as follows:

$(x,y) \vDash \bigcirc A$ iff $(y,x) \vDash A$

$(x,y) \vDash \bigcirc_2 A$ iff $(x,x) \vDash A$

$(x,y) \vDash \bigcirc_1 A$ iff $(y,y) \vDash A$

Remark. Segerberg used a different notation for these modalities.

Formally we define the Segerberg square of a frame $F=(W,R_1,\ldots,R_n)$ as the (2n+3)-frame

$$_{Seg}F^2 := (F^2, f_\circ, f_1, f_2)$$

(where $f_\circ, f_1, f_2$ are the functions on W^2 described above).

Respectively, the Segerberg square of an n-modal logic L_1 is defined the logic of the Segerberg squares of its frames

$$_{Seg}(L_1)^2 := \mathbf{L}(\{{}_{Seg}F^2 \mid F \vDash L_1\})$$

Tomorrow (or Sucessor) logic

SL := **K** + $\Diamond p \leftrightarrow \Box p$

This well-known logic is also due to Segerberg (1967). It is complete w.r.t. the frame

(the successor relation on natural numbers).

Every logic of a frame with a functional accessibility relation is an extension of **SL**.

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AXIOMATIZING SEGERBERG SQUARES

Soundness Every Segerberg square validates the following formulas

The corresponding semantic conditions for an arbitrary (2n+3)-frame $(V, X_1, \ldots, X_n, Y_1, \ldots, Y_n, f_\bigcirc, f, f)$

are in the right column; here fg denotes the composition of functions: (fg)(x)=f(g(x))

(I) The SL-axioms for the circles ○, , .

(II) (Sg1) $\bigcirc\bigcirc p \leftrightarrow p$ $f_\bigcirc f_\bigcirc = 1$ (the identity function on V)

The "symmetry" $f_\bigcirc$ is an involution.

(Sg2) $\bigcirc_1\bigcirc_1 p \leftrightarrow \bigcirc_1 p$ $f_1 f_1 = f_1$

(Sg2') The same for 2.

Both projections f_i are idempotent transformations of the square. In fact (Sg2') follows from (Sg1), (Sg2), (Sg3).

(Sg3) $\bigcirc\bigcirc_1 p \leftrightarrow \bigcirc_2 p$ $f_1 f_\circ = f_2$

(Sg4) $\bigcirc_1 \bigcirc p \leftrightarrow \bigcirc_1 p$ $f_\circ f_1 = f_1$

In Segerberg squares (Sg4) means that the image of f consists of self-symmetric points (or: every diagonal point is self-symmetric). But in the general case not all self-symmetric points are in $f[V]$.

(Sg3), (Sg4) imply that

$f_\bigcirc f_1 f_\bigcirc = f_2$, i.e., the involution $f_\bigcirc$ conjugates the projections f_1 and f_2.

(Sg3) shows that $\bigcirc_2$ is expressible in terms of $\bigcirc$, $\bigcirc_1$.

(Sg5) $\bigcirc\Box_i \bigcirc p \leftrightarrow \blacksquare_i p$ $aR_{i1}b$ iff $f_\circ(a)R_{i2}f_\circ(b)$

Symmetry is an isomorphism between R_{i1} and R_{i2}

(Sg6) $\bigcirc_1 \square_i (\blacksquare_i p \rightarrow \bigcirc_2 p)$ $f(a)R_{i1}b$ implies $bR_{i2}f(b)$

In Segerberg squares:

If $(y,y)R_{i1}(x,y)$ (i.e. yR_ix), then $(x,y)R_{i2}(x,x)$.

(Sg7) $\bigcirc_1 p \rightarrow \square_i \bigcirc_1 p$ $aR_{i1}b$ only if $f_1(a) = f_1(b)$

Horizontally accessible points are in the same horizontal row.

(Sg8) $\bigcirc_1 \blacksquare_i \bigcirc_1 p \leftrightarrow \blacksquare_i \bigcirc_1 p$ $(\exists b\ aR_{i2}b)$ iff $(\exists b\ f_1(a)R_{i2}b)$

In Segerberg squares: vertical seriality is equivalent for (y,y) and (x,y).

Def. For a modal logic L_1, put

$_{Seg}[L_1, L_1] :=$

$[L_1, L_1]$ + **SL*SL*SL** (for $\bigcirc$, $\bigcirc_1$, $\bigcirc_2$) + {(Sg1),..., (Sg8)}.

Theorem 1 (Completeness) If a logic L_1 is Horn axiomatizable, then $_{\text{Seg}}[L_1, L_1] = {}_{\text{Seg}}(L_1)^2$

Theorem 2 $_{\text{Seg}}(\mathbf{K_n})^2$ has the product fmp.

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Tarski's axioms for relation algebras

$$a \circ (b \cup c) = (a \circ b) \cup (a \circ c)$$

$$a \circ I = a$$

$$(a^{-1})^{-1} = a$$

$$(a \cup b)^{-1} = a^{-1} \cup b^{-1}$$

$$(a \circ b)^{-1} = b^{-1} \circ a^{-1}$$

$$a^{-1} \circ (-(a \circ b)) \leq -b$$

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