Tensor products of logics containing **S4**

Ilya Shapirovsky

Institute of Information Transmission Problems, Moscow, Russia

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The product of Kripke frames $\mathsf{F}_1 = (W, R)$, $\mathsf{F}_2 = (U, S)$ is the frame $\mathsf{F}_1 \times \mathsf{F}_2 = (W \times U, R^\times, S^\times)$, where

$$\begin{aligned}
(w_1, w_2) R^\times (v_1, v_2) &\Leftrightarrow w_1 R v_1 \ \& \ w_2 = v_2, \\
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The product of Kripke frames $\mathsf{F}_1 = (W, R_1, \ldots, R_n)$, $\mathsf{F}_2 = (V, S_1, \ldots, S_k)$ is the $(n+k)$-frame $\mathsf{F}_1 \times \mathsf{F}_2 = (W \times V, R_1^\times, \ldots, R_n^\times, S_1^\times, \ldots, S_k^\times)$, where

$$\begin{array}{lcl} (w_1, w_2) R_i^\times (v_1, v_2) & \Leftrightarrow & w_1 R_i v_1 \ \& \ w_2 = v_2, \\ (w_1, w_2) S_j^\times (v_1, v_2) & \Leftrightarrow & w_1 = v_1 \ \& \ w_2 S_j v_2. \end{array}$$

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For classes of Kripke frames $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{F} \times \mathfrak{G} := \{F \times G \mid F \in \mathfrak{F},\ G \in \mathfrak{G}\}$.
For logics $\mathrm{L}_1, \mathrm{L}_2$,

$$
\mathrm{L}_1 \times \mathrm{L}_2 := \mathrm{Log}(\mathrm{Fr}(\mathrm{L}_1) \times \mathrm{Fr}(\mathrm{L}_2)),
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where $\mathrm{Fr}(\mathrm{L})$ is the class of all L-frames.

Two "bad" logical properties of the product operation

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- Products of Kripke incomplete logics

If two logics $\mathrm{L}_1, \mathrm{L}_1'$ have the same frames, then for any L_2

$$\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1' \times \mathrm{L}_2.$$

In particular, if a logic L_1 is consistent, but the class of its frames is empty (e.g. L_1 is the Thomason's bimodal logic), then $\mathrm{L}_1 \times \mathrm{L}_2$ is inconsistent for any L_2.

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- Logical non-invariance

  It may happen that $\mathrm{Log}(\mathfrak{F}) = \mathrm{Log}(\mathfrak{F}')$, $\mathrm{Log}(\mathfrak{G}) = \mathrm{Log}(\mathfrak{G}')$ while

  $$\mathrm{Log}(\mathfrak{F} \times \mathfrak{G}) \neq \mathrm{Log}(\mathfrak{F}' \times \mathfrak{G}').$$

  For example, **S4** $= \mathrm{Log}(\mathfrak{F})$, where $\mathfrak{F}$ is the class of all finite preorders, but **S4** $\times$ **S4** $\neq \mathrm{Log}(\mathfrak{F} \times \mathfrak{F})$.

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  Note that logical invariance holds for direct products of elementary theories (A. Mostowski, 1952).

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Nevertheless, a possible answer was already known by that time:

[Y. Hasimoto. Normal products of modal logics. Advances in Modal Logic, volume 3. 2002.]

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[D. Gabbay, I. Shapirovsky, and V. Shehtman. Products of Modal Logics and Tensor Products of Modal Algebras. Journal of Applied Logic. In press.]

Main construction (Hasimoto)

Definition

A set $U \times V$, where $U \subseteq X$, $V \subseteq Y$, is called a *rectangle* in $X \times Y$. A *chequered subset* of $X \times Y$ is a finite union of rectangles.

Proposition

The set of all chequered subsets of $W_1 \times W_2$ is closed under Boolean operations. Moreover, if A_i is a subalgebra of 2^{W_i}, $i = 1, 2$, then the set of all finite unions of rectangles $V_1 \times V_2$, where $V_i \in A_i$, is closed under Boolean operations as well.

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For nonempty sets X, Y let $ch(X, Y)$ be the Boolean algebra of all chequered subsets of $X \times Y$. If A, B are subalgebras of 2^X, 2^Y respectively, let $ch_{AB}(X, Y)$ be the Boolean algebra of finite unions of rectangles U, V, where $U \in A$, $V \in B$.

Proposition

Let $F_1 = (W_1, R_1)$, $F_2 = (W_2, R_2)$, $F_1 \times F_2 = (W_1 \times W_2, R_1^\times, R_2^\times)$. *Then*

(1) *for any rectangle* $U \times V$ *we have*

$$R_1^{\times -1}(U \times V) = R_1^{-1}(U) \times V, \quad R_2^{\times -1}(U \times V) = U \times R_2^{-1}(V);$$

(2) *if* (F_1, A_1) *and* (F_2, A_2) *are general 1-frames, then* $(F_1 \times F_2, ch_{A_1 A_2}(W_1, W_2))$ *is a general 2-frame.*

Every Boolean algebra can be regarded as a Boolean ring, where the ring multiplication is the meet and the ring addition is the symmetric difference:

$$xy := x \wedge y, \quad x + y := (x \wedge \neg y) \vee (y \wedge \neg x).$$

A Boolean ring is a commutative associative algebra over the two-element field $\mathbf{F}_2$ with an idempotent multiplication; so the standard construction of a tensor product of associative algebras is applicable here.

Viz., the tensor product of algebras A, B is a pair $(A \otimes B, \pi)$, where $A \otimes B$ is an algebra, $\pi : (a, b) \mapsto a \otimes b$ is a bilinear map $A \times B \longrightarrow A \otimes B$ with the following universal property: every bilinear map $f : A \times B \longrightarrow C$, where C is an $\mathbf{F}_2$-space, uniquely factors through π, i.e., $f = g \cdot \pi$ for a unique linear $g : A \otimes B \longrightarrow C$.

X, Y are nonempty sets, A, B are subalgebras of 2^X, 2^Y respectively.
$ch_{AB}(X, Y)$ is the Boolean algebra of finite unions of rectangles U, V, where $U \in A$, $V \in B$.

Observation

$Ch_{AB}(X, Y)$ is the tensor product of A and B.
More precisely, $(ch_{AB}(X, Y), \pi|(A \times B))$ is the tensor product of A and B, where $\pi : 2^X \times 2^Y \to Ch(X, Y)$ such that $\pi(U, V) := U \times V$.
In particular, $(ch(X, Y), \pi)$ is the tensor product of 2^X and 2^Y.

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Theorem (Gabbay, Shehtman, Sh)

If $(A_1, \Diamond_1)$, $(A_2, \Diamond_2)$ *are normal 1-modal algebras, then there exists a unique 2-modal algebra structure on* $A_1 \otimes A_2$ *with diamond operations* $\Diamond_1^\times, \Diamond_2^\times$ *such that for any* $a \in A_1,\ b \in A_2$

$$\Diamond_1^\times(a \otimes b) = \Diamond_1 a \otimes b,\ \Diamond_2^\times(a \otimes b) = a \otimes \Diamond_2 b.$$

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$$\Diamond_1^\times(a \otimes b) = \Diamond_1 a \otimes b,\ \Diamond_2^\times(a \otimes b) = a \otimes \Diamond_2 b.$$

Put $(A_1, \Diamond_1) \otimes (A_2, \Diamond_2) := (A_1 \otimes A_2, \Diamond_1^\times, \Diamond_2^\times)$.

Definition

The *tensor product of general frames*:

$$(\mathsf{F}_1, A_1) \otimes (\mathsf{F}_2, A_2) := (\mathsf{F}_1 \times \mathsf{F}_2, A_1 \otimes A_2).$$

In particular, the *tensor product of Kripke frames*

$$\mathsf{F}_1 \otimes \mathsf{F}_2 := (\mathsf{F}_1 \times \mathsf{F}_2, ch(W_1, W_2)).$$

For classes of algebras (general frames) $\mathfrak{A}$, $\mathfrak{B}$, put
$\mathfrak{A} \otimes \mathfrak{B} := \{A \otimes B \mid A \in \mathfrak{A},\ B \in \mathfrak{B}\}$.

Definition

The *tensor product* of logics L_1 and L_2 is the logic
$\mathrm{L}_1 \otimes \mathrm{L}_2 := \mathrm{Log}(\mathrm{Alg}(\mathrm{L}_1) \otimes \mathrm{Alg}(\mathrm{L}_2))$.

Since every modal algebra is an algebra of a general frame, we have
$\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}(\mathrm{GFr}(\mathrm{L}_1) \otimes \mathrm{GFr}(\mathrm{L}_2))$.

Proposition (Hasimoto)

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- *If* L_1 *and* L_2 *are consistent, then* $\mathrm{L}_1 \otimes \mathrm{L}_2$ *is conservative over* L_1 *and* L_2*.*

Logical invariance

Theorem (Hasimoto)

For classes of 1-algebras (general frames, Kripke frames) $\mathfrak{A}$, $\mathfrak{B}$,
$\mathrm{Log}(\mathfrak{A}) \otimes \mathrm{Log}(\mathfrak{B}) = \mathrm{Log}(\mathfrak{A} \otimes \mathfrak{B})$.

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F_{L} denotes the canonical frame of a logic L, and $(\mathsf{F}_{\mathrm{L}}, A_{\mathrm{L}})$ denotes its general canonical frame.

Corollary

For any $\mathrm{L}_1,\ \mathrm{L}_2$, $\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}((\mathsf{F}_{\mathrm{L}_1}, A_{\mathrm{L}_1}) \otimes (\mathsf{F}_{\mathrm{L}_2}, A_{\mathrm{L}_2}))$.

Corollary

If $\mathrm{L}_1,\ \mathrm{L}_2$ *are canonical, then* $\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}(\mathsf{F}_{\mathrm{L}_1} \otimes \mathsf{F}_{\mathrm{L}_2})$.

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$$\big((\mathsf{F}_1, A_1) \otimes (\mathsf{F}_2, A_2)\big) \;\otimes\; (\mathsf{F}_3, A_3) \;\;\cong\;\; (\mathsf{F}_1, A_1) \;\otimes\; \big((\mathsf{F}_2, A_2) \otimes (\mathsf{F}_3, A_3)\big).$$

Corollary

$(\mathrm{L}_1 \otimes \mathrm{L}_2) \otimes \mathrm{L}_3 = \mathrm{L}_1 \otimes (\mathrm{L}_2 \otimes \mathrm{L}_3)$.

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Corollary

$(\mathrm{L}_1 \otimes \mathrm{L}_2) \otimes \mathrm{L}_3 = \mathrm{L}_1 \otimes (\mathrm{L}_2 \otimes \mathrm{L}_3)$.

Problem

$(\mathrm{L}_1 \times \mathrm{L}_2) \times \mathrm{L}_3 \stackrel{?}{=} \mathrm{L}_1 \times (\mathrm{L}_2 \times \mathrm{L}_3)$.
In particular, $(\mathbf{K}^2 \times \mathbf{K}) \times \mathbf{K} \stackrel{?}{=} \mathbf{K}^2 \times (\mathbf{K} \times \mathbf{K})$.

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For classes of 1-algebras (general frames, Kripke frames) $\mathfrak{A}$, $\mathfrak{B}$,
$\mathrm{Log}(\mathfrak{A}) \otimes \mathrm{Log}(\mathfrak{B}) = \mathrm{Log}(\mathfrak{A} \otimes \mathfrak{B})$.

Proposition

$ch(X, Y) = 2^{X \times Y}$ *iff* X *or* Y *is finite.*

Corollary

If $\mathrm{L}_1, \ \mathrm{L}_2$ *are Kripke complete, then*

$$\mathrm{L}_1 \times \mathrm{L}_2 \subseteq \mathrm{L}_1 \otimes \mathrm{L}_2 \subseteq \mathrm{L}_1 \times_{fin} \mathrm{L}_2,$$

where $\mathrm{L}_1 \times_{fin} \mathrm{L}_2 := \mathrm{Log}(\mathrm{Fr}_{fin}(\mathrm{L}_1) \times \mathrm{Fr}_{fin}(\mathrm{L}_2))$, $\mathrm{Fr}_{fin}(\mathrm{L})$ *is the class of all finite* L*-frames.*

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$\mathrm{Log}(\mathfrak{A}) \otimes \mathrm{Log}(\mathfrak{B}) = \mathrm{Log}(\mathfrak{A} \otimes \mathfrak{B})$.

A logic $\mathrm{L}_1 \times \mathrm{L}_2$ has the *product fmp* if $\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \times_{fin} \mathrm{L}_2$.

Corollary

Let $\mathrm{L}_1,\ \mathrm{L}_2$ *be Kripke complete logics.* $\mathrm{L}_1 \times \mathrm{L}_2$ *has the product fmp iff*

$$\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{L}_1 \times_{fin} \mathrm{L}_2;$$

it follows that if $\mathrm{L}_1 \times \mathrm{L}_2$ *has the product fmp, then for any* $\mathfrak{F}_i$ *such that* $\mathrm{L}_i = \mathrm{Log}(\mathfrak{F}_i)$, $i = 1, 2$, *we have*

$$\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{Log}(\mathfrak{F}_1 \times \mathfrak{F}_2).$$

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For classes of 1-algebras (general frames, Kripke frames) $\mathfrak{A}$, $\mathfrak{B}$,
$\mathrm{Log}(\mathfrak{A}) \otimes \mathrm{Log}(\mathfrak{B}) = \mathrm{Log}(\mathfrak{A} \otimes \mathfrak{B})$.

Corollary

If L_1 *and* L_2 *have the fmp, then:*

$$\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{L}_1 \times_{\mathit{fin}} \mathrm{L}_2;$$

$$\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2 \textit{ iff } \mathrm{L}_1 \times \mathrm{L}_2 \textit{ has the product fmp.}$$

Products with tabular logics (Gabbay, Shehtman, Sh)

A *tabular* logic is the logic of a finite frame.

Corollary

If L_1 *is Kripke complete,* L_2 *is tabular, then* $\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2$.

Corollary

For a class of frames $\mathfrak{F}$*, and a finite frame* G*,*
$\mathrm{Log}(\mathfrak{F}) \times \mathrm{Log}(\mathsf{G}) = \mathrm{Log}(\mathfrak{F} \times \{\mathsf{G}\})$.

Corollary

If L_1 *has the fmp and* L_2 *is tabular, then* $\mathrm{L}_1 \times \mathrm{L}_2$ *has the product fmp.*

Corollary

The modal product of tabular logics is tabular: if F *and* G *are finite, then*

$$\mathrm{Log}(\mathsf{F}) \times \mathrm{Log}(\mathsf{G}) = \mathrm{Log}(\mathsf{F} \times \mathsf{G}).$$

Products with tabular logics (Gabbay, Shehtman, Sh)

A *tabular* logic is the logic of a finite frame.

Theorem

Suppose L_2 *is tabular. Then:*

1. *if* L_1 *admits filtration, then* $\mathrm{L}_1 \times \mathrm{L}_2$ *has the exponential product fmp;*
2. *if* L_1 *is Kripke complete and decidable, then* $\mathrm{L}_1 \times \mathrm{L}_2$ *is decidable.*

Some problems

- Do there exist $\mathrm{L}_1, \mathrm{L}_2$ such that $\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2$, but $\mathrm{L}_1 \times \mathrm{L}_2$ lacks the product fmp and $\mathrm{L}_1, \mathrm{L}_2$ are non-tabular?
- Do there exist $\mathrm{L}_1, \mathrm{L}_2$ such that $\mathrm{L}_1 \times \mathrm{L}_2$ is undecidable, but $\mathrm{L}_1 \otimes \mathrm{L}_2$ is decidable?
- Do there exist $\mathrm{L}_1, \mathrm{L}_2$ such that $\mathrm{L}_1 \times \mathrm{L}_2$ is not finitely axiomatizable, but $\mathrm{L}_1 \otimes \mathrm{L}_2$ is finitely axiomatizable?

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- Does Kripke completeness transfer from L_1 and L_2 to $\mathrm{L}_1 \otimes \mathrm{L}_2$?

  If L_1 and L_2 are Kripke complete then $\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{Log}(\mathrm{Fr}(\mathrm{L}_1) \otimes \mathrm{Fr}(\mathrm{L}_2))$. The latter is the logic of a class of general frames.

Signature preserving products

For Kripke 1-frames $F_1 = (W, R)$, $F_2 = (U, S)$, their $\times$-product is the 2-frame $F_1 \times F_2 = (W \times U, R^\times, S^\times)$.

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$F_1 \times_{dir} F_2 := (W \times U, R^\times \circ S^\times)$ is the direct product of F_1 and F_2:

$$(w_1, w_2)\ (R^\times \circ S^\times)\ (v_1, v_2) \quad \Leftrightarrow \quad w_1 R v_1 \ \&\ w_2 S v_2.$$

Direct products of logics containing **S4**

L_1, $\mathrm{L}_2 \supseteq$ **S4**.

Proposition

$\mathbf{S4} \subseteq \mathrm{L}_1 \times_{dir} \mathrm{L}_2 \subseteq \mathrm{L}_1 \cap \mathrm{L}_2$.

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Example

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- $\mathbf{S4} \times_{dir} \mathbf{S4} = \mathbf{S4}$;
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- If φ is preserved in direct products then $(\mathbf{S4} + \varphi) \times_{dir} (\mathbf{S4} + \varphi) = \mathbf{S4} + \varphi$;
- $\mathbf{S4.3} \times_{dir} \mathbf{S4.3} = \mathbf{S4.2} \times_{dir} \mathbf{S4.3} = \mathbf{S4.2}$.

$\mathbf{S4.2} = \mathbf{S4.2} \times_{dir} \mathbf{S4.2} \subseteq \mathbf{S4.3} \times_{dir} \mathbf{S4.3} \subseteq \mathrm{Log}((\mathbb{R}, \leq) \times_{dir} (\mathbb{R}, \leq))$.

Direct products of logics containing **S4**

$\mathrm{L}_1, \mathrm{L}_2 \supseteq \mathbf{S4}$.

Proposition

$\mathbf{S4} \subseteq \mathrm{L}_1 \times_{dir} \mathrm{L}_2 \subseteq \mathrm{L}_1 \cap \mathrm{L}_2$.

Example

- $\mathbf{S4} \times_{dir} \mathbf{S4} = \mathbf{S4}$;
- $\mathbf{S4.2} \times_{dir} \mathbf{S4.2} = \mathbf{S4.2}$;
- If φ is preserved in direct products then $(\mathbf{S4} + \varphi) \times_{dir} (\mathbf{S4} + \varphi) = \mathbf{S4} + \varphi$;
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$\mathbf{S4.2} = \mathbf{S4.2} \times_{dir} \mathbf{S4.2} \subseteq \mathbf{S4.3} \times_{dir} \mathbf{S4.3} \subseteq \mathrm{Log}((\mathbb{R}, \leq) \times_{dir} (\mathbb{R}, \leq))$.
The latter logic is **S4.2**, since it is the logic of the *causal future relation* in Minkowski plane (Goldblatt, 80; Shehtman, 83).

$(W, R) \times_{dir} (U, S) := (W \times U, R^{\times} \circ S^{\times})$.

Definition

For general frames, put

$$(\mathsf{F}_1, A_1) \otimes_{dir} (\mathsf{F}_2, A_2) := (\mathsf{F}_1 \times_{dir} \mathsf{F}_2, A_1 \otimes A_2);$$

for algebras, put $(A_1, \Diamond_1) \otimes_{dir} (A_2, \Diamond_2) := (A_1 \otimes A_2, \Diamond_1^{\times} \Diamond_2^{\times})$.

$\mathrm{L}_1 \otimes_{dir} \mathrm{L}_2 := \mathrm{Log}(\mathrm{Alg}(\mathrm{L}_1) \otimes_{dir} \mathrm{Alg}(\mathrm{L}_2))$ $(= \mathrm{Log}(\mathrm{GFr}(\mathrm{L}_1) \otimes_{dir} \mathrm{GFr}(\mathrm{L}_2)))$.

Theorem (Hasimoto)

For classes of algebras (general frames) $\mathfrak{C}_1$, $\mathfrak{C}_2$,

$$\mathrm{Log}(\mathfrak{C}_1) \otimes_{dir} \mathrm{Log}(\mathfrak{C}_2) = \mathrm{Log}(\mathfrak{C}_1 \otimes_{dir} \mathfrak{C}_2).$$

The commutator of L_1 and L_2:

$$[\mathrm{L}_1, \mathrm{L}_2] := \mathrm{L}_1 * \mathrm{L}_2 + \Diamond_1\Diamond_2 p \leftrightarrow \Diamond_2\Diamond_1 p + \Diamond_1\Box_2 p \to \Diamond_2\Box_1 p,$$

where $\mathrm{L}_1 * \mathrm{L}_2$ is the fusion of L_1 and L_2.

Proposition

If $\mathrm{L}_1, \mathrm{L}_2 \supseteq \mathbf{S4}$, *then* $\mathrm{L}_1 \otimes \mathrm{L}_2 \supseteq [\mathbf{S4}, \mathbf{S4}]$.

Proof.

$\mathrm{L}_1 \otimes \mathrm{L}_2$ contains the fusion $\mathrm{L}_1 * \mathrm{L}_2$, so it contains $\mathbf{S4} * \mathbf{S4}$. Also,

$$\mathrm{L}_1 \otimes \mathrm{L}_2 \supseteq \mathbf{K} \otimes \mathbf{K} = \mathbf{K} \times \mathbf{K} = [\mathbf{K}, \mathbf{K}].$$

□

Proposition

If a bimodal logic contains $[\mathbf{S4}, \mathbf{S4}]$, *then* $\Diamond_1\Diamond_2$ *is an* **S4***-operator.*

Corollary

If $\mathrm{L}_1, \mathrm{L}_2 \supseteq \mathbf{S4}$, *then* $\mathrm{L}_1 \otimes_{dir} \mathrm{L}_2 \supseteq \mathbf{S4}$.

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- Does Kripke completeness transfer from L_1 and L_2 to $\mathrm{L}_1 \otimes_{dir} \mathrm{L}_2$?
- For $\mathrm{L}_1, \mathrm{L}_2 \supseteq$ **S4**, does topological completeness transfer from L_1 and L_2 to $\mathrm{L}_1 \otimes_{dir} \mathrm{L}_2$?

Some more problems

- Main question. $\mathfrak{X}_1, \mathfrak{X}_1', \mathfrak{X}_2, \mathfrak{X}_2'$ are topological spaces, $\mathrm{Log}(\mathfrak{X}_1) = \mathrm{Log}(\mathfrak{X}_1')$, $\mathrm{Log}(\mathfrak{X}_2) = \mathrm{Log}(\mathfrak{X}_2')$.

$$\mathrm{Log}(\mathfrak{X}_1 \times \mathfrak{X}_2) \stackrel{?}{=} \mathrm{Log}(\mathfrak{X}_1' \times X_2').$$

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$$\mathrm{Log}(\mathfrak{X}_1 \times \mathfrak{X}_2) \stackrel{?}{=} \mathrm{Log}(\mathfrak{X}'_1 \times X'_2).$$

- How $\otimes_{dir}$ interacts with the product topology? Let $\mathrm{L}_1 = \mathrm{Log}(\mathfrak{X}_1)$, $\mathrm{L}_2 = \mathrm{Log}(\mathfrak{X}_2)$.

$$\mathrm{L}_1 \otimes_{dir} \mathrm{L}_2 \quad ??? \quad \mathrm{Log}(\mathfrak{X}_1 \times \mathfrak{X}_2)$$

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$$\mathrm{L}_1 \otimes_{dir} \mathrm{L}_2 \quad ??? \quad \mathrm{Log}(\mathfrak{X}_1 \times \mathfrak{X}_2)$$

- Let $(X_1 \times X_2, \tau_1, \tau_2)$ be the *bitopological product* of $\mathfrak{X}_1 = (X_1, \tau_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2)$. In general, $\tau_1 \circ \tau_2$ is not an **S4**-operator (e.g., if $\mathrm{Log}(X_1 \times X_2, \tau_1, \tau_2) = \mathbf{S4} * \mathbf{S4}$).

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- Main question. $\mathfrak{X}_1, \mathfrak{X}_1', \mathfrak{X}_2, \mathfrak{X}_2'$ are topological spaces, $\mathrm{Log}(\mathfrak{X}_1) = \mathrm{Log}(\mathfrak{X}_1'),\ \mathrm{Log}(\mathfrak{X}_2) = \mathrm{Log}(\mathfrak{X}_2')$.

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Fact: $((X_1 \times X_2, \tau_1, \tau_2), Ch(X_1, X_2)) \vDash [\mathbf{S4}, \mathbf{S4}]$, so $((X_1 \times X_2, \tau_1 \circ \tau_2), Ch(X_1, X_2)) \vDash \mathbf{S4}$.

$$\mathrm{Log}((X_1 \times X_2, \tau_1 \circ \tau_2), Ch(X_1, X_2)) \quad ??? \quad \mathrm{Log}(\mathfrak{X}_1 \times \mathfrak{X}_2)$$

Thank you!