

# A New Topological Semantics for (Dynamic) Doxastic Logic

### Aybüke Özgün

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Joint work with Alexandru Baltag, Nick Bezhanishvili and Sonja Smets.

Some History and Motivation	Stalnaker's Logic <b>KB</b>	Our Work 0000 000000	Future Work

### **Background Notions**

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## **Background Notions**

Topological space: (X, τ)

For any  $A \subseteq X$ , we denote,

- interior operator: Int(A)
- closure operator:  $Cl(A) := X \setminus Int(X \setminus A)$
- derived set operator: d(A)
- co-derived set operator:  $t(A) := X \setminus d(X \setminus A)$

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# Topological Semantics for Knowledge

(McKinsey and Tarski, 1944)

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# Topological Semantics for Knowledge

(McKinsey and Tarski, 1944)

$$(\mathcal{L}_{\mathcal{K}}) \varphi ::= \mathbf{p} \mid \neg \varphi \mid \varphi \land \varphi \mid \mathbf{K} \varphi$$

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# Topological Semantics for Knowledge

(McKinsey and Tarski, 1944)

 $(\mathcal{L}_{\mathcal{K}}) \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid \mathcal{K}\varphi$ 

 $\mathit{K} arphi :=$  the agent knows arphi

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# Topological Semantics for Knowledge

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$$(\mathcal{L}_{\mathcal{K}}) \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid \mathcal{K}\varphi$$

 $\mathit{K} arphi :=$  the agent knows arphi

A topological model is a tuple  $\mathcal{M} = (X, \tau, \nu)$  where X and  $\tau$  as before and  $\nu$  : Prop  $\rightarrow \mathcal{P}(X)$  is valuation function.

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$$oldsymbol{x} \in \llbracket oldsymbol{K} arphi 
rbrace^{\mathcal{M}}$$
 iff  $\exists oldsymbol{U} \in au(oldsymbol{x} \in oldsymbol{U} \land oldsymbol{U} \subseteq \llbracket arphi 
rbrace^{\mathcal{M}})$ 

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$$oldsymbol{x} \in \llbracket {\mathcal K} arphi 
brace {\mathcal M}^{\mathcal M} ext{ iff } \exists oldsymbol{U} \in au(oldsymbol{x} \in oldsymbol{U} \, \wedge oldsymbol{U} \subseteq \llbracket arphi 
brace {\mathcal M}^{\mathcal M})$$

i.e.,

$$\llbracket K \varphi \rrbracket = \operatorname{Int}(\llbracket \varphi \rrbracket)$$

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# Topological Semantics for Knowledge

(McKinsey and Tarski, 1944)

$$(\mathcal{L}_{\mathcal{K}}) \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid \mathcal{K}\varphi$$

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brace {\mathcal M}^{\mathcal M})$$

i.e.,

$$\llbracket K\varphi \rrbracket = \operatorname{Int}(\llbracket \varphi \rrbracket)$$
$$\llbracket \langle K \rangle \varphi \rrbracket = \operatorname{Cl}(\llbracket \varphi \rrbracket) \quad (\langle K \rangle \varphi := \neg K \neg \varphi)$$

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Why knowledge is interpreted as interior?			

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Why knowledge is interpreted as interior?

#### Theorem (McKinsey and Tarski, 1944)

**S4** is sound and complete wrt the class of all topological spaces.

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### Theorem (McKinsey and Tarski, 1944)

**S4** is sound and complete wrt the class of all topological spaces.

Hence, topologically,

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### Theorem (McKinsey and Tarski, 1944)

**S4** is sound and complete wrt the class of all topological spaces.

Hence, topologically, knowledge is *truthful* 

 $K\varphi \rightarrow \varphi,$ 

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#### Theorem (McKinsey and Tarski, 1944)

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Hence, topologically, knowledge is *truthful* 

$$K\varphi \to \varphi,$$

positively introspective

$$K\varphi \rightarrow KK\varphi,$$

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### Theorem (McKinsey and Tarski, 1944)

**S4** is sound and complete wrt the class of all topological spaces.

Hence, topologically, knowledge is *truthful* 

$$K\varphi \to \varphi,$$

positively introspective

 $K\varphi \rightarrow KK\varphi,$ 

but not necessarily negatively introspective

$$\neg K\varphi \rightarrow K\neg K\varphi.$$

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Why knowledge is interpreted as interior?			

It provides a deeper insight into the evidential-based interpretation of knowledge:

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Why knowledge is interpreted as interior?			

It provides a deeper insight into the evidential-based interpretation of knowledge:

• open sets  $U \in \tau$  are pieces of evidence, and

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Why knowledge is interpreted as interior?			

It provides a deeper insight into the evidential-based interpretation of knowledge:

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Why knowledge is interpreted as interior?			

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- open sets  $U \in \tau$  are pieces of evidence, and
- open neighborhoods of the actual world are pieces of truthful evidence.

Recall:

$$x \in \llbracket K \varphi \rrbracket^{\mathcal{M}} \text{ iff } \exists U \in \tau (x \in U \land U \subseteq \llbracket \varphi \rrbracket^{\mathcal{M}})$$

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Belief via Topology			

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Belief via Topology			

$$x \in \llbracket \langle B \rangle \varphi \rrbracket^{\mathcal{M}} \text{ iff } \forall U \in \tau (x \in U \to U \setminus \{x\} \cap \llbracket \varphi \rrbracket^{\mathcal{M}} \neq \emptyset)$$

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$$x \in \llbracket \langle B \rangle \varphi \rrbracket^{\mathcal{M}} \text{ iff } \forall U \in \tau (x \in U \to U \setminus \{x\} \cap \llbracket \varphi \rrbracket^{\mathcal{M}} \neq \emptyset)$$

$$\llbracket \langle \boldsymbol{B} \rangle \varphi \rrbracket = \boldsymbol{d}(\llbracket \varphi \rrbracket)$$

i.e.,

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Belief via Topology			

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i.e.,  
$$\llbracket \langle B \rangle \varphi \rrbracket = d(\llbracket \varphi \rrbracket)$$

$$\llbracket \langle \mathbf{D} / \boldsymbol{\varphi} \rrbracket = \mathbf{Q} (\llbracket \boldsymbol{\varphi} \rrbracket)$$

$$x \in \llbracket B\varphi \rrbracket^{\mathcal{M}} \text{ iff } \exists U \in \tau (x \in U \land U \setminus \{x\} \subseteq \llbracket \varphi \rrbracket^{\mathcal{M}})$$

$$\llbracket B\varphi \rrbracket = t(\llbracket \varphi \rrbracket)$$

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i.e.,

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Belief via Topology			

• it entails *the necessity of error*: there is at least one false belief in all worlds of every topological model.

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Belief via Topology			

• it entails *the necessity of error*: there is at least one false belief in all worlds of every topological model.

2 it supports "knowledge as (justified) true belief" view:

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Belief via Topology			

• it entails *the necessity of error*: there is at least one false belief in all worlds of every topological model.

2 it supports "knowledge as (justified) true belief" view:

$$x \in \llbracket B\varphi \rrbracket^{\mathcal{M}} \text{ iff } \exists U \in \tau (x \in U \land U \setminus \{x\} \subseteq \llbracket \varphi \rrbracket^{\mathcal{M}})$$

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Belief via Topology			

- it entails *the necessity of error*: there is at least one false belief in all worlds of every topological model.
- 2 it supports "knowledge as (justified) true belief" view:

$$\begin{array}{l} x \in \llbracket B \varphi \rrbracket^{\mathcal{M}} \text{ iff } \exists U \in \tau (x \in U \land U \setminus \{x\} \subseteq \llbracket \varphi \rrbracket^{\mathcal{M}}) \\ x \in \llbracket K \varphi \rrbracket^{\mathcal{M}} \text{ iff } \exists U \in \tau (x \in U \land U \subseteq \llbracket \varphi \rrbracket^{\mathcal{M}}) \end{array}$$

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Belief via Topology			

• it entails *the necessity of error*: there is at least one false belief in all worlds of every topological model.

2 it supports "knowledge as (justified) true belief" view:

 $x \in \llbracket B\varphi \rrbracket^{\mathcal{M}} \text{ iff } \exists U \in \tau (x \in U \land U \setminus \{x\} \subseteq \llbracket \varphi \rrbracket^{\mathcal{M}})$  $x \in \llbracket K\varphi \rrbracket^{\mathcal{M}} \text{ iff } \exists U \in \tau (x \in U \land U \subseteq \llbracket \varphi \rrbracket^{\mathcal{M}})$ 

$$\mathbf{K}\varphi := \mathbf{B}\varphi \wedge \varphi$$

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Belief via Topology			

• it entails *the necessity of error*: there is at least one false belief in all worlds of every topological model.

2 it supports "knowledge as (justified) true belief" view:

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$$\boldsymbol{K}\varphi := \boldsymbol{B}\varphi \wedge \varphi$$

**3** KD45 is sound and complete wrt *dense-in-itself*  $T_D$  spaces where d(A) is open for every  $A \subseteq X$  (i.e., *DSO*-spaces).

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Given the interior-based topological semantics for knowledge:

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Given the interior-based topological semantics for knowledge:

how can we construct a topological semantics for belief which can also address the problem of understanding the relation between knowledge and belief?

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Belief via Topology			



Given the interior-based topological semantics for knowledge:

- how can we construct a topological semantics for belief which can also address the problem of understanding the relation between knowledge and belief?
- A how to extend this setting to dynamic belief revision?
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| Stalnaker's Logi            | c <b>KB</b>                 |                            |             |

 $(\mathcal{L}_{\mathsf{K}\mathsf{B}}) \varphi ::= \mathbf{p} \mid \neg \varphi \mid \varphi \land \varphi \mid \mathsf{K}\varphi \mid \mathsf{B}\varphi$ 



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$$(\mathcal{L}_{\mathcal{K}\mathcal{B}}) \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid \mathcal{K}\varphi \mid \mathcal{B}\varphi$$

Epistemic-Doxastic Axioms	
$ig K(arphi  ightarrow \psi)  ightarrow (Karphi  ightarrow K\psi)$	Knowledge is additive
${m K}arphi ightarrow arphi$	Knowledge implies truth
${\it K}arphi  ightarrow {\it K}{\it K}arphi$	Positive introspection for K
$m{B}arphi  ightarrow  eg m{B}  eg arphi$	Consistency of belief
${m B}arphi  o {m K} {m B}arphi$	(Strong) positive introspection of B
$ eg B arphi  ightarrow {m K}  eg B arphi$	(Strong) negative introspection of <i>B</i>
$oldsymbol{K}arphi ightarrowoldsymbol{B}arphi$	Knowledge implies Belief
$m{B}arphi  ightarrow m{B} K arphi$	Full Belief
Inference Rules	
From $\varphi$ and $\varphi \rightarrow \psi$ infer $\psi$ .	Modus Ponens
From $\varphi$ infer $K\varphi$ .	Necessitation

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$$(\mathcal{L}_{\mathcal{K}\mathcal{B}}) \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid \mathcal{K}\varphi \mid \mathcal{B}\varphi$$

Epistemic-Doxastic Axioms	
$egin{array}{c} K(arphi  ightarrow \psi)  ightarrow (K arphi  ightarrow K \psi) \end{array}$	Knowledge is additive
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$m{B}arphi  ightarrow  eg m{B}  eg arphi$	Consistency of belief
$m{B}arphi  ightarrow m{K} m{B}arphi$	(Strong) positive introspection of B
$ eg B arphi  ightarrow {m K}  eg B arphi$	(Strong) negative introspection of <i>B</i>
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${\cal B}arphi  o {\cal B} {\cal K}arphi$	Full Belief
Inference Rules	
From $\varphi$ and $\varphi \rightarrow \psi$ infer $\psi$ .	Modus Ponens
From $\varphi$ infer $K\varphi$ .	Necessitation

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Epistemic-Doxast	tic Axioms		
$K(\varphi  ightarrow \psi)  ightarrow (K$	$\varphi \rightarrow K\psi$ )	Knowledge is additive	
$K \varphi \rightarrow \varphi$	>	Knowledge implies truth	า
Karphi  ightarrow Kk	$\langle \varphi    $	Positive introspection for	ĸ
$Barphi  ightarrow  eg B \phi$	$\neg \varphi$	Consistency of belief	
$Barphi  ightarrow  extsf{KE}$	$B \varphi$ (4)	Strong) positive introspectio	n of B
$\neg B \varphi \rightarrow K \neg$	$B\varphi$ (S	Strong) negative introspectic	on of <i>B</i>
$K \varphi \rightarrow B$	$\varphi$	Knowledge implies Belie	əf
$B\varphi \rightarrow Bk$	6	Full Belief	

Inference RulesFrom  $\varphi$  and  $\varphi \rightarrow \psi$  infer  $\psi$ .Modus PonensFrom  $\varphi$  infer  $K\varphi$ .Necessitation

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	Epistemic-Doxastic Axioms			
	$\boxed{\mathbf{K}(\varphi \to \psi) \to (\mathbf{K}\varphi \to \mathbf{K}\psi)}$	Knowled	dge is additive	
	$K \varphi  ightarrow \varphi$	Knowledg	ge implies truth	
	Karphi  ightarrow KKarphi	Positive int	trospection for I	ĸ
	$egin{array}{c} Barphi ightarrow  eg B eg arphi ightarrow  eg B eg arphi ightarrow  eg B eg arphi ightarrow  eg arphi ightarrow $	Consist	ency of belief	
	$egin{array}{c} Barphi ightarrow {m K} Barphi \end{array}$	(Strong) positi	ve introspectior	ı of B
	$\neg B arphi  ightarrow K \neg B arphi$	(Strong) negati	ive introspection	n of B
	Karphi  ightarrow Barphi	Knowledg	je implies Beliet	f
	Barphi  ightarrow BKarphi	Fi	ull Belief	
	Inference Rules			
	From $\varphi$ and $\varphi \rightarrow \psi$ infer $\psi$ .	Mod	us Ponens	

 Belief as subjective certainty: an agent "fully" believes φ iff she believes that she knows it.

Necessitation

From  $\varphi$  infer  $K\varphi$ .

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(Full) belief can be defined in terms of knowledge as

$$B\varphi \leftrightarrow \langle K \rangle K\varphi$$

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(Full) belief can be defined in terms of knowledge as

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**2 KD45** as the logic of belief

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(Full) belief can be defined in terms of knowledge as

$$B\varphi \leftrightarrow \langle K \rangle K\varphi$$

**2 KD45** as the logic of belief

**3 S4.2** as the logic of knowledge

Some History and Motivation	Stalnaker's Logic <b>KB</b>	Our Work	Future Work

#### Theorem (Folklore)

**S4.2** is sound and complete wrt the class of extremally disconnected spaces (under the interior semantics).

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#### Theorem (Folklore)

**S4.2** is sound and complete wrt the class of extremally disconnected spaces (under the interior semantics).

A space  $(X, \tau)$  is called extremally disconnected (e.d.) if the closure of each open subset of X is open.

re Work

Topological Semantics for Full Belief

# Our Proposal: Topological semantics for full belief

A New Topological Semantics for (Dynamic) Doxastic Logic

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# Our Proposal: Topological semantics for full belief

 $\mathsf{RECALL}: \vdash_{\mathsf{KB}} B\varphi \leftrightarrow \langle \mathsf{K} \rangle \mathsf{K} \varphi$ 





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## Our Proposal: Topological semantics for full belief

 $\mathsf{RECALL}: \vdash_{\mathsf{KB}} B\varphi \leftrightarrow \langle K \rangle K\varphi$ 

Given an extremally disconnected space  $(X, \tau)$ , we interpret belief as *the closure of the interior operator*:

 $\llbracket B\varphi \rrbracket = \operatorname{Cl}(\operatorname{Int}(\llbracket \varphi \rrbracket))$ 

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A New Topological Semantics for (Dynamic) Doxastic Logic

# The Most General Extensional Semantics for Full Belief

• An extensional model:  $(X, K, B, \nu)$ 

$$\begin{bmatrix} \mathcal{K}\varphi \end{bmatrix}^{\mathcal{M}} = \mathbf{K} \llbracket \varphi \end{bmatrix}^{\mathcal{M}} \\ \begin{bmatrix} \mathcal{B}\varphi \end{bmatrix}^{\mathcal{M}} = \mathbf{B} \llbracket \varphi \end{bmatrix}^{\mathcal{M}}$$

#### Theorem

An extensional semantics (X, K, B) validates all the axioms and rules of Stalnaker's system **KB** iff it is a topological extensional semantics given by an extremally disconnected topology  $\tau$  on X, such that K = Int and B = Cl(Int).

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## Unimodal Cases: S4.2 and KD45

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## Unimodal Cases: S4.2 and KD45



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## Unimodal Cases: S4.2 and KD45





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## Unimodal Cases: S4.2 and KD45



 Known: S4.2 is sound and complete wrt the class of extremally disconnected spaces.

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## Unimodal Case: Completeness for KD45

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## Unimodal Case: Completeness for KD45

$$(\mathcal{L}_{\mathcal{B}}) \varphi ::= \mathbf{p} \mid \neg \varphi \mid \varphi \land \varphi \mid \mathbf{B}\varphi$$

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# Unimodal Case: Completeness for **KD45**

$$(\mathcal{L}_{\mathcal{B}}) \varphi ::= \mathbf{p} \mid \neg \varphi \mid \varphi \land \varphi \mid \mathbf{B}\varphi$$

#### Theorem

**KD45** is sound and complete wrt the class of extremally disconnected spaces.

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# Unimodal Case: Completeness for **KD45**

$$(\mathcal{L}_{B}) \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid B\varphi$$

#### Theorem

**KD45** is sound and complete wrt the class of extremally disconnected spaces.

• The class of extremally disconnected spaces is *strictly* larger than the class of *DSO*-spaces.

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Dynamic Belief Revision: Updates			

 $\varphi := \boldsymbol{\rho} \mid \neg \varphi \mid \varphi \land \varphi \mid \boldsymbol{K} \varphi \mid \boldsymbol{B} \varphi \mid \langle ! \varphi \rangle \varphi$ 

Some History and Motivation oo ooo	Stalnaker's Logic <b>KB</b>	Our Work ○○○○ ●○○○○○	Future Work
Dynamic Belief Revision: Updates			

$$\varphi := p \mid \neg \varphi \mid \varphi \land \varphi \mid K\varphi \mid B\varphi \mid \langle !\varphi \rangle \varphi$$

⟨!φ⟩ψ := φ is true and after the agent learns φ, ψ becomes true.

Some History and Motivation	Stalnaker's Logic <b>KB</b>	Our Work ○○○○ ●○○○○○	Future Work
Dynamic Belief Revision: Updates			

$$\varphi := \pmb{\rho} \mid \neg \varphi \mid \varphi \land \varphi \mid \pmb{K}\varphi \mid \pmb{B}\varphi \mid \langle !\varphi \rangle \varphi$$

⟨!φ⟩ψ := φ is true and after the agent learns φ, ψ becomes true.

$$\mathcal{M} = (X, \tau, \nu)$$
  $\mathcal{M}_{\varphi} = (\llbracket \varphi \rrbracket, \tau_{\llbracket \varphi \rrbracket}, \nu_{\llbracket \varphi \rrbracket})$ 

 $\langle |\varphi \rangle$ 





• 
$$\llbracket \varphi \rrbracket = \llbracket \varphi \rrbracket^{\mathcal{N}}$$

• 
$$\tau_{\llbracket \varphi \rrbracket} = \{ U \cap \llbracket \varphi \rrbracket : U \in \tau \}$$

•  $\nu_{\llbracket \varphi \rrbracket}(p) = \nu(p) \cap \llbracket \varphi \rrbracket$  for any  $p \in \operatorname{Prop}$ 

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Some History and Motivation	Stalnaker's Logic <b>KB</b>	Our Work ○○○○ ○●○○○○	Future Work
Dynamic Belief Revision: Updates			

$$\varphi := \pmb{\rho} \mid \neg \varphi \mid \varphi \land \varphi \mid \pmb{K}\varphi \mid \pmb{B}\varphi \mid \langle !\varphi \rangle \varphi$$

⟨!φ⟩ψ := φ is true and after the agent learns φ, ψ becomes true.

$$\mathcal{M} = (X, \tau, \nu) \qquad \qquad \mathcal{M}_{\varphi} = (\llbracket \varphi \rrbracket, \tau_{\llbracket \varphi \rrbracket}, \nu_{\llbracket \varphi \rrbracket})$$



Given a topological model *M* = (*X*, *τ*, *ν*), the additional semantic clause reads

$$x \in \llbracket \langle ! \varphi \rangle \psi \rrbracket^{\mathcal{M}}$$
 iff  $x \in \llbracket \varphi \rrbracket$  and  $x \in \llbracket \psi \rrbracket^{\mathcal{M}_{\varphi}}$ 

Our Work

Dynamic Belief Revision: Updates

# Updates on E.D. spaces: inconsistent beliefs!

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## Updates on E.D. spaces: inconsistent beliefs!







Figure: not e.d.  $(\llbracket \neg p \rrbracket^{\mathcal{M}}, \tau_{\neg p})$ 

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## Updates on E.D. spaces: inconsistent beliefs!





•  $x_1 \models Bq \land B \neg q$  (against *Consistency of Beliefs*)

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## Updates on E.D. spaces: inconsistent beliefs!





- $x_1 \models Bq \land B \neg q$  (against *Consistency of Beliefs*)
- $x_1 \not\models Bq \land B \neg q \rightarrow B(q \land \neg q)$  (against *Normality*)

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Some History and Motivation	Stalnaker's Logic <b>KB</b>	Our Work ○○○○ ○○○●○○	Future Work
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Hereditarily Extremally Disconnected Space (h.e.d.)
 All Tapalagian Space

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2 All Topological Spaces

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## Updates on H.E.D. spaces

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#### Updates on H.E.D. spaces





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## Updates on H.E.D. spaces





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#### Updates on H.E.D. spaces



• **S4.3** is the logic of h.e.d. spaces (under the interior semantics).

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Dynamic Belief Revision: Updates

### Updates on all topological spaces

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# Updates on all topological spaces

Axioms of S4
$$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$$
 $K\varphi \rightarrow \varphi$  $K\varphi \rightarrow KK\varphi$ 

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# Updates on all topological spaces

Axioms of S4
$$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$$
 $K\varphi \rightarrow \varphi$  $K\varphi \rightarrow KK\varphi$ 

Axioms of wKD45
$$B(\varphi \rightarrow \psi) \rightarrow (B\varphi \rightarrow B\psi)$$
 $B\varphi \rightarrow \langle B \rangle \varphi$  $B\varphi \rightarrow BB\varphi$  $B\langle B \rangle B\varphi \rightarrow B\varphi$ 

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Future Work

Updates on all topological spaces

Axioms of S4Axioms of wKD45
$$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$$
  
 $K\varphi \rightarrow \varphi$   
 $K\varphi \rightarrow KK\varphi$  $B(\varphi \rightarrow \psi) \rightarrow (B\varphi \rightarrow B\psi)$   
 $B\varphi \rightarrow \langle B \rangle \varphi$   
 $B\varphi \rightarrow BB\varphi$   
 $B\langle B \rangle B\varphi \rightarrow B\varphi$ 

• **S4** is the logic of all topological spaces.

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Some History and Motivation	Stalnaker's Logic <b>KB</b>	Our Work 0000 000000	Future Work

#### Future Work

- Multi-agent case
- Action models
- Different Dynamic attitudes
- Relation with topo-logic and effort modality

Some History and Motivation oo ooo	Stalnaker's Logic <b>KB</b>	Our Work 0000 000000	Future Work

#### Thank you!

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