On the completion of pointfree function rings

Jorge Picado

Department of Mathematics

University of Coimbra

PORTUGAL

Centre for Mathematics
University of Coimbra

— joint work with J. Gutiérrez García and I. Mozo Carollo (Bilbao, Spain)

POINTFREE TOPOLOGY

POINTFREE TOPOLOGY

topology in the category $\mathbf{Loc}$ $\left[\begin{array}{l} \text{locales} \\ \\ \text{localic maps} \end{array} \right.$

POINTFREE TOPOLOGY

topology in the category $\mathbf{Loc}$ [locales / localic maps]

$$\mathbf{Loc} = \mathbf{Frm}^{op}$$

POINTFREE TOPOLOGY

topology in the category $\mathbf{Loc}$
- locales
- localic maps

$\mathbf{Loc} = \mathbf{Frm}^{op}$

the category $\mathbf{Frm}$
- frames: $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$
- frame homomorphisms: preserve $\bigvee, 0$ and $\wedge, 1$

POINTFREE TOPOLOGY

topology in the category $\mathbf{Loc}$ [locales / localic maps]

$\mathbf{Loc} = \mathbf{Frm}^{op}$

the category $\mathbf{Frm}$ [

frames: $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$

frame homomorphisms: preserve

$\bigvee, 0$

$\wedge, 1$]

Algebraic category: *presentations by generators and relations.*

REAL NUMBERS POINTFREELY

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

REAL NUMBERS POINTFREELY

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

The frame of reals:

$$\mathfrak{L}(\mathbb{R}) = \mathbf{Frm}\langle\, (p,\text{—}), (\text{—},q)\, (p,q \in \mathbb{Q}) \mid$$

REAL NUMBERS POINTFREELY

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

The frame of reals:

$\mathfrak{L}(\mathbb{R}) = \mathbf{Frm}\langle\, (p,-), (-,q)\, (p, q \in \mathbb{Q}) \mid$ (R1) $(p,-) \wedge (-,q) = 0$ for $p \geqslant q$,

REAL NUMBERS POINTFREELY

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

The frame of reals:

$$\mathfrak{L}(\mathbb{R}) = \mathbf{Frm}\big\langle\, (p,\text{—}), (\text{—},q) (p,q \in \mathbb{Q}) \;\big|\; \text{(R1)}\ (p,\text{—}) \wedge (\text{—},q) = 0 \ \text{ for } p \geqslant q,$$

$$\text{(R2)}\ (p,\text{—}) \vee (\text{—},q) = 1 \ \text{ for } p < q,$$

REAL NUMBERS POINTFREELY

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

The frame of reals:

$$\mathfrak{L}(\mathbb{R}) = \mathbf{Frm}\langle (p,-), (-,q)\,(p,q \in \mathbb{Q}) \mid \text{(R1) } (p,-) \wedge (-,q) = 0 \text{ for } p \geqslant q,$$

$$\text{(R2) } (p,-) \vee (-,q) = 1 \text{ for } p < q,$$

$$\text{(R3) } (p,-) = \bigvee_{r>p} (r,-),$$

$$(-,q) = \bigvee_{s<q} (-,s),$$

REAL NUMBERS POINTFREELY

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

The frame of reals:

$$\mathfrak{L}(\mathbb{R}) = \mathbf{Frm}\big\langle\, (p,-), (-,q)\, (p,q \in \mathbb{Q}) \;\big|\; \text{(R1) } (p,-) \wedge (-,q) = 0 \text{ for } p \geqslant q,$$

$$\text{(R2) } (p,-) \vee (-,q) = 1 \text{ for } p < q,$$

$$\text{(R3) } (p,-) = \textstyle\bigvee_{r>p} (r,-),$$

$$(-,q) = \textstyle\bigvee_{s<q} (-,s),$$

$$\text{(R4) } \textstyle\bigvee_{p\in\mathbb{Q}} (p,-) = 1,$$

$$\textstyle\bigvee_{q\in\mathbb{Q}} (-,q) = 1 \,\big\rangle$$

CONTINUOUS REAL FUNCTIONS POINTFREELY

$f\colon \mathfrak{L}(\mathbb{R}) \to L$

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

CONTINUOUS REAL FUNCTIONS POINTFREELY

$f\colon \mathfrak{L}(\mathbb{R}) \to L$

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

- The spectrum $\Sigma\mathfrak{L}(\mathbb{R}) \simeq \mathbb{R}$ (usual space of reals).

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

- The spectrum $\Sigma\mathfrak{L}(\mathbb{R}) \simeq \mathbb{R}$ (usual space of reals).
- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

  there is a natural isomorphism $\mathbf{Top}(X, \Sigma L) \xrightarrow{\sim} \mathbf{Frm}(L, \mathfrak{O}X)$.

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

- The spectrum $\Sigma\mathfrak{L}(\mathbb{R}) \simeq \mathbb{R}$ (usual space of reals).
- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

  there is a natural isomorphism $\mathbf{Top}(X, \Sigma L) \xrightarrow{\sim} \mathbf{Frm}(L, \mathfrak{O}X)$.

- For $L = \mathfrak{L}(\mathbb{R})$: $\boxed{\mathsf{C}(X) = \mathbf{Top}(X, \mathbb{R}) \xrightarrow{\sim} \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), \mathfrak{O}X)}$

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

- The spectrum $\Sigma\mathfrak{L}(\mathbb{R}) \simeq \mathbb{R}$ (usual space of reals).
- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

  there is a natural isomorphism $\mathbf{Top}(X, \Sigma L) \xrightarrow{\sim} \mathbf{Frm}(L, \mathfrak{O}X)$.

- For $L = \mathfrak{L}(\mathbb{R})$: $\mathsf{C}(X) = \mathbf{Top}(X, \mathbb{R}) \xrightarrow{\sim} \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), \mathfrak{O}X)$

Natural extension:

$$\mathsf{C}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), L)$$

ALGEBRAIC OPERATIONS IN $\mathsf{C}(L)$

$f: \mathfrak{L}(\mathbb{R}) \to L$

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

$\mathsf{C}(L)$ is a lattice-ordered ring

ALGEBRAIC OPERATIONS IN $\mathsf{C}(L)$

$f\colon \mathfrak{L}(\mathbb{R}) \to L$

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

$\mathsf{C}(L)$ is a lattice-ordered ring partially ordered by

$$f \leqslant g \quad \text{iff} \quad f(p,—) \leqslant g(p,—)\ \forall p \in \mathbb{Q}$$

$$\text{iff} \quad g(—,q) \leqslant f(—,q)\ \forall q \in \mathbb{Q}.$$

ALGEBRAIC OPERATIONS IN $\mathrm{C}(L)$

$f\colon \mathfrak{L}(\mathbb{R}) \to L$

[B. Banaschewski, *The real numbers in pointfree topology*, 1997]

$\mathrm{C}(L)$ is a lattice-ordered ring partially ordered by

$$f \leqslant g \quad \text{iff} \quad f(p,\text{—}) \leqslant g(p,\text{—}) \ \forall p \in \mathbb{Q}$$
$$\text{iff} \quad g(\text{—},q) \leqslant f(\text{—},q) \ \forall q \in \mathbb{Q}.$$

For $\diamond = +, \cdot, \wedge, \vee$:

$$(f \diamond g)(p,\text{—}) = \bigvee_{r \diamond s > p} f(r,\text{—}) \wedge g(s,\text{—}) = \cdots$$

$$(f \diamond g)(\text{—},q) = \bigvee_{r \diamond s < q} f(\text{—},r) \wedge g(\text{—},s) = \cdots$$

[JGG & JP, *Rings of real functions in pointfree topology*, TAA 2011]

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete:

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete:

non-void sets of continuous real functions in $\mathsf{C}(L)$,

bounded above, need not have a least upper bound in $\mathsf{C}(L)$.

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete:

non-void sets of continuous real functions in $\mathsf{C}(L)$,

bounded above, need not have a least upper bound in $\mathsf{C}(L)$.

BACKGROUND:

Definition. A poset P is Dedekind complete (=conditionally complete)

if every bounded $\emptyset \neq A \subseteq P$ has a supremum and an infimum in P.

(more useful since we are dealing with l.o. groups with no $\perp$ and $\top$)

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete:

non-void sets of continuous real functions in $\mathsf{C}(L)$,

bounded above, need not have a least upper bound in $\mathsf{C}(L)$.

BACKGROUND:

A Dedekind completion of P is a join- and meet-dense embedding

$$\Phi\colon P \to \mathcal{D}(P)$$

in a Dedekind complete poset.

$$\forall\, \hat{p} \in \mathcal{D}(P) \quad \boxed{\hat{p} = \bigvee^{\mathcal{D}(P)} \{\Phi(p) \mid \Phi(p) \leqslant \hat{p}\} = \bigwedge^{\mathcal{D}(P)} \{\Phi(p) \mid \Phi(p) \geqslant \hat{p}\}}$$

THE PROBLEM: ORDER COMPLETENESS OF $\mathrm{C}(L)$

$\mathrm{C}(L)$ fails to be Dedekind complete. But ... WHY?

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete. But ... WHY?

Let $\{f_i\}_{i\in I} \subseteq \mathsf{C}(L)$ and $f \in \mathsf{C}(L)$ such that $f_i \leqslant f$ for all $i \in I$.

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete. But ... WHY?

Let $\{f_i\}_{i\in I} \subseteq \mathsf{C}(L)$ and $f \in \mathsf{C}(L)$ such that $f_i \leqslant f$ for all $i \in I$.

Natural candidate:

$$h(p,—) = \bigvee_{i\in I} f_i(p,—) \quad \text{and} \quad h(—,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(—,s)\Big)$$

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete. But ... WHY?

$$h(p,—) = \bigvee_{i\in I} f_i(p,—) \quad \text{and} \quad h(—,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(—,s)\Big)$$

$$h \in \mathsf{C}(L) \Leftrightarrow \begin{cases} \text{(R1) if } p \geqslant q, \text{ then } h(p,—) \wedge h(—,q) = 0 \\ \text{(R2) if } p < q, \text{ then } h(p,—) \vee h(—,q) = 1 \\ \text{(R3) } h(p,—) = \bigvee_{r>p} h(r,—) \text{ and } h(—,q) = \bigvee_{s<q} h(—,s) \\ \text{(R4) } \bigvee_{p\in\mathbb{Q}} h(p,—) = 1 = \bigvee_{q\in\mathbb{Q}} h(—,q) \end{cases}$$

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete. But ... WHY?

$$h(p,—) = \bigvee_{i\in I} f_i(p,—) \quad \text{and} \quad h(—,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(—,s)\Big)$$

$h \in \mathsf{C}(L) \Leftrightarrow$

- (R1) if $p \geqslant q$, then $h(p,—) \wedge h(—,q) = \overset{\text{(R1) for } f_i}{\cdots} = 0$ ✓
- (R2) if $p < q$, then $h(p,—) \vee h(—,q) = 1$
- (R3) $h(p,—) = \bigvee_{r>p} h(r,—)$ and $h(—,q) = \bigvee_{s<q} h(—,s)$
- (R4) $\bigvee_{p\in\mathbb{Q}} h(p,—) = 1 = \bigvee_{q\in\mathbb{Q}} h(—,q)$

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete. But ... WHY?

$$h(p,—) = \bigvee_{i\in I} f_i(p,—) \quad \text{and} \quad h(—,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(—,s)\Big)$$

$h \in \mathsf{C}(L) \Leftrightarrow$

- (R1) if $p \geqslant q$, then $h(p,—) \wedge h(—,q) \overset{\text{(R1) for } f_i}{=\cdots} = 0$ ✓
- (R2) if $p < q$, then $h(p,—) \vee h(—,q) = 1$
- (R3) $h(p,—) = \bigvee_{r>p} h(r,—)$ and $h(—,q) = \bigvee_{s<q} h(—,s)$ ✓
- (R4) $\bigvee_{p\in\mathbb{Q}} h(p,—) = 1 = \bigvee_{q\in\mathbb{Q}} h(—,q)$

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete. But ... WHY?

$$h(p,—) = \bigvee_{i\in I} f_i(p,—) \quad \text{and} \quad h(—,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(—,s)\Big)$$

$$h \in \mathsf{C}(L) \Leftrightarrow \begin{cases} \text{(R1) if } p \geqslant q, \text{ then } h(p,—) \wedge h(—,q) \overset{\text{(R1) for } f_i}{=} \cdots = 0 & ✓ \\ \text{(R2) if } p < q, \text{ then } h(p,—) \vee h(—,q) = 1 & \\ \text{(R3) } h(p,—) = \bigvee_{r>p} h(r,—) \text{ and } h(—,q) = \bigvee_{s<q} h(—,s) & ✓ \\ \text{(R4) } \bigvee_{p\in\mathbb{Q}} h(p,—) = 1 = \bigvee_{q\in\mathbb{Q}} h(—,q) & ✓ \end{cases}$$

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete. But ... WHY?

$$h(p,—) = \bigvee_{i\in I} f_i(p,—) \quad \text{and} \quad h(—,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(—,s)\Big)$$

$$h \in \mathsf{C}(L) \Leftrightarrow \begin{cases} \text{(R1) if } p \geqslant q, \text{ then } h(p,—) \wedge h(—,q) \overset{\text{(R1) for } f_i}{=\cdots} = 0 & ✓ \\ \text{(R2) if } p < q, \text{ then } h(p,—) \vee h(—,q) = 1 & ✗ \\ \text{(R3) } h(p,—) = \bigvee_{r>p} h(r,—) \text{ and } h(—,q) = \bigvee_{s<q} h(—,s) & ✓ \\ \text{(R4) } \bigvee_{p\in\mathbb{Q}} h(p,—) = 1 = \bigvee_{q\in\mathbb{Q}} h(—,q) & ✓ \end{cases}$$

(R2) if $p < q$, then $h(p,—) \vee h(—,q) \neq 1$ in general.

THE PROBLEM: ORDER COMPLETENESS OF $\mathsf{C}(L)$

$\mathsf{C}(L)$ fails to be Dedekind complete. But ... WHY?

$$h(p,-) = \bigvee_{i\in I} f_i(p,-) \quad \text{and} \quad h(-,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(-,s)\Big)$$

$$h \in \mathsf{C}(L) \Leftrightarrow \begin{cases} \text{(R1) if } p \leqslant q, \text{ then } h(-,p) \wedge h(q,-) \overset{\text{(R1) for } f_i}{=\cdots} = 0 & ✓ \\ \text{(R2) if } p > q, \text{ then } h(-,p) \vee h(q,-) = 1 & ✗ \\ \text{(R3) } h(p,-) = \bigvee_{r>p} h(r,-) \text{ and } h(-,q) = \bigvee_{s<q} h(-,s) & ✓ \\ \text{(R4) } \bigvee_{p\in\mathbb{Q}} h(p,-) = 1 = \bigvee_{q\in\mathbb{Q}} h(-,q) & ✓ \end{cases}$$

$\mathsf{C}(L)$ fails to be Dedekind complete because of (R2)!

Approach I:

Partial real functions

J. Gutiérrez García, I. Mozo Carollo & JP
On the Dedekind completion of function rings, **Forum Math., in press.**

IDEA: DELETE RELATION (R2)

Generators	$(p, -), (-, q), \quad p, q \in \mathbb{Q}$
Relations	(R1) $(p, -) \wedge (-, q) = 0$ whenever $p \geqslant q$
	~~(R2) $(p, -) \vee (-, q) = 1$ whenever $p < q$~~
	(R3) $(p, -) = \bigvee_{r>p}(r, -)$ and $(-, q) = \bigvee_{s<q}(-, s)$
	(R4) $\bigvee_{p\in\mathbb{Q}}(p, -) = 1 = \bigvee_{q\in\mathbb{Q}}(-, q)$

IDEA: DELETE RELATION (R2)

The frame of partial reals $\mathfrak{L}(\mathbb{IR})$

Generators	$(p, —), (—, q), \quad p, q \in \mathbb{Q}$
Relations	(R1) $(p, —) \wedge (—, q) = 0$ whenever $p \geqslant q$
	~~(R2) $(p, —) \vee (—, q) = 1$ whenever $p < q$~~
	(R3) $(p, —) = \bigvee_{r>p}(r, —)$ and $(—, q) = \bigvee_{s<q}(—, s)$
	(R4) $\bigvee_{p\in\mathbb{Q}}(p, —) = 1 = \bigvee_{q\in\mathbb{Q}}(—, q)$

IDEA: DELETE RELATION (R2)

The frame of partial reals $\mathfrak{L}(\mathbb{IR})$

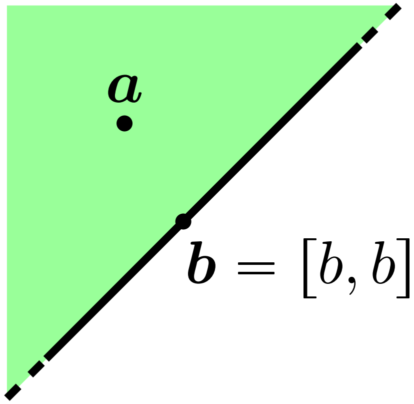
Generators	$(p, —), (—, q), \quad p, q \in \mathbb{Q}$
Relations	(R1) $(p, —) \wedge (—, q) = 0$ whenever $p \geqslant q$
	~~(R2) $(p, —) \vee (—, q) = 1$ whenever $p < q$~~
	(R3) $(p, —) = \bigvee_{r>p}(r, —)$ and $(—, q) = \bigvee_{s<q}(—, s)$
	(R4) $\bigvee_{p\in\mathbb{Q}}(p, —) = 1 = \bigvee_{q\in\mathbb{Q}}(—, q)$

$\Sigma\mathfrak{L}(\mathbb{IR})$ is the partial real line:

The frame of partial reals $\mathfrak{L}(\mathbb{IR})$

Generators	$(p, —), (—, q), \quad p, q \in \mathbb{Q}$
Relations	(R1) $(p, —) \wedge (—, q) = 0$ whenever $p \geqslant q$
	~~(R2) $(p, —) \vee (—, q) = 1$ whenever $p < q$~~
	(R3) $(p, —) = \bigvee_{r>p}(r, —)$ and $(—, q) = \bigvee_{s<q}(—, s)$
	(R4) $\bigvee_{p\in\mathbb{Q}}(p, —) = 1 = \bigvee_{q\in\mathbb{Q}}(—, q)$

$\Sigma\mathfrak{L}(\mathbb{IR})$ is the partial real line:



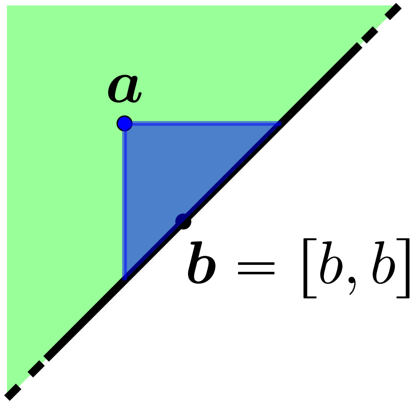
$\mathbb{IR} = \{\boldsymbol{a} := [\underline{a}, \overline{a}] \subseteq \mathbb{R} \mid \underline{a}, \overline{a} \in \mathbb{R}$ and $\underline{a} \leqslant \overline{a}\}$

IDEA: DELETE RELATION (R2)

The frame of partial reals $\mathfrak{L}(\mathbb{IR})$

Generators	$(p, —), (—, q), \quad p, q \in \mathbb{Q}$
Relations	(R1) $(p, —) \wedge (—, q) = 0$ whenever $p \geqslant q$
	~~(R2) $(p, —) \vee (—, q) = 1$ whenever $p < q$~~
	(R3) $(p, —) = \bigvee_{r>p} (r, —)$ and $(—, q) = \bigvee_{s<q} (—, s)$
	(R4) $\bigvee_{p \in \mathbb{Q}} (p, —) = 1 = \bigvee_{q \in \mathbb{Q}} (—, q)$

$\Sigma\mathfrak{L}(\mathbb{IR})$ is the partial real line:



$\mathbb{IR} = \{\boldsymbol{a} := [\underline{a}, \overline{a}] \subseteq \mathbb{R} \mid \underline{a}, \overline{a} \in \mathbb{R}$ and $\underline{a} \leqslant \overline{a}\}$

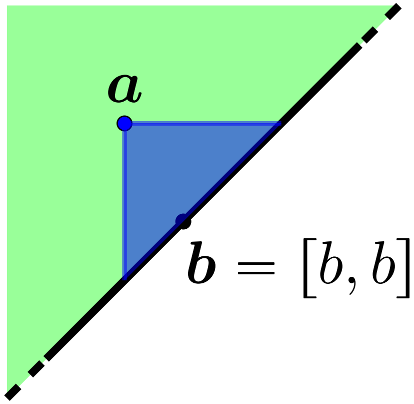
$\boldsymbol{a} \sqsubseteq \boldsymbol{b}$ iff $[\underline{a}, \overline{a}] \supseteq [\underline{b}, \overline{b}]$ (interval-domain)

IDEA: DELETE RELATION (R2)

The frame of partial reals $\mathfrak{L}(\mathbb{IR})$

Generators	$(p, —), (—, q), \quad p, q \in \mathbb{Q}$
Relations	(R1) $(p, —) \wedge (—, q) = 0$ whenever $p \geqslant q$
	~~(R2) $(p, —) \vee (—, q) = 1$ whenever $p < q$~~
	(R3) $(p, —) = \bigvee_{r>p} (r, —)$ and $(—, q) = \bigvee_{s<q} (—, s)$
	(R4) $\bigvee_{p \in \mathbb{Q}} (p, —) = 1 = \bigvee_{q \in \mathbb{Q}} (—, q)$

$\Sigma\mathfrak{L}(\mathbb{IR})$ is the partial real line:

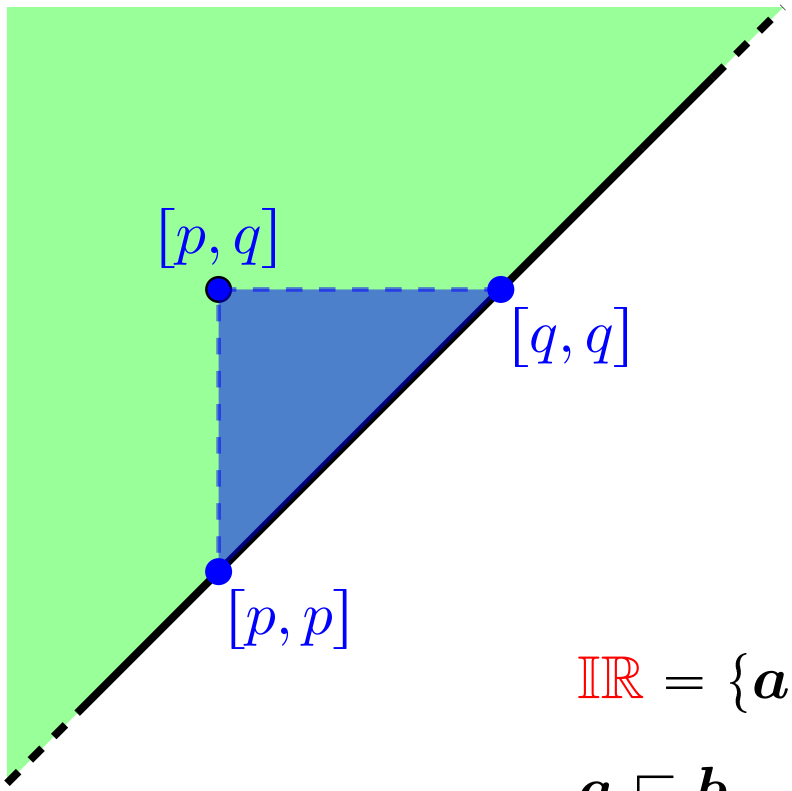


$\mathbb{IR} = \{\boldsymbol{a} := [\underline{a}, \overline{a}] \subseteq \mathbb{R} \mid \underline{a}, \overline{a} \in \mathbb{R}$ and $\underline{a} \leqslant \overline{a}\}$

$\boldsymbol{a} \sqsubseteq \boldsymbol{b}$ iff $[\underline{a}, \overline{a}] \supseteq [\underline{b}, \overline{b}]$ (interval-domain)

$\Sigma\mathfrak{L}(\mathbb{IR}) \simeq \mathbb{IR}$ with the Scott topology on $(\mathbb{IR}, \sqsubseteq)$

The frame of partial reals $\mathfrak{L}(\mathbb{IR})$



$\mathbb{IR} = \{\boldsymbol{a} := [\underline{a}, \overline{a}] \subseteq \mathbb{R} \mid \underline{a}, \overline{a} \in \mathbb{R}$ and $\underline{a} \leqslant \overline{a}\}$

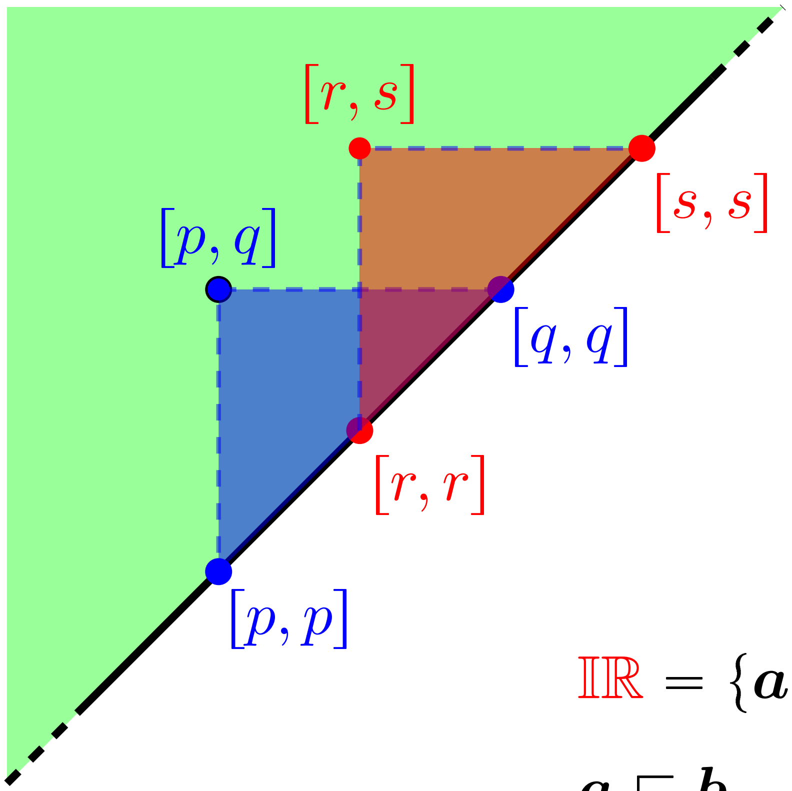
$\boldsymbol{a} \sqsubseteq \boldsymbol{b}$ iff $[\underline{a}, \overline{a}] \supseteq [\underline{b}, \overline{b}]$ (interval-domain)

$\Sigma\mathfrak{L}(\mathbb{IR}) \simeq \mathbb{IR}$ with the Scott topology on $(\mathbb{IR}, \sqsubseteq)$

Basic open sets: $\Uparrow[p, q] \equiv \{\boldsymbol{a} \in \mathbb{IR} \mid p < \underline{a} \leqslant \overline{a} < q\}$

IDEA: DELETE RELATION (R2)

The frame of partial reals $\mathfrak{L}(\mathbb{IR})$



$\mathbb{IR} = \{\boldsymbol{a} := [\underline{a}, \overline{a}] \subseteq \mathbb{R} \mid \underline{a}, \overline{a} \in \mathbb{R}$ and $\underline{a} \leqslant \overline{a}\}$

$\boldsymbol{a} \sqsubseteq \boldsymbol{b}$ iff $[\underline{a}, \overline{a}] \supseteq [\underline{b}, \overline{b}]$ (interval-domain)

$\Sigma\mathfrak{L}(\mathbb{IR}) \simeq \mathbb{IR}$ with the Scott topology on $(\mathbb{IR}, \sqsubseteq)$

Basic open sets: $\Uparrow[p, q] \equiv \{\boldsymbol{a} \in \mathbb{IR} \mid p < \underline{a} \leqslant \overline{a} < q\}$

PARTIAL CONTINUOUS REAL FUNCTIONS POINTFREELY

- The spectrum $\Sigma\mathfrak{L}(\mathbb{IR}) \simeq \mathbb{IR}$ (interval-domain space).

PARTIAL CONTINUOUS REAL FUNCTIONS POINTFREELY

- The spectrum $\Sigma\mathfrak{L}(\mathbb{IR}) \simeq \mathbb{IR}$ (interval-domain space).

- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

  there is a natural isomorphism $\mathbf{Top}(X, \Sigma L) \xrightarrow{\sim} \mathbf{Frm}(L, \mathfrak{O}X)$.

PARTIAL CONTINUOUS REAL FUNCTIONS POINTFREELY

- The spectrum $\Sigma\mathfrak{L}(\mathbb{IR}) \simeq \mathbb{IR}$ (interval-domain space).

- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

  there is a natural isomorphism $\mathbf{Top}(X, \Sigma L) \xrightarrow{\sim} \mathbf{Frm}(L, \mathfrak{O}X)$.

- For $L = \mathfrak{L}(\mathbb{IR})$:

$$\mathsf{IC}(X) = \mathbf{Top}(X, \mathbb{IR}) \xrightarrow{\sim} \mathbf{Frm}(\mathfrak{L}(\mathbb{IR}), \mathfrak{O}X)$$

- The spectrum $\Sigma\mathfrak{L}(\mathbb{IR}) \simeq \mathbb{IR}$ (interval-domain space).

- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

  there is a natural isomorphism $\mathbf{Top}(X, \Sigma L) \xrightarrow{\sim} \mathbf{Frm}(L, \mathfrak{O}X)$.

- For $L = \mathfrak{L}(\mathbb{IR})$: $\mathsf{IC}(X) = \mathbf{Top}(X, \mathbb{IR}) \xrightarrow{\sim} \mathbf{Frm}(\mathfrak{L}(\mathbb{IR}), \mathfrak{O}X)$

Natural extension: $\mathsf{IC}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{IR}), L)$

$$f\colon \mathfrak{L}(\mathbb{IR}) \to L$$

The frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

Generators	$(p,—),(—,q),\quad p,q\in\mathbb{Q}$
Relations	(R1) $(p,—)\wedge(—,q)=0$ whenever $p\geqslant q$
	(R2) $(p,—)\vee(—,q)=1$ whenever $p<q$
	(R3) $(p,—)=\bigvee_{r>p}(r,—)$ and $(—,q)=\bigvee_{s<q}(—,s)$
	~~(R4) $\bigvee_{p\in\mathbb{Q}}(p,—)=1=\bigvee_{q\in\mathbb{Q}}(—,q)$~~

A SIDE NOTE

The frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$

Generators	$(p,—),(—,q), \quad p,q\in\mathbb{Q}$
Relations	(R1) $(p,—)\wedge(—,q)=0$ whenever $p\geqslant q$
	(R2) $(p,—)\vee(—,q)=1$ whenever $p<q$
	(R3) $(p,—)=\bigvee_{r>p}(r,—)$ and $(—,q)=\bigvee_{s<q}(—,s)$
	~~(R4) $\bigvee_{p\in\mathbb{Q}}(p,—)=1=\bigvee_{q\in\mathbb{Q}}(—,q)$~~

Similarly, we have the extended continuous real functions:

$$\overline{\mathsf{C}}(L)=\mathbf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}),L)$$

B. BANASCHEWSKI, J. GUTIÉRREZ GARCÍA & J. P.
Extended real functions in pointfree topology, *J. Pure Appl. Algebra* **216 (2012)**

DEDEKIND COMPLETENESS OF $\mathsf{C}(L)$

Let $\{f_i\}_{i\in I} \subseteq \mathsf{C}(L)$ and $f \in \mathsf{C}(L)$ such that $f_i \leqslant f$ for all $i \in I$.

Does there exist $\bigvee_{i\in I} f_i$ in $\mathsf{C}(L)$?

DEDEKIND COMPLETENESS OF $\mathsf{IC}(L)$

Let $\{f_i\}_{i\in I} \subseteq \mathsf{IC}(L)$ and $f \in \mathsf{IC}(L)$ such that $f_i \leqslant f$ for all $i \in I$.

Does there exist $\bigvee_{i\in I} f_i$ in $\mathsf{IC}(L)$?

Natural candidate (again):

$$h(p,-) = \bigvee_{i\in I} f_i(p,-) \quad \text{and} \quad h(-,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(-,s)\Big)$$

DEDEKIND COMPLETENESS OF $\mathsf{IC}(L)$

Let $\{f_i\}_{i\in I} \subseteq \mathsf{IC}(L)$ and $f \in \mathsf{IC}(L)$ such that $f_i \leqslant f$ for all $i \in I$.

Does there exist $\bigvee_{i\in I} f_i$ in $\mathsf{IC}(L)$?

Natural candidate (again):

$$h(p,-) = \bigvee_{i\in I} f_i(p,-) \quad \text{and} \quad h(-,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(-,s)\Big)$$

$$h \in \mathsf{IC}(L) \Leftrightarrow \begin{cases} \text{(R1) if } p \leqslant q, \text{ then } h(-,p) \wedge h(q,-) = 0 \\ \text{(R3) } h(p,-) = \bigvee\limits_{r>p} h(r,-) \text{ and } h(-,q) = \bigvee\limits_{s<q} h(-,s) \\ \text{(R4) } \bigvee_{p\in\mathbb{Q}} h(p,-) = 1 = \bigvee_{q\in\mathbb{Q}} h(-,q) \end{cases}$$

DEDEKIND COMPLETENESS OF $\mathsf{IC}(L)$

Let $\{f_i\}_{i\in I} \subseteq \mathsf{IC}(L)$ and $f \in \mathsf{IC}(L)$ such that $f_i \leqslant f$ for all $i \in I$.

Does there exist $\bigvee_{i\in I} f_i$ in $\mathsf{IC}(L)$?

Natural candidate (again):

$$h(p,—) = \bigvee_{i\in I} f_i(p,—) \quad \text{and} \quad h(—,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(—,s)\Big)$$

$h \in \mathsf{IC}(L) \Leftrightarrow$

- (R1) if $p \leqslant q$, then $h(—,p) \wedge h(q,—) \overset{\text{(R1) for } f_i}{=} \cdots = 0$ ✓
- (R3) $h(p,—) = \bigvee_{r>p} h(r,—)$ and $h(—,q) = \bigvee_{s<q} h(—,s)$ ✓
- (R4) $\bigvee_{p\in\mathbb{Q}} h(p,—) = 1 = \bigvee_{q\in\mathbb{Q}} h(—,q)$ ✓

DEDEKIND COMPLETENESS OF $\mathsf{IC}(L)$

Let $\{f_i\}_{i\in I} \subseteq \mathsf{IC}(L)$ and $f \in \mathsf{IC}(L)$ such that $f_i \leqslant f$ for all $i \in I$.

Does there exist $\bigvee_{i\in I} f_i$ in $\mathsf{IC}(L)$?

Natural candidate (again):

$$h(p,\text{—}) = \bigvee_{i\in I} f_i(p,\text{—}) \quad \text{and} \quad h(\text{—},q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(\text{—},s)\Big)$$

$h \in \mathsf{IC}(L) \Leftrightarrow$

- (R1) if $p \leqslant q$, then $h(\text{—},p) \wedge h(q,\text{—}) \overset{\text{(R1) for } f_i}{=} \cdots = 0$ ✓
- (R3) $h(p,\text{—}) = \bigvee_{r>p} h(r,\text{—})$ and $h(\text{—},q) = \bigvee_{s<q} h(\text{—},s)$ ✓
- (R4) $\bigvee_{p\in\mathbb{Q}} h(p,\text{—}) = 1 = \bigvee_{q\in\mathbb{Q}} h(\text{—},q)$ ✓

Hence $h \in \mathsf{IC}(L)$. Moreover, $h = \bigvee^{\mathsf{IC}(L)}_{i\in I} f_i$.

DEDEKIND COMPLETENESS OF $\mathsf{IC}(L)$

Let $\{f_i\}_{i\in I} \subseteq \mathsf{IC}(L)$ and $f \in \mathsf{IC}(L)$ such that $f_i \leqslant f$ for all $i \in I$.

Does there exist $\bigvee_{i\in I} f_i$ in $\mathsf{IC}(L)$?

Natural candidate (again):

$$h(p,—) = \bigvee_{i\in I} f_i(p,—) \quad \text{and} \quad h(—,q) = \bigvee_{s<q}\Big(\bigwedge_{i\in I} f_i(—,s)\Big)$$

$h \in \mathsf{IC}(L) \Leftrightarrow$

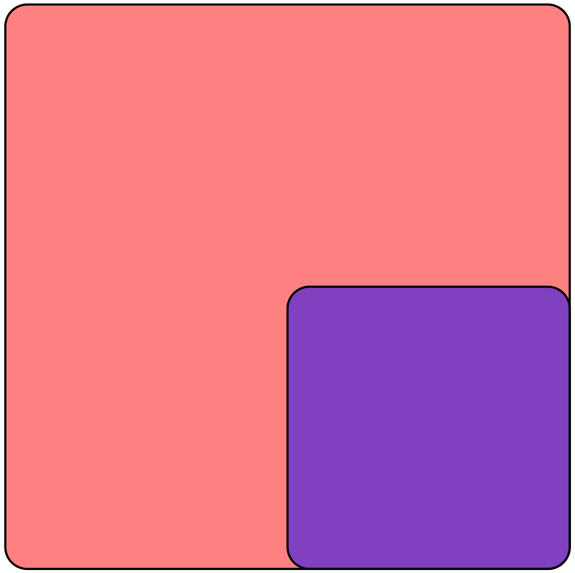
- (R1) if $p \leqslant q$, then $h(—,p) \wedge h(q,—) \overset{\text{(R1) for } f_i}{=\cdots} = 0$ ✓
- (R3) $h(p,—) = \bigvee_{r>p} h(r,—)$ and $h(—,q) = \bigvee_{s<q} h(—,s)$ ✓
- (R4) $\bigvee_{p\in\mathbb{Q}} h(p,—) = 1 = \bigvee_{q\in\mathbb{Q}} h(—,q)$ ✓

THEOREM. $\mathsf{IC}(L)$ is Dedekind complete.

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

Of course, we may consider $\mathsf{C}(L)$ as a subset of $\mathsf{IC}(L)$:

$\mathsf{IC}(L)$

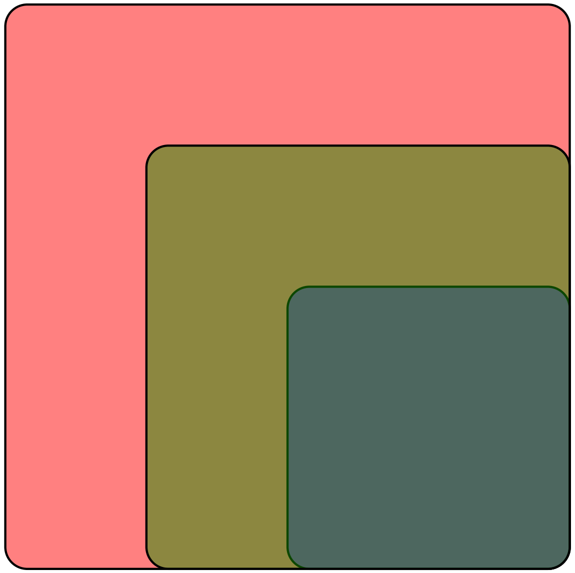


$$\mathsf{C}(L) = \{h \in \mathsf{IC}(L) \mid h(p, \text{—}) \vee h(\text{—}, q) = 1 \text{ for every } p < q\}$$

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

Of course, we may consider $\mathsf{C}(L)$ as a subset of $\mathsf{IC}(L)$:

$\mathsf{IC}(L)$



$\mathcal{D}(\mathsf{C}(L)) = ?$

$$\mathsf{C}(L) = \{h \in \mathsf{IC}(L) \mid h(p,—) \vee h(—,q) = 1 \text{ for every } p < q\}$$

Since $\mathsf{IC}(L)$ is Dedekind complete, it must contain the Dedekind completion of $\mathsf{C}(L)$.

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

From now on: $L =$ *completely regular frame*

(no loss of generality [Banaschewski & Hong, 2003])

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

From now on: $L =$ *completely regular frame*

(no loss of generality [Banaschewski & Hong, 2003])

Remark

$$\text{(R2) } \underbrace{h(p,\text{—}) \vee h(\text{—},q) = 1}_{\forall p<q} \quad \Rightarrow \quad \text{(R2}'\text{)} \begin{cases} h(p,\text{—})^* \leqslant h(\text{—},q) \\ h(\text{—},q)^* \leqslant h(p,\text{—}) \end{cases} \quad \forall p < q$$

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

From now on: $L =$ *completely regular frame*

(no loss of generality [Banaschewski & Hong, 2003])

Remark

(R2) $\underbrace{h(p,—) \vee h(—,q) = 1}_{\forall p<q} \quad \Rightarrow \quad$ (R2′) $\begin{cases} h(p,—)^* \leqslant h(—,q) \\ h(—,q)^* \leqslant h(p,—) \end{cases} \quad \forall p < q$

THEOREM. The Dedekind completion $\mathcal{D}(\mathsf{C}(L))$ of $\mathsf{C}(L)$ is given by

$\{h \in \mathsf{IC}(L) \mid$ (1) $\exists f, g \in \mathsf{C}(L)\colon f \leqslant h \leqslant g$

(2) $h(p,—)^* \leqslant h(—,q)$ and $h(—,q)^* \leqslant h(p,—)\ \forall p < q\}$

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

From now on: $L =$ *completely regular frame*

(no loss of generality [Banaschewski & Hong, 2003])

Remark

(R2) $\underbrace{h(p,\text{—}) \vee h(\text{—},q) = 1}_{\forall p<q} \quad \Rightarrow \quad$ (R2′) $\begin{cases} h(p,\text{—})^* \leqslant h(\text{—},q) \\ h(\text{—},q)^* \leqslant h(p,\text{—}) \end{cases} \quad \forall p < q$

If L is extremally disconnected ($a^* \vee a^{**} = 1, \ \forall a$) then (R2) $\Leftrightarrow$ (R2′).

THEOREM. The Dedekind completion $\mathcal{D}(\mathsf{C}(L))$ of $\mathsf{C}(L)$ is given by

$\{h \in \mathsf{IC}(L) \mid$ (1) $\exists f, g \in \mathsf{C}(L): f \leqslant h \leqslant g$

(2) $h(p,\text{—})^* \leqslant h(\text{—},q)$ and $h(\text{—},q)^* \leqslant h(p,\text{—}) \ \forall p < q\}$

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

From now on: $L =$ *completely regular frame*

(no loss of generality [Banaschewski & Hong, 2003])

Remark

(R2) $\underbrace{h(p, —) \vee h(—, q) = 1}_{\forall p<q}$ $\Rightarrow$ (R2′) $\begin{cases} h(p, —)^* \leqslant h(—, q) \\ h(—, q)^* \leqslant h(p, —) \end{cases}$ $\forall p < q$

If L is extremally disconnected ($a^* \vee a^{**} = 1, \ \forall a$) then (R2) $\Leftrightarrow$ (R2′).

COROLLARY. $\mathsf{C}(L)$ is Dedekind complete iff L is extr. disconnected.
[redacted]

[Banaschewski & Hong, 2003]

VARIANTS OF $C(L)$

We have similar results for

VARIANTS OF $\mathsf{C}(L)$

We have similar results for

- $\mathsf{C}^*(L)$: bounded functions ($\exists p\colon f(-p,p)=1$)

$$\mathcal{D}(\mathsf{C}^*(L)) = \mathcal{D}(\mathsf{C}(L)) \cap \mathsf{IC}^*(L)$$

VARIANTS OF $\mathsf{C}(L)$

We have similar results for

- $\mathsf{C}^*(L)$: bounded functions ($\exists p\colon f(-p,p)=1$)

$$\boxed{\mathcal{D}(\mathsf{C}^*(L)) = \mathcal{D}(\mathsf{C}(L)) \cap \mathsf{IC}^*(L)}$$

- $\mathsf{C}(L,\mathbb{Z})$: integer functions ($f(p,q) = f(\lfloor p \rfloor, \lceil q \rceil)\ \forall p,q$)

$$\boxed{\mathcal{D}(\mathsf{C}(L,\mathbb{Z})) = \mathcal{D}(\mathsf{C}(L)) \cap \mathsf{IC}(L,\mathbb{Z})}$$

VARIANTS OF $\mathsf{C}(L)$

We have similar results for

- $\mathsf{C}^*(L)$: bounded functions ($\exists p\colon f(-p,p)=1$)

$$\mathcal{D}(\mathsf{C}^*(L)) = \mathcal{D}(\mathsf{C}(L)) \cap \mathsf{IC}^*(L)$$

- $\mathsf{C}(L,\mathbb{Z})$: integer functions ($f(p,q) = f(\lfloor p \rfloor, \lceil q \rceil)\ \forall p,q$)

$$\mathcal{D}(\mathsf{C}(L,\mathbb{Z})) = \mathcal{D}(\mathsf{C}(L)) \cap \mathsf{IC}(L,\mathbb{Z})$$

- $\overline{\mathsf{C}}(L)$: extended functions

• • •

Approach II:

Semicontinuous real functions

J. GUTIÉRREZ GARCÍA, I. MOZO CAROLLO & JP
Normal semicontinuity and
the Dedekind completion of pointfree function rings, **2014, submitted.**

- Any $f : X \longrightarrow \mathbb{R}$

- Any $f : (X, \mathcal{P}(X)) \longrightarrow (\mathbb{R}, \mathfrak{T})$ is continuous.

- Any $f\colon (X,\mathcal{P}(X)) \longrightarrow (\mathbb{R},\mathfrak{T})$ is continuous.

  i.e. $F(X) \simeq \mathsf{Top}((X,\mathcal{P}(X)),(\mathbb{R},\mathfrak{T}))$

- Any $f : (X, \mathcal{P}(X)) \longrightarrow (\mathbb{R}, \mathfrak{T})$ is continuous.

  i.e. $F(X) \simeq \mathsf{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathfrak{T}))$

- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

$$F(X) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{P}(X))$$

- Any $f : (X, \mathcal{P}(X)) \longrightarrow (\mathbb{R}, \mathfrak{T})$ is continuous.

  i.e. $F(X) \simeq \mathsf{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathfrak{T}))$

- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

$$F(X) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \underbrace{\mathcal{P}(X)})$$

lattice of subspaces of X

- Any $f: (X, \mathcal{P}(X)) \longrightarrow (\mathbb{R}, \mathfrak{T})$ is continuous.

  i.e. $F(X) \simeq \mathsf{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathfrak{T}))$

- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

$$F(X) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \underbrace{\mathcal{P}(X)})$$ lattice of subspaces of X

$$\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \underbrace{\mathcal{S}(L)})$$ lattice of sublocales of L

- Any $f: (X, \mathcal{P}(X)) \longrightarrow (\mathbb{R}, \mathfrak{T})$ is continuous.

  i.e. $F(X) \simeq \mathsf{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \mathfrak{T}))$

- From the (dual) adjunction $\mathbf{Top} \underset{\Sigma}{\overset{\mathfrak{O}}{\rightleftarrows}} \mathbf{Frm}$

$$F(X) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \underbrace{\mathcal{P}(X)}_{\text{lattice of subspaces of } X})$$

$$\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \underbrace{\mathcal{S}(L)}_{\text{lattice of sublocales of } L})$$

Natural extension:

$$\boxed{\mathsf{F}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))}$$

BUT in the pointfree setting the situation is somewhat distinct:

BUT in the pointfree setting the situation is somewhat distinct:

- In spaces, $\mathcal{P}(X)$ is e.d. and thus $\mathsf{F}(X)$ is Dedekind complete.

  So one may find $\mathcal{D}(\mathsf{C}(X))$ inside $\mathsf{F}(X)$

  (classical approach of [Dilworth, Horne]).

BUT in the pointfree setting the situation is somewhat distinct:

- In spaces, $\mathcal{P}(X)$ is e.d. and thus $\mathsf{F}(X)$ is Dedekind complete.

  So one may find $\mathcal{D}(\mathsf{C}(X))$ inside $\mathsf{F}(X)$

  (classical approach of [Dilworth, Horne]).

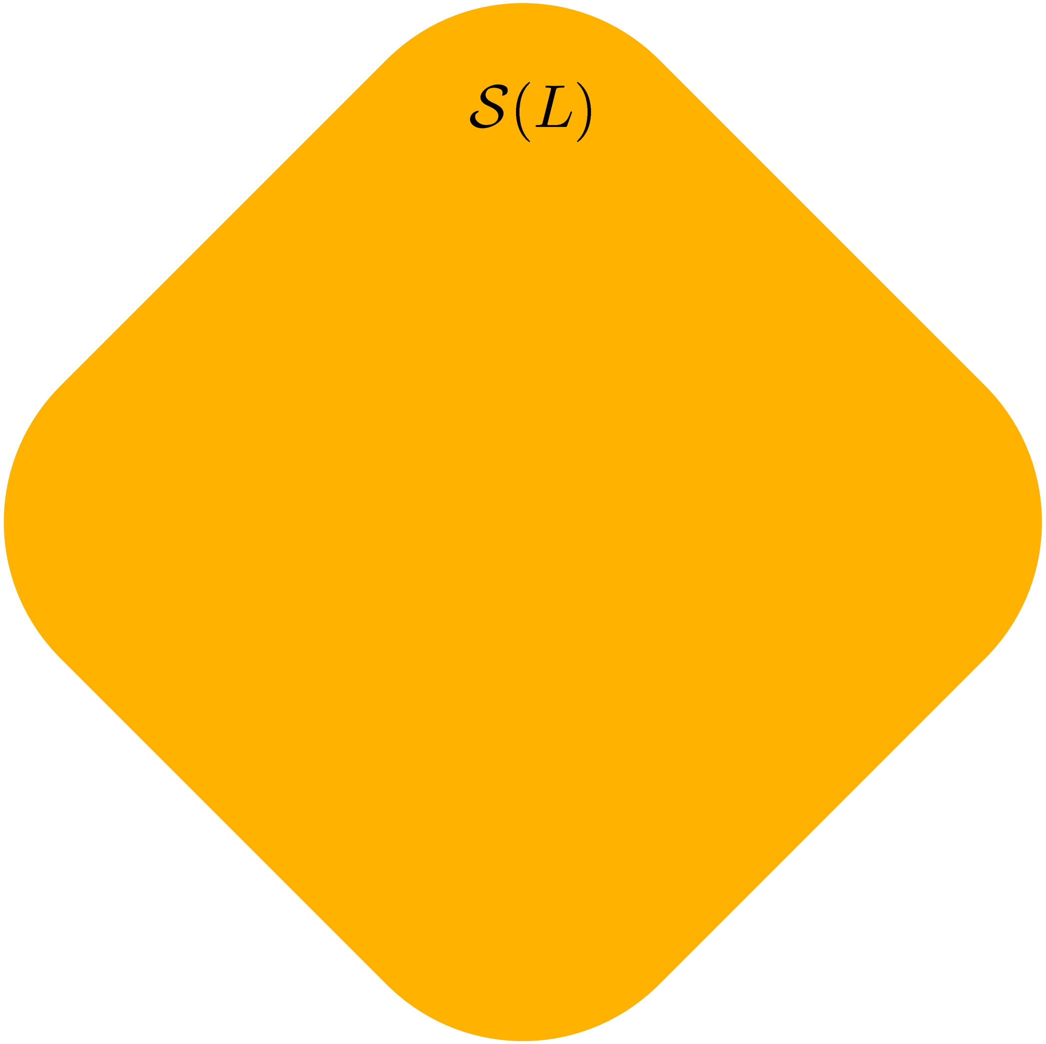
- $\mathcal{S}(L)$ is NOT e.d. in general and so

  $\mathsf{F}(L)$ is NOT necessarily Dedekind complete...

It is a co-frame!

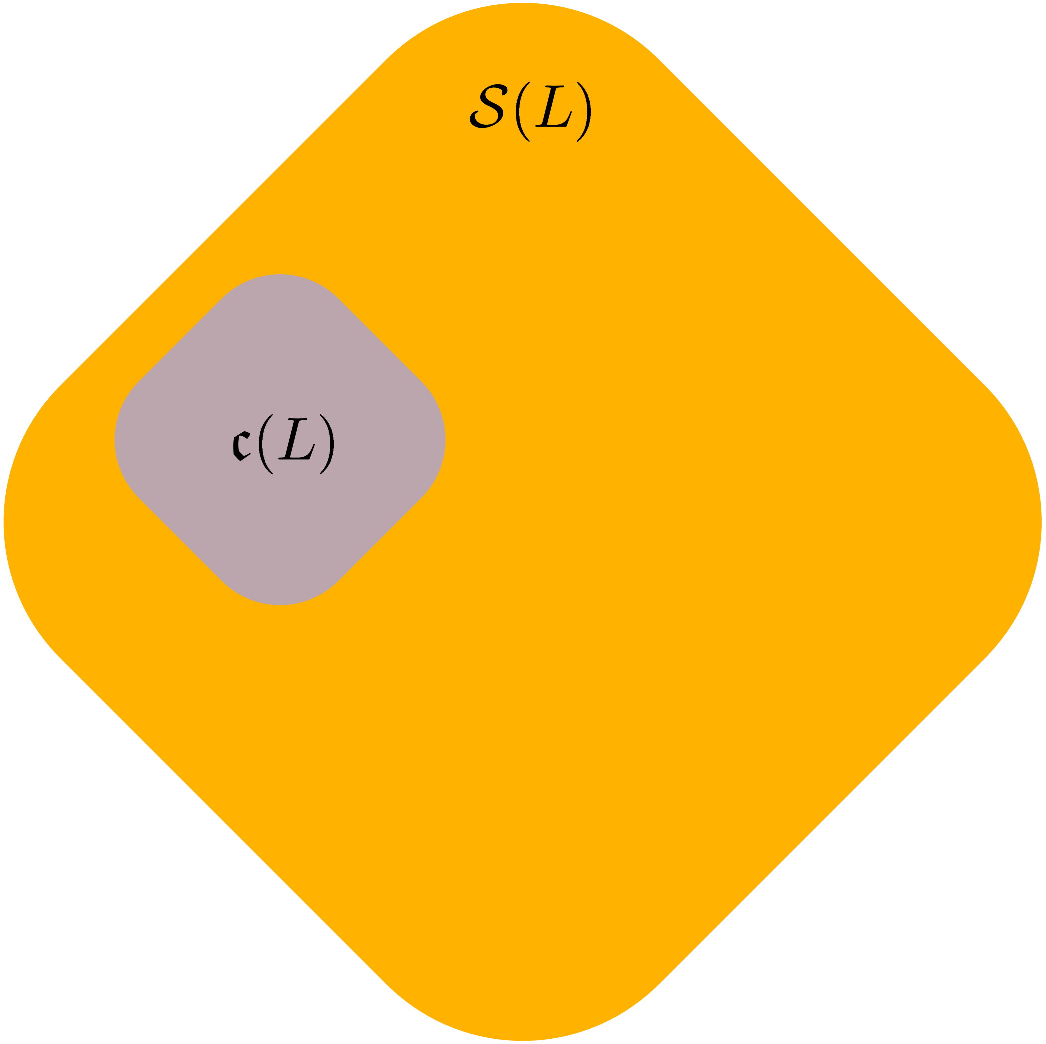
It is a co-frame!

$\mathcal{S}(L)$: the DUAL FRAME



It is a co-frame!

$\mathcal{S}(L)$: the DUAL FRAME

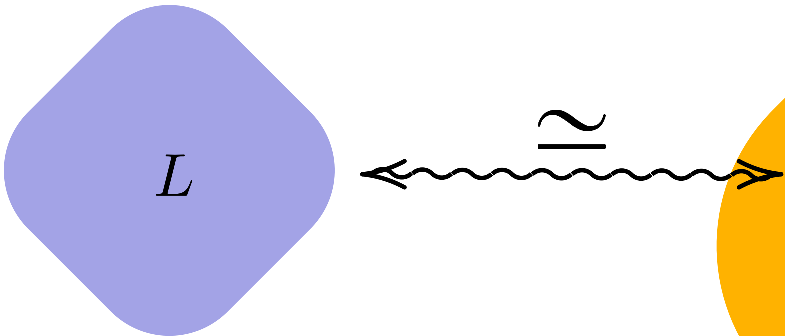


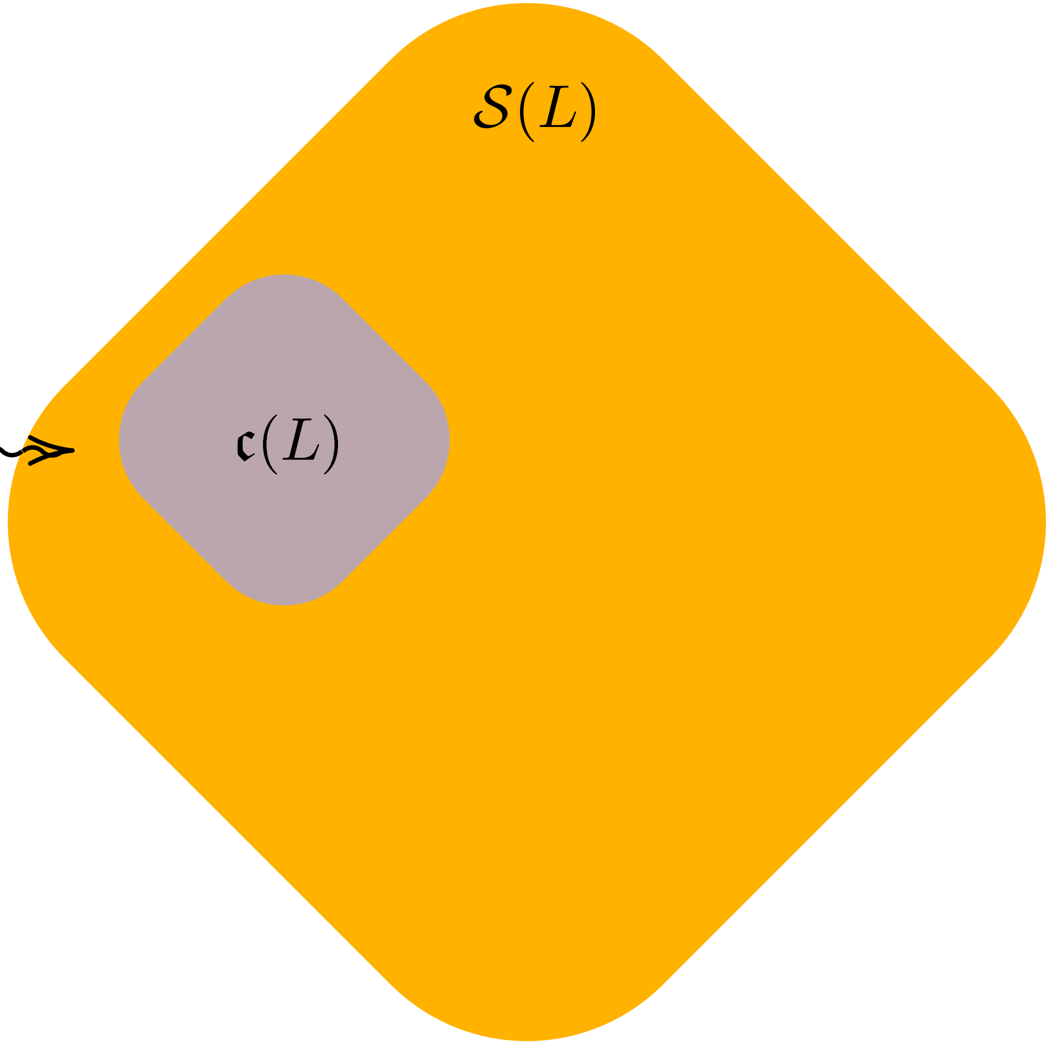
$$\mathfrak{c}(L) := \{\mathfrak{c}(a) \mid a \in L\}$$

Background: THE SUBLOCALE LATTICE

It is a co-frame!

$\mathcal{S}(L)$: the DUAL FRAME





$$\mathfrak{c}(L) := \{\mathfrak{c}(a) \mid a \in L\}$$

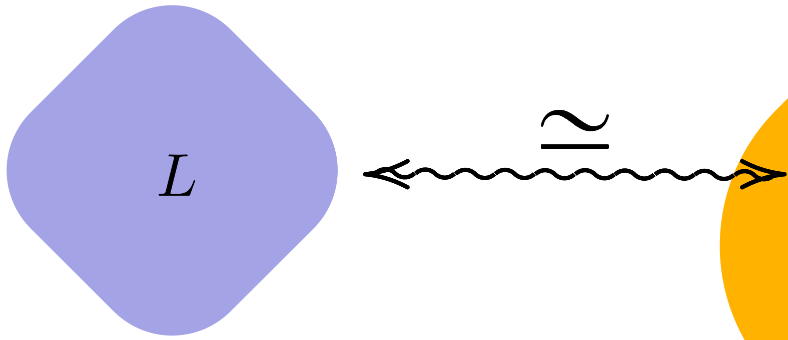
$$\bigvee_{i \in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i \in I} a_i)$$

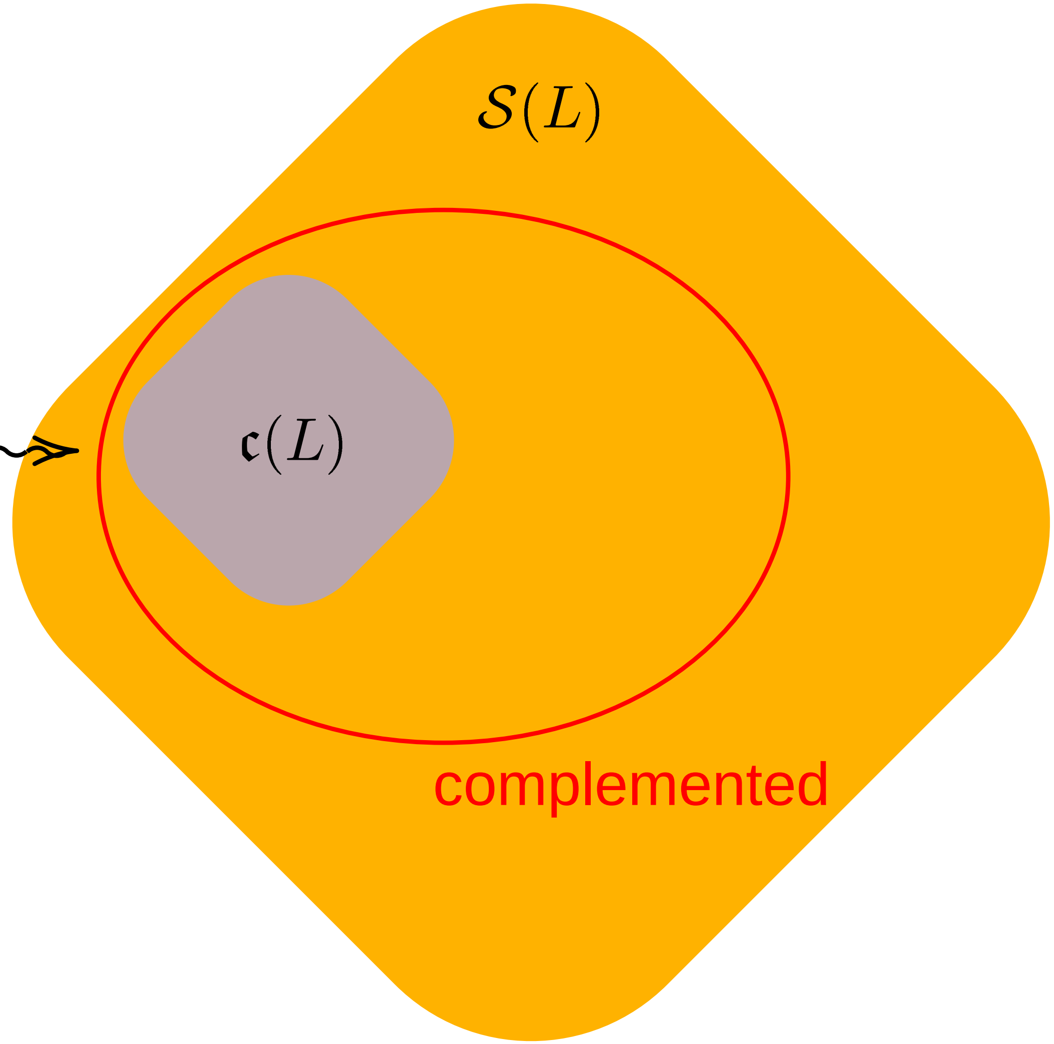
$$\mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$$

Background: THE SUBLOCALE LATTICE

It is a co-frame!

$\mathcal{S}(L)$: the DUAL FRAME





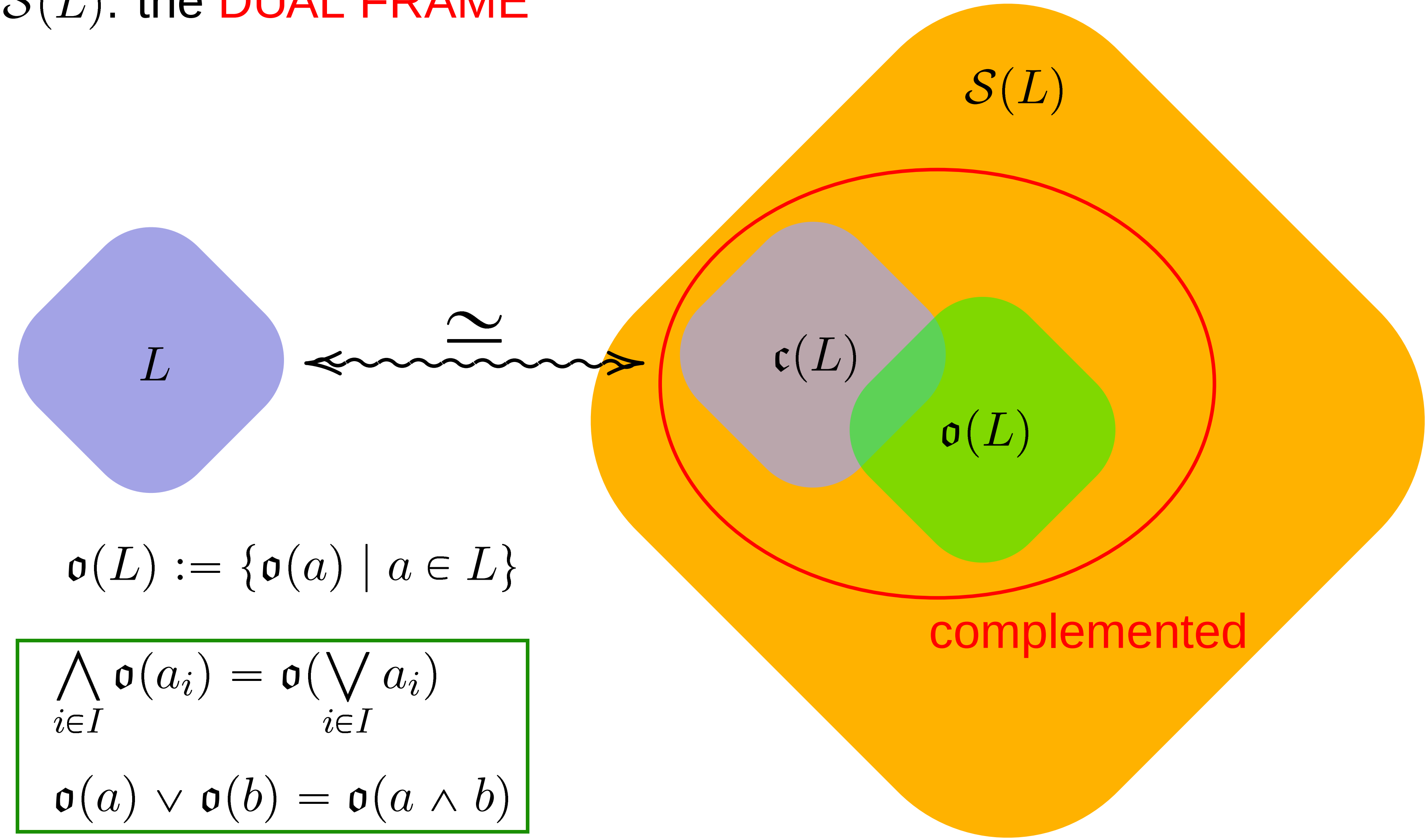
$$\mathfrak{c}(L) := \{\mathfrak{c}(a) \mid a \in L\}$$

$$\bigvee_{i \in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i \in I} a_i)$$

$$\mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$$

It is a co-frame!

$\mathcal{S}(L)$: the DUAL FRAME



$$\mathfrak{o}(L) := \{\mathfrak{o}(a) \mid a \in L\}$$

$$\bigwedge_{i \in I} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i \in I} a_i)$$

$$\mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$$

(GENERAL) REAL FUNCTIONS POINTFREELY

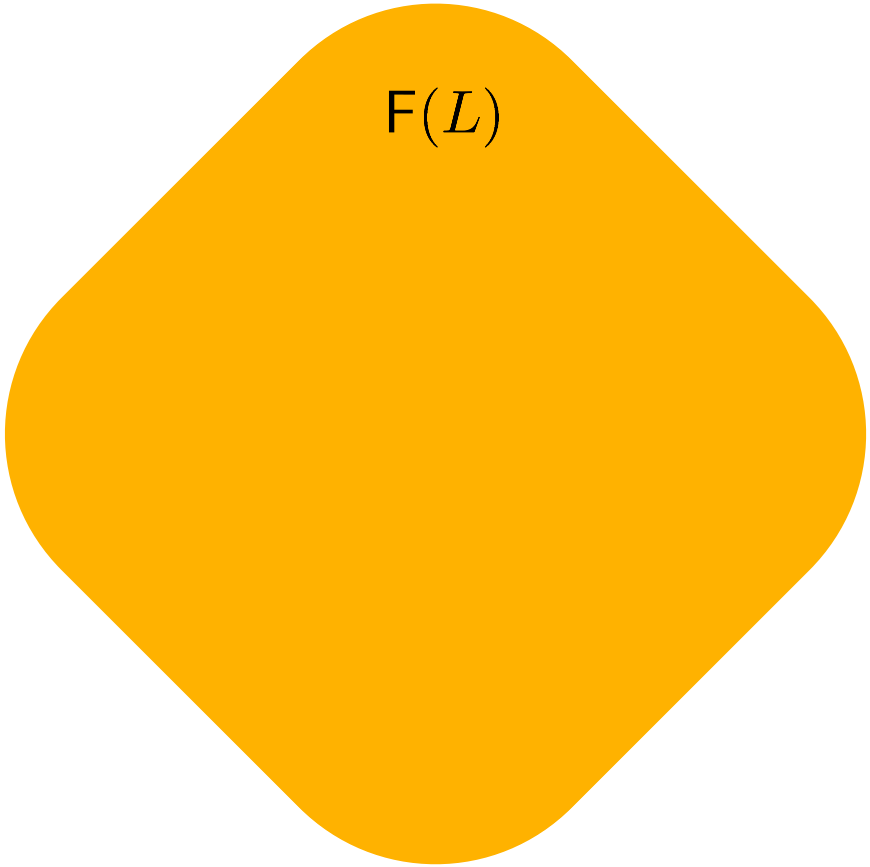
$f\colon \mathfrak{L}(\mathbb{R}) \longrightarrow \mathcal{S}(L)$

$\mathcal{S}(L)$ is rich enough to allow to segregate the specific classes of real functions we are interested with in a satisfactory manner:

(GENERAL) REAL FUNCTIONS POINTFREELY

$f\colon \mathfrak{L}(\mathbb{R}) \longrightarrow \mathcal{S}(L)$

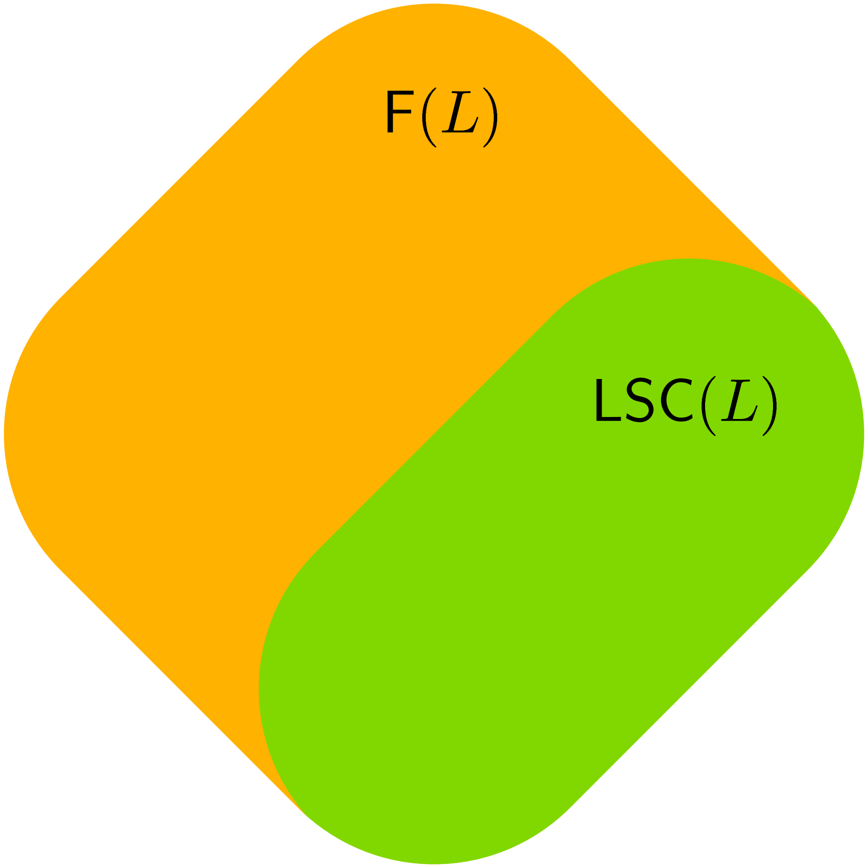
$\mathcal{S}(L)$ is rich enough to allow to segregate the specific classes of real functions we are interested with in a satisfactory manner:



(GENERAL) REAL FUNCTIONS POINTFREELY

$f\colon \mathfrak{L}(\mathbb{R}) \longrightarrow \mathcal{S}(L)$

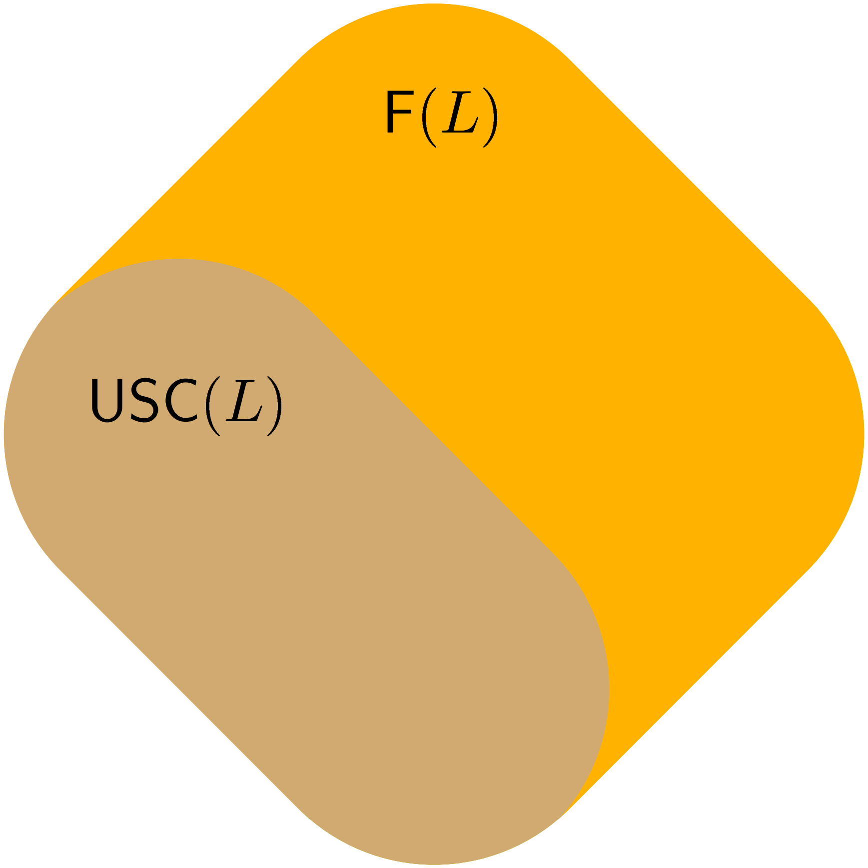
$\mathcal{S}(L)$ is rich enough to allow to segregate the specific classes of real functions we are interested with in a satisfactory manner:



$f(p, —) \in \mathfrak{c}(L)$

$f\colon \mathfrak{L}(\mathbb{R}) \longrightarrow \mathcal{S}(L)$

$\mathcal{S}(L)$ is rich enough to allow to segregate the specific classes of real functions we are interested with in a satisfactory manner:



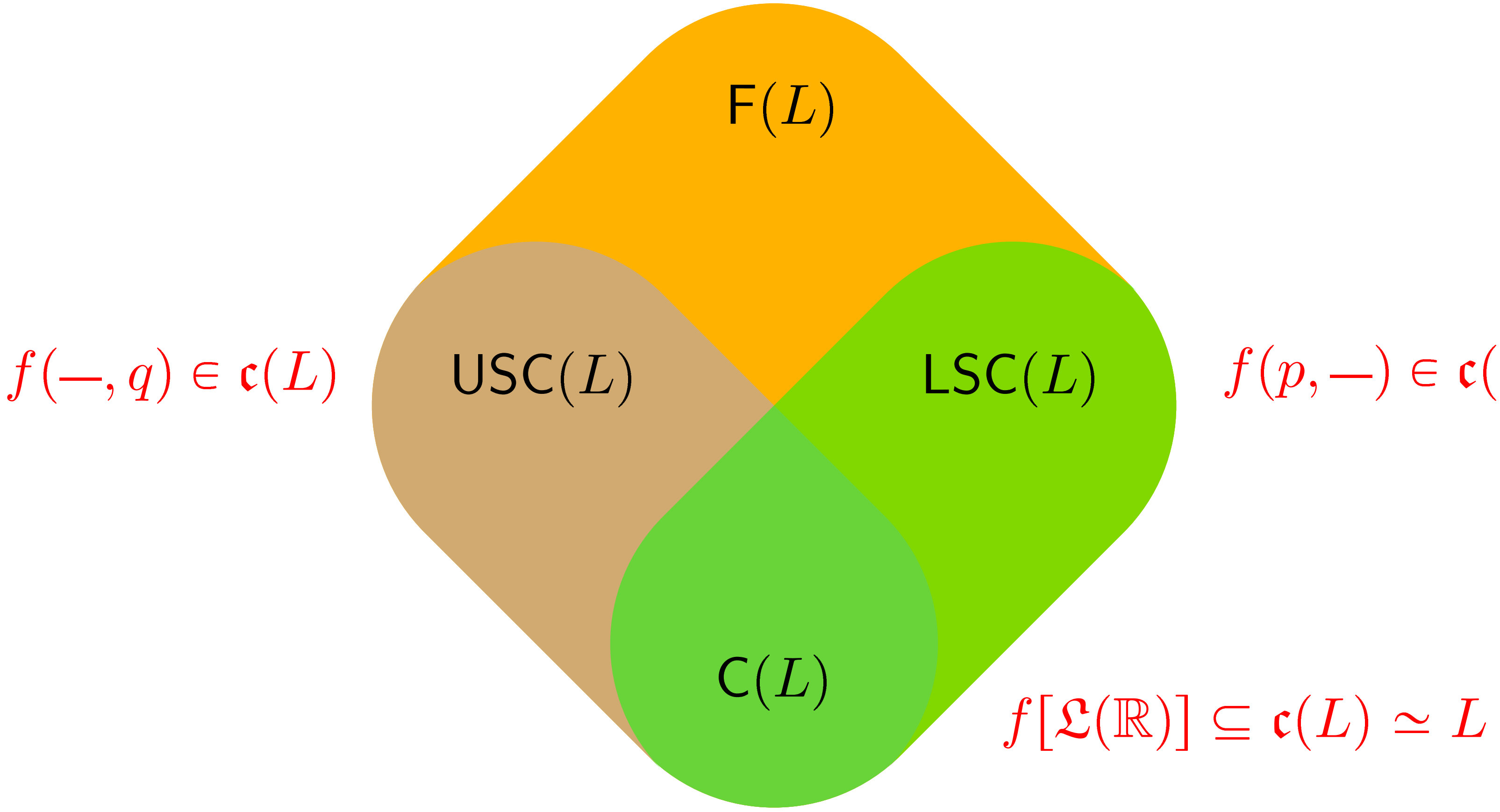
$f(-, q) \in \mathfrak{c}(L)$

$f(p, -) \in \mathfrak{c}(L)$

(GENERAL) REAL FUNCTIONS POINTFREELY

$f\colon \mathfrak{L}(\mathbb{R}) \longrightarrow \mathcal{S}(L)$

$\mathcal{S}(L)$ is rich enough to allow to segregate the specific classes of real functions we are interested with in a satisfactory manner:

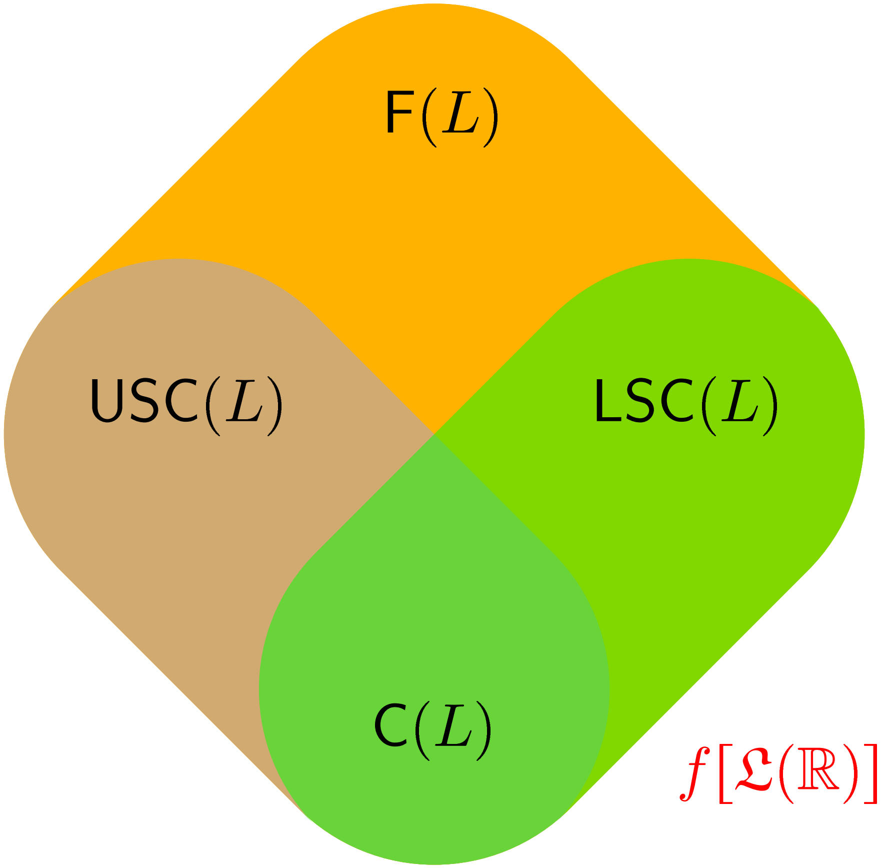


(GENERAL) REAL FUNCTIONS POINTFREELY

$f\colon \mathfrak{L}(\mathbb{R}) \longrightarrow \mathcal{S}(L)$

$\mathcal{S}(L)$ is rich enough to allow to segregate the specific classes of real functions we are interested with in a satisfactory manner:

$f(-, q) \in \mathfrak{c}(L)$



$f(p, -) \in \mathfrak{c}(L)$

$f[\mathfrak{L}(\mathbb{R})] \subseteq \mathfrak{c}(L) \simeq L$

J. GUTIÉRREZ GARCÍA, T. KUBIAK & J. P.
Localic real functions: a general setting, *J. Pure Appl. Algebra* **213 (2009) 1064-1074**

REGULARIZATIONS OF A REAL FUNCTION

$f \in \mathsf{F}(L)$

- lower regularization f°

REGULARIZATIONS OF A REAL FUNCTION

$f \in \mathsf{F}(L)$

- lower regularization f°

$$f^\circ(p,\text{—}) = \bigvee_{r>p} \overline{f(r,\text{—})}$$

$$f^\circ(\text{—},q) = \bigvee_{s<q} \overline{f(s,\text{—})}^{\,*}$$

REGULARIZATIONS OF A REAL FUNCTION

$f \in \mathsf{F}(L)$

- lower regularization f°

$$f^\circ(p,—) = \bigvee_{r>p} \overline{f(r,—)}$$

$$f^\circ(—,q) = \bigvee_{s<q} \overline{f(s,—)}^{\,*}$$

Then: $f^\circ \in \overline{\mathsf{LSC}}(L)$

REGULARIZATIONS OF A REAL FUNCTION

$f \in \mathsf{F}(L)$

- lower regularization f°

$$f^\circ(p,-) = \bigvee_{r>p} \overline{f(r,-)}$$

$$f^\circ(-,q) = \bigvee_{s<q} \overline{f(s,-)}^{\,*}$$

Then: $f^\circ \in \overline{\mathsf{LSC}}(L)$, $f^\circ \leqslant f$

REGULARIZATIONS OF A REAL FUNCTION

$f \in \mathsf{F}(L)$

- lower regularization f°

$$f^\circ(p, —) = \bigvee_{r>p} \overline{f(r, —)}$$

$$f^\circ(—, q) = \bigvee_{s<q} \overline{f(s, —)}^{\,*}$$

Then: $f^\circ \in \overline{\mathsf{LSC}}(L)$, $f^\circ \leqslant f$, $f^\circ = \bigvee\{g \in \overline{\mathsf{LSC}}(L) \mid g \leqslant f\}$.

REGULARIZATIONS OF A REAL FUNCTION

$f \in \mathsf{F}(L)$

- lower regularization f°

$$f^\circ(p,—) = \bigvee_{r>p} \overline{f(r,—)}$$

$$f^\circ(—,q) = \bigvee_{s<q} \overline{f(s,—)}^{\,*}$$

Then: $f^\circ \in \overline{\mathsf{LSC}}(L)$, $f^\circ \leqslant f$, $f^\circ = \bigvee\{g \in \overline{\mathsf{LSC}}(L) \mid g \leqslant f\}$.

If $\{g \in \mathsf{LSC}(L) \mid g \leqslant f\} \neq \varnothing$ then $f^\circ \in \mathsf{LSC}(L)$.

REGULARIZATIONS OF A REAL FUNCTION

$f \in \mathsf{F}(L)$

- lower regularization f°

$$f^\circ(p, -) = \bigvee_{r>p} \overline{f(r, -)}$$

$$f^\circ(-, q) = \bigvee_{s<q} \overline{f(s, -)}^{\,*}$$

Then: $f^\circ \in \overline{\mathsf{LSC}}(L)$, $f^\circ \leqslant f$, $f^\circ = \bigvee\{g \in \overline{\mathsf{LSC}}(L) \mid g \leqslant f\}$.

If $\{g \in \mathsf{LSC}(L) \mid g \leqslant f\} \neq \varnothing$ then $f^\circ \in \mathsf{LSC}(L)$.

- Dually: the upper regularization $f^- = -(-f)^\circ$

J. GUTIÉRREZ GARCÍA, T. KUBIAK & J. P.
Lower and upper regularizations of frame semicontinuous real functions, *Alg. Univ.* (2009)

VARIANTS OF BOUNDEDNESS

$f \in \mathsf{F}(L)$

- BOUNDED: $\exists\, p < q \colon f(p, —) = 1 = f(—, q)$ $\mathsf{F}^*(L)$

$\Leftrightarrow \quad \exists\, p < q \colon \boldsymbol{p} \leqslant f \leqslant \boldsymbol{q}$

VARIANTS OF BOUNDEDNESS

$f \in \mathsf{F}(L)$

- CONTINUOUSLY BOUNDED: $\exists\, h_1, h_2 \in \mathsf{C}(L) \colon h_1 \leqslant f \leqslant h_2$ $\quad \mathsf{F}^{cb}(L)$

- BOUNDED: $\exists\, p < q \colon f(p, \text{—}) = 1 = f(\text{—}, q)$ $\quad \mathsf{F}^{*}(L)$

  $\Leftrightarrow \quad \exists\, p < q \colon \boldsymbol{p} \leqslant f \leqslant \boldsymbol{q}$

VARIANTS OF BOUNDEDNESS

$f \in \mathsf{F}(L)$

- CONTINUOUSLY BOUNDED: $\exists\, h_1, h_2 \in \mathsf{C}(L) \colon h_1 \leqslant f \leqslant h_2$ $\quad \mathsf{F}^{cb}(L)$

$$\cup\!\shortmid$$

- BOUNDED: $\exists\, p < q \colon f(p, —) = 1 = f(—, q)$ $\quad \mathsf{F}^*(L)$

$$\Leftrightarrow \quad \exists\, p < q \colon \boldsymbol{p} \leqslant f \leqslant \boldsymbol{q}$$

VARIANTS OF BOUNDEDNESS

$f \in \mathsf{F}(L)$

- LOCALLY BOUNDED: $\bigvee_{p\in\mathbb{Q}} \overline{f(p,\text{—})} = 1 = \bigvee_{q\in\mathbb{Q}} \overline{f(\text{—},q)}$ $\quad\mathsf{F}^{lb}(L)$

- CONTINUOUSLY BOUNDED: $\exists\, h_1, h_2 \in \mathsf{C}(L) \colon h_1 \leqslant f \leqslant h_2$ $\quad\mathsf{F}^{cb}(L)$

$\cup\!\!|$

- BOUNDED: $\exists\, p < q \colon f(p,\text{—}) = 1 = f(\text{—},q)$ $\quad\mathsf{F}^{*}(L)$

$\Leftrightarrow \quad \exists\, p < q \colon \boldsymbol{p} \leqslant f \leqslant \boldsymbol{q}$

VARIANTS OF BOUNDEDNESS

$f \in \mathsf{F}(L)$

$\bigcup\!|$

- LOCALLY BOUNDED: $\bigvee_{p\in\mathbb{Q}} \overline{f(p,—)} = 1 = \bigvee_{q\in\mathbb{Q}} \overline{f(—,q)}$ $\mathsf{F}^{lb}(L)$

$\bigcup\!|$

- CONTINUOUSLY BOUNDED: $\exists\, h_1, h_2 \in \mathsf{C}(L) \colon h_1 \leqslant f \leqslant h_2$ $\mathsf{F}^{cb}(L)$

$\bigcup\!|$

- BOUNDED: $\exists\, p < q \colon f(p,—) = 1 = f(—,q)$ $\mathsf{F}^{*}(L)$

$$\Leftrightarrow \quad \exists\, p < q \colon \boldsymbol{p} \leqslant f \leqslant \boldsymbol{q}$$

VARIANTS OF BOUNDEDNESS

$f \in \mathsf{F}(L)$

$f^\circ, f^- \in \mathsf{F}(L)$ (and thus also belong to $\mathsf{F}^{lb}(L)$) $\quad \bigcup\!\mid$

$\Updownarrow$

- LOCALLY BOUNDED: $\bigvee_{p\in\mathbb{Q}} \overline{f(p,—)} = 1 = \bigvee_{q\in\mathbb{Q}} \overline{f(—,q)}$ $\quad \mathsf{F}^{lb}(L)$

$\bigcup\!\mid$

- CONTINUOUSLY BOUNDED: $\exists\, h_1, h_2 \in \mathsf{C}(L) \colon h_1 \leqslant f \leqslant h_2$ $\quad \mathsf{F}^{cb}(L)$

$\bigcup\!\mid$

- BOUNDED: $\exists\, p < q \colon f(p,—) = 1 = f(—,q)$ $\quad \mathsf{F}^*(L)$

$\Leftrightarrow \quad \exists\, p < q \colon \boldsymbol{p} \leqslant f \leqslant \boldsymbol{q}$

VARIANTS OF BOUNDEDNESS

$f \in \mathsf{F}(L)$

$f^\circ, f^- \in \mathsf{F}(L)$ (and thus also belong to $\mathsf{F}^{lb}(L)$) $\quad \bigcup|$

$\Updownarrow$

- LOCALLY BOUNDED: $\bigvee_{p\in\mathbb{Q}} \overline{f(p,\text{—})} = 1 = \bigvee_{q\in\mathbb{Q}} \overline{f(\text{—},q)}$ $\quad \mathsf{F}^{lb}(L)$

L is a cb-frame iff $\quad ||$

- CONTINUOUSLY BOUNDED: $\exists\, h_1, h_2 \in \mathsf{C}(L) \colon h_1 \leqslant f \leqslant h_2$ $\quad \mathsf{F}^{cb}(L)$

$\bigcup|$

- BOUNDED: $\exists\, p < q \colon f(p,\text{—}) = 1 = f(\text{—},q)$ $\quad \mathsf{F}^*(L)$

$\Leftrightarrow \quad \exists\, p < q \colon \boldsymbol{p} \leqslant f \leqslant \boldsymbol{q}$

VARIANTS OF BOUNDEDNESS

$f \in \mathsf{F}(L)$

$f^\circ, f^- \in \mathsf{F}(L)$ (and thus also belong to $\mathsf{F}^{lb}(L)$) $\quad \bigcup\!|$

$\Updownarrow$

- LOCALLY BOUNDED: $\bigvee_{p\in\mathbb{Q}} \overline{f(p,\text{—})} = 1 = \bigvee_{q\in\mathbb{Q}} \overline{f(\text{—},q)}$ $\quad \mathsf{F}^{lb}(L)$

L is a cb-frame iff $\quad ||$

(every $f \in \mathsf{F}^{lb}(L)$ is bounded above by a continuous g)

- CONTINUOUSLY BOUNDED: $\exists\, h_1, h_2 \in \mathsf{C}(L) \colon h_1 \leqslant f \leqslant h_2$ $\quad \mathsf{F}^{cb}(L)$

$\bigcup\!|$

- BOUNDED: $\exists\, p < q \colon f(p,\text{—}) = 1 = f(\text{—},q)$ $\quad \mathsf{F}^*(L)$

$\Leftrightarrow \quad \exists\, p < q \colon \boldsymbol{p} \leqslant f \leqslant \boldsymbol{q}$

NORMAL FUNCTIONS

[classically: Dilworth, 1950]

$f \in \mathsf{F}(L)$ is normal lsc if $f^- \in \mathsf{F}(L)$ and $(f^-)^\circ = f$. $\mathsf{NLSC}(L)$

NORMAL FUNCTIONS

[classically: Dilworth, 1950]

$f \in \mathsf{F}(L)$ is normal lsc if $f^- \in \mathsf{F}(L)$ and $(f^-)^\circ = f$. $\mathsf{NLSC}(L)$

normal usc if $f^\circ \in \mathsf{F}(L)$ and $(f^\circ)^- = f$. $\mathsf{NUSC}(L)$

NORMAL FUNCTIONS

[classically: Dilworth, 1950]

$f \in \mathsf{F}(L)$ is normal lsc if $f^- \in \mathsf{F}(L)$ and $(f^-)^\circ = f$. $\mathsf{NLSC}(L)$

normal usc if $f^\circ \in \mathsf{F}(L)$ and $(f^\circ)^- = f$. $\mathsf{NUSC}(L)$

Important role here:

Weak cb-frames: every $f \in \mathsf{LSC}^{lb}(L)$ is bounded above by a continuous g

NORMAL FUNCTIONS

[classically: Dilworth, 1950]

$f \in \mathsf{F}(L)$ is normal lsc if $f^- \in \mathsf{F}(L)$ and $(f^-)^\circ = f$. NLSC(L)

normal usc if $f^\circ \in \mathsf{F}(L)$ and $(f^\circ)^- = f$. NUSC(L)

Important role here:

Weak cb-frames: every $f \in \mathsf{LSC}^{lb}(L)$ is bounded above by a continuous g

$$\Leftrightarrow \mathsf{LSC}^{cb}(L) = \mathsf{LSC}^{lb}(L) \Leftrightarrow \mathsf{USC}^{cb}(L) = \mathsf{USC}^{lb}(L) \cdots$$

NORMAL FUNCTIONS

[classically: Dilworth, 1950]

$f \in \mathsf{F}(L)$ is normal lsc if $f^- \in \mathsf{F}(L)$ and $(f^-)^\circ = f$. $\mathsf{NLSC}(L)$

normal usc if $f^\circ \in \mathsf{F}(L)$ and $(f^\circ)^- = f$. $\mathsf{NUSC}(L)$

Important role here:

Weak cb-frames: every $f \in \mathsf{LSC}^{lb}(L)$ is bounded above by a continuous g

$$\Leftrightarrow \mathsf{LSC}^{cb}(L) = \mathsf{LSC}^{lb}(L) \Leftrightarrow \mathsf{USC}^{cb}(L) = \mathsf{USC}^{lb}(L) \cdots$$

PROPOSITION. TFAE for a frame L:

(1) L is weak cb.

(2) $\mathsf{NLSC}^{cb}(L) = \mathsf{NLSC}^{lb}(L) = \mathsf{NLSC}(L)$.

(3) $\mathsf{NUSC}^{cb}(L) = \mathsf{NUSC}^{lb}(L) = \mathsf{NUSC}(L)$.

NORMAL FUNCTIONS

[classically: Dilworth, 1950]

$f \in \mathsf{F}(L)$ is normal lsc if $f^- \in \mathsf{F}(L)$ and $(f^-)^\circ = f$. $\quad$ $\mathsf{NLSC}(L)$

normal usc if $f^\circ \in \mathsf{F}(L)$ and $(f^\circ)^- = f$. $\quad$ $\mathsf{NUSC}(L)$

Important role here:

Weak cb-frames: every $f \in \mathsf{LSC}^{lb}(L)$ is bounded above by a continuous g

$$\Leftrightarrow \mathsf{LSC}^{cb}(L) = \mathsf{LSC}^{lb}(L) \Leftrightarrow \mathsf{USC}^{cb}(L) = \mathsf{USC}^{lb}(L) \cdots$$

PROPOSITION. TFAE for a frame L:

(1) L is extremally disconnected.

(2) $\mathsf{NLSC}(L) = \mathsf{C}(L)$.

(3) $\mathsf{NUSC}(L) = \mathsf{C}(L)$.

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

Note: for any directed poset P with no $\bot$, $\mathcal{D}(P) = \{\mathcal{A} \subseteq P \mid \mathcal{A}^{ul} = \mathcal{A},\ \varnothing \neq \mathcal{A} \neq P\}$.

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

Note: for any directed poset P with no $\perp$, $\mathcal{D}(P) = \{\mathcal{A} \subseteq P \mid \mathcal{A}^{ul} = \mathcal{A},\ \varnothing \neq \mathcal{A} \neq P\}$.

- $\mathcal{A} \in \mathcal{D}(\mathsf{C}(L)) \rightsquigarrow \left(\bigvee^{\overline{\mathsf{F}}(L)} \mathcal{A} \right)^{-\circ} \in \mathsf{NLSC}^{cb}(L)$

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

Note: for any directed poset P with no $\bot$, $\mathcal{D}(P) = \{\mathcal{A} \subseteq P \mid \mathcal{A}^{ul} = \mathcal{A},\ \varnothing \neq \mathcal{A} \neq P\}$.

- $\mathcal{A} \in \mathcal{D}(\mathsf{C}(L)) \quad \rightsquigarrow \quad \Big(\bigvee^{\overline{\mathsf{F}}(L)} \mathcal{A}\Big)^{-\circ} \in \mathsf{NLSC}^{cb}(L)$
- $\{g \in \mathsf{C}(L) \mid g \geqslant f\} \in \mathcal{D}(\mathsf{C}(L)) \quad \leftsquigarrow \quad f \in \mathsf{NLSC}^{cb}(L)$

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

Note: for any directed poset

P with no $\perp$, $\mathcal{D}(P) = \{\mathcal{A} \subseteq P \mid \mathcal{A}^{ul} = \mathcal{A},\ \varnothing \neq \mathcal{A} \neq P\}$.

- $\mathcal{A} \in \mathcal{D}(\mathsf{C}(L)) \rightsquigarrow \Big(\bigvee^{\overline{\mathsf{F}}(L)} \mathcal{A} \Big)^{-\circ} \in \mathsf{NLSC}^{cb}(L)$
- $\{g \in \mathsf{C}(L) \mid g \geqslant f\} \in \mathcal{D}(\mathsf{C}(L)) \leftsquigarrow f \in \mathsf{NLSC}^{cb}(L)$

$\simeq$

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

Note: for any directed poset

P with no $\bot$, $\mathcal{D}(P) = \{\mathcal{A} \subseteq P \mid \mathcal{A}^{ul} = \mathcal{A},\ \varnothing \neq \mathcal{A} \neq P\}$.

- $\mathcal{A} \in \mathcal{D}(\mathsf{C}(L)) \rightsquigarrow \left(\bigvee^{\overline{\mathsf{F}}(L)} \mathcal{A}\right)^{-\circ} \in \mathsf{NLSC}^{cb}(L)$
- $\{g \in \mathsf{C}(L) \mid g \geqslant f\} \in \mathcal{D}(\mathsf{C}(L)) \leftsquigarrow f \in \mathsf{NLSC}^{cb}(L)$

$\simeq$

THEOREM. Let L be a completely regular frame. Then:

(1) $\mathcal{D}(\mathsf{C}(L)) \simeq \mathsf{NLSC}^{cb}(L)$. (2) $\mathcal{D}(\mathsf{C}^*(L)) \simeq \mathsf{NLSC}^*(L)$.

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

Note: for any directed poset

P with no $\perp$, $\mathcal{D}(P) = \{\mathcal{A} \subseteq P \mid \mathcal{A}^{ul} = \mathcal{A},\ \emptyset \neq \mathcal{A} \neq P\}$.

- $\mathcal{A} \in \mathcal{D}(\mathsf{C}(L))$ $\rightsquigarrow$ $\left(\bigvee^{\overline{\mathsf{F}}(L)} \mathcal{A}\right)^{-\circ} \in \mathsf{NLSC}^{cb}(L)$
- $\{g \in \mathsf{C}(L) \mid g \geqslant f\} \in \mathcal{D}(\mathsf{C}(L))$ $\leftsquigarrow$ $f \in \mathsf{NLSC}^{cb}(L)$

$\simeq$

THEOREM. Let L be a completely regular frame. Then:

(1) $\mathcal{D}(\mathsf{C}(L)) \simeq \mathsf{NLSC}^{cb}(L)$. (2) $\mathcal{D}(\mathsf{C}^*(L)) \simeq \mathsf{NLSC}^*(L)$.

COROLLARIES.

(1) If L is weak cb then $\mathcal{D}(\mathsf{C}(L)) \simeq \mathsf{NLSC}(L)$.

THE DEDEKIND COMPLETION OF $\mathsf{C}(L)$

Note: for any directed poset

P with no $\perp$, $\mathcal{D}(P) = \{\mathcal{A} \subseteq P \mid \mathcal{A}^{ul} = \mathcal{A},\ \varnothing \neq \mathcal{A} \neq P\}$.

- $\mathcal{A} \in \mathcal{D}(\mathsf{C}(L)) \rightsquigarrow \Big(\bigvee^{\overline{\mathsf{F}}(L)} \mathcal{A}\Big)^{-\circ} \in \mathsf{NLSC}^{cb}(L)$
- $\{g \in \mathsf{C}(L) \mid g \geqslant f\} \in \mathcal{D}(\mathsf{C}(L)) \leftsquigarrow f \in \mathsf{NLSC}^{cb}(L)$

$\simeq$

THEOREM. Let L be a completely regular frame. Then:

(1) $\mathcal{D}(\mathsf{C}(L)) \simeq \mathsf{NLSC}^{cb}(L)$. (2) $\mathcal{D}(\mathsf{C}^*(L)) \simeq \mathsf{NLSC}^*(L)$.

COROLLARIES.

(1) If L is weak cb then $\mathcal{D}(\mathsf{C}(L)) \simeq \mathsf{NLSC}(L)$.

(2) L is extremally disconnected iff $\mathsf{C}(L)$ is Dedekind complete.

THE COMPLETION AS A FUNCTION RING

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$?

THE COMPLETION AS A FUNCTION RING

bounded case

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$?

YES!

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$? YES!

$M = \mathfrak{B}(L) = \{a \in L \mid a^{**} = a\}$

Booleanization of L

(the largest dense quotient of L)

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$? YES!

$M = \mathfrak{B}(L) = \{a \in L \mid a^{**} = a\}$

Booleanization of L

(the largest dense quotient of L)

- $f \in \mathsf{NLSC}^*(L) \rightsquigarrow \varphi_f \in \mathsf{C}^*(\mathfrak{B}(L))$:

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$? YES!

$M = \mathfrak{B}(L) = \{a \in L \mid a^{**} = a\}$

Booleanization of L

(the largest dense quotient of L)

- $f \in \mathsf{NLSC}^*(L) \rightsquigarrow \varphi_f \in \mathsf{C}^*(\mathfrak{B}(L))$:

$$\varphi_f(p, -) = \bigvee_{r>p}^{\mathfrak{B}(L)} \left(\bigwedge f(r, -)\right)^{**}$$

$$\varphi_f(-, q) = \bigvee_{s<q}^{\mathfrak{B}(L)} \left(\bigwedge f(s, -)\right)^{*}$$

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$? YES!

$M = \mathfrak{B}(L) = \{a \in L \mid a^{**} = a\}$

Booleanization of L

(the largest dense quotient of L)

- $f \in \mathsf{NLSC}^*(L) \rightsquigarrow \varphi_f \in \mathsf{C}^*(\mathfrak{B}(L))$:

$$\varphi_f(p, \text{—}) = \bigvee_{r>p}^{\mathfrak{B}(L)} \left(\bigwedge f(r, \text{—})\right)^{**}$$

$$\varphi_f(\text{—}, q) = \bigvee_{s<q}^{\mathfrak{B}(L)} \left(\bigwedge f(s, \text{—})\right)^{*}$$

THEOREM. Let L be a completely regular frame. Then

$$\mathcal{D}(\mathsf{C}^*(L)) \simeq \mathsf{C}^*(\mathfrak{B}(L)).$$

THE COMPLETION AS A FUNCTION RING

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$?

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$?

NO! in general (counter-examples: even among spatial frames...)

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$?

NO! in general (counter-examples: even among spatial frames...)

YES! for weak cb-frames.

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$?

NO! in general (counter-examples: even among spatial frames...)

YES! for weak cb-frames.

Now, the role of $\mathfrak{B}(L)$ is taken by

$\mathfrak{G}(L)$ Gleason cover of L

(the largest essential extension of L)

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$?

NO! in general (counter-examples: even among spatial frames...)

YES! for weak cb-frames.

Now, the role of $\mathfrak{B}(L)$ is taken by

$\mathfrak{G}(L)$ Gleason cover of L

(the largest essential extension of L)

For every completely regular frame L, there exists a (unique) completely regular and extremally disconnected frame $\mathfrak{G}(L)$ and a proper essential embedding $\gamma_L\colon L \hookrightarrow \mathfrak{G}(L)$.

[B. Banaschewski, 1988]

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$?

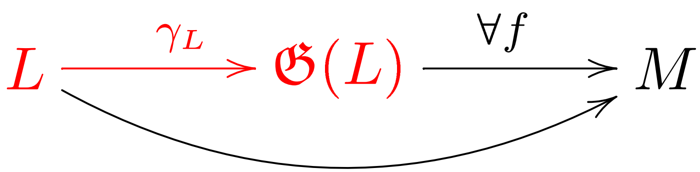
NO! in general (counter-examples: even among spatial frames...)

YES! for weak cb-frames.

Now, the role of $\mathfrak{B}(L)$ is taken by

$\mathfrak{G}(L)$ **Gleason cover of L**

(the largest essential extension of L)



embedding $\Rightarrow$ f is an embedding

QUESTION: Is $\mathcal{D}(\mathsf{C}(L))$ isomorphic to some $\mathsf{C}(M)$?

NO! in general (counter-examples: even among spatial frames...)

YES! for weak cb-frames.

Now, the role of $\mathfrak{B}(L)$ is taken by

$\mathfrak{G}(L)$ Gleason cover of L

(the largest essential extension of L)

THEOREM. Let L be a completely regular, weak cb-frame.

Then $\mathcal{D}(\mathsf{C}(L)) \simeq \mathsf{NLSC}(L) \simeq \mathsf{C}(\mathfrak{G}(L))$.

THE COMPLETION AS A FUNCTION RING

$\mathsf{NLSC}(L) \simeq \mathsf{C}(\mathfrak{G}(L))$

SKETCH OF PROOF:

THE COMPLETION AS A FUNCTION RING

$\mathsf{NLSC}(L) \simeq \mathsf{C}(\mathfrak{G}(L))$

SKETCH OF PROOF: $h\colon L \to M$ proper essential embedding, M: e.d.

THE COMPLETION AS A FUNCTION RING

$\mathsf{NLSC}(L) \simeq \mathsf{C}(\mathfrak{G}(L))$

SKETCH OF PROOF: $h\colon L \to M$ proper essential embedding, M: e.d.

h_* (localic right adjoint)

THE COMPLETION AS A FUNCTION RING

$\mathsf{NLSC}(L) \simeq \mathsf{C}(\mathfrak{G}(L))$

SKETCH OF PROOF: $h\colon L \to M$ proper essential embedding, M: e.d.

h_* (localic right adjoint)

- $h^{\leftarrow}(f) \in \mathsf{NLSC}(L) \leftsquigarrow f \in \mathsf{LSC}(M)$:

THE COMPLETION AS A FUNCTION RING

$\mathsf{NLSC}(L) \simeq \mathsf{C}(\mathfrak{G}(L))$

SKETCH OF PROOF: $h\colon L \to M$ proper essential embedding, M: e.d.

h_* (localic right adjoint)

- $h^{\leftarrow}(f) \in \mathsf{NLSC}(L) \quad \leftsquigarrow \quad f \in \mathsf{LSC}(M)$:

$$h^{\leftarrow}(f)(p,-) = \bigvee_{r>p} h_*[f(r,-)]$$

$$h^{\leftarrow}(f)(-,q) = \bigvee_{s<q} (h_*[f(s,-)])^*$$

THE COMPLETION AS A FUNCTION RING

$\mathsf{NLSC}(L) \simeq \mathsf{C}(\mathfrak{G}(L))$

SKETCH OF PROOF: $h\colon L \to M$ proper essential embedding, M: e.d.

h_* (localic right adjoint)

- $h^{\leftarrow}(f) \in \mathsf{NLSC}(L) \;\leftsquigarrow\; f \in \mathsf{LSC}(M)$:

$$h^{\leftarrow}(f)(p,—) = \bigvee_{r>p} h_*[f(r,—)]$$

$$h^{\leftarrow}(f)(—,q) = \bigvee_{s<q} (h_*[f(s,—)])^*$$

- $g \in \mathsf{NLSC}(L) \;\rightsquigarrow\; h^{\rightarrow}(g) \in \mathsf{LSC}(M)$:

$$h^{\rightarrow}(g)(p,—) = \bigvee_{r>p} \mathfrak{c}(h(\bigwedge g(r,—))^{**})$$

$$h^{\rightarrow}(g)(—,q) = \bigvee_{s<q} \mathfrak{c}(h(\bigwedge g(s,—))^{*})$$

THE COMPLETION AS A FUNCTION RING

$\mathsf{NLSC}(L) \simeq \mathsf{C}(\mathfrak{G}(L))$

SKETCH OF PROOF: $h\colon L \to M$ proper essential embedding, M: e.d.

h_* (localic right adjoint)

- $h^{\leftarrow}(f) \in \mathsf{NLSC}(L) \;\leftarrow\!\!\sim\!\sim\; f \in \mathsf{LSC}(M)$:

$$h^{\leftarrow}(f)(p,-) = \bigvee_{r>p} h_*[f(r,-)]$$

$$h^{\leftarrow}(f)(-,q) = \bigvee_{s<q} (h_*[f(s,-)])^*$$

- $g \in \mathsf{NLSC}(L) \;\sim\!\sim\!\!\rightarrow\; h^{\rightarrow}(g) \in \mathsf{LSC}(M)$:

$$h^{\rightarrow}(g)(p,-) = \bigvee_{r>p} \mathfrak{c}(h(\bigwedge g(r,-))^{**})$$

$$h^{\rightarrow}(g)(-,q) = \bigvee_{s<q} \mathfrak{c}(h(\bigwedge g(s,-))^{*})$$

- Galois adjunction: $\boldsymbol{h}^{\leftarrow}(-) \dashv \boldsymbol{h}^{\rightarrow}(-)$.

THE COMPLETION AS A FUNCTION RING

$\mathsf{NLSC}(L) \simeq \mathsf{C}(\mathfrak{G}(L))$

SKETCH OF PROOF: $h\colon L \to M$ proper essential embedding, M: e.d.

h_* (localic right adjoint)

- $h^{\leftarrow}(f) \in \mathsf{NLSC}(L)$ ⇜ $f \in \mathsf{LSC}(M)$:

$$h^{\leftarrow}(f)(p,—) = \bigvee_{r>p} h_*[f(r,—)]$$

$$h^{\leftarrow}(f)(—,q) = \bigvee_{s<q} (h_*[f(s,—)])^*$$

- $g \in \mathsf{NLSC}(L)$ ⇝ $h^{\rightarrow}(g) \in \mathsf{LSC}(M)$:

$$h^{\rightarrow}(g)(p,—) = \bigvee_{r>p} \mathfrak{c}(h(\bigwedge g(r,—))^{**})$$

$$h^{\rightarrow}(g)(—,q) = \bigvee_{s<q} \mathfrak{c}(h(\bigwedge g(s,—))^{*})$$

- Galois adjunction: $\boldsymbol{h}^{\leftarrow}(-) \dashv \boldsymbol{h}^{\rightarrow}(-)$. It yields the equivalence

$\mathsf{NLSC}(L) \simeq \mathsf{C}(M)$.