A New Topological Semantics for (Dynamic) Doxastic Logic

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Background Notions

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- Topological space: (X, τ)

For any $A \subseteq X$, we denote,

- interior operator: $\mathrm{Int}(A)$
- closure operator: $\mathrm{Cl}(A) := X \setminus \mathrm{Int}(X \setminus A)$
- derived set operator: $d(A)$
- co-derived set operator: $t(A) := X \setminus d(X \setminus A)$

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i.e.,

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$$[\![\langle K\rangle\varphi]\!] = \mathrm{Cl}([\![\varphi]\!]) \quad (\langle K\rangle\varphi := \neg K\neg\varphi)$$

Why knowledge is interpreted as interior?

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but **not** necessarily *negatively introspective*

$$\neg K\varphi \rightarrow K\neg K\varphi.$$

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Recall:

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Belief via Topology

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3. **KD45** is sound and complete wrt *dense-in-itself* T_D spaces where $d(A)$ is open for every $A \subseteq X$ (i.e., *DSO*-spaces).

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2. how to extend this setting to dynamic belief revision?

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Epistemic-Doxastic Axioms	
$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	Knowledge is additive
$K\varphi \to \varphi$	Knowledge implies truth
$K\varphi \to KK\varphi$	Positive introspection for K
$B\varphi \to \neg B\neg\varphi$	Consistency of belief
$B\varphi \to KB\varphi$	(Strong) positive introspection of B
$\neg B\varphi \to K\neg B\varphi$	(Strong) negative introspection of B
$K\varphi \to B\varphi$	Knowledge implies Belief
$B\varphi \to BK\varphi$	Full Belief

Inference Rules	
From φ and $\varphi \to \psi$ infer ψ.	Modus Ponens
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- Belief as *subjective certainty*: *an agent "fully" believes φ iff she believes that she knows it.*

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3. **S4.2** as the logic of knowledge

Theorem (Folklore)

S4.2 *is sound and complete wrt the class of extremally disconnected spaces (under the interior semantics).*

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A space (X, τ) is called extremally disconnected (e.d.) if the closure of each open subset of X is open.

Our Proposal: Topological semantics for full belief

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Given an extremally disconnected space (X, τ), we interpret belief as *the closure of the interior operator*:

$$[\![B\varphi]\!] = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))$$

The Most General Extensional Semantics for Full Belief

- An extensional model: $(X, \mathrm{K}, \mathrm{B}, \nu)$

$$
\begin{aligned}
[\![K\varphi]\!]^{\mathcal{M}} &= \mathrm{K}[\![\varphi]\!]^{\mathcal{M}} \\
[\![B\varphi]\!]^{\mathcal{M}} &= \mathrm{B}[\![\varphi]\!]^{\mathcal{M}}.
\end{aligned}
$$

Theorem

An extensional semantics $(X, \mathrm{K}, \mathrm{B})$ *validates all the axioms and rules of Stalnaker's system* **KB** *iff it is a topological extensional semantics given by an extremally disconnected topology* τ *on* X*, such that* $\mathrm{K} = \mathrm{Int}$ *and* $\mathrm{B} = \mathrm{Cl}(\mathrm{Int})$*.*

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- Known: **S4.2** is sound and complete wrt the class of extremally disconnected spaces.

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Theorem

KD45 *is sound and complete wrt the class of extremally disconnected spaces.*

- The class of extremally disconnected spaces is *strictly* larger than the class of *DSO*-spaces.

Updates

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi \mid \langle !\varphi \rangle \varphi$$

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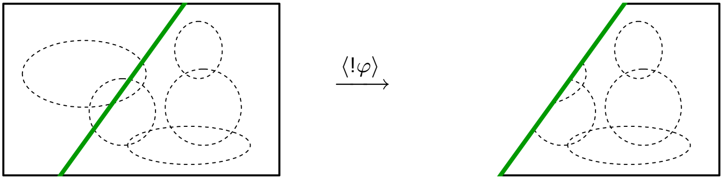
$\mathcal{M} = (X, \tau, \nu)$ $\mathcal{M}_\varphi = ([\![\varphi]\!], \tau_{[\![\varphi]\!]}, \nu_{[\![\varphi]\!]})$



- $[\![\varphi]\!] = [\![\varphi]\!]^{\mathcal{M}}$
- $\tau_{[\![\varphi]\!]} = \{U \cap [\![\varphi]\!] : U \in \tau\}$
- $\nu_{[\![\varphi]\!]}(p) = \nu(p) \cap [\![\varphi]\!]$ for any $p \in \mathrm{Prop}$

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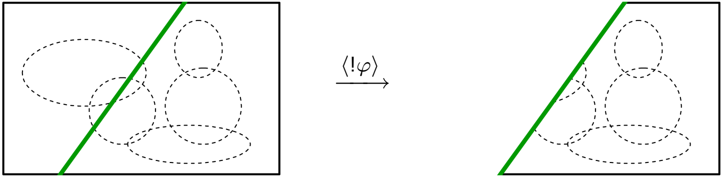

- Given a topological model $\mathcal{M} = (X, \tau, \nu)$, the additional semantic clause reads

$$x \in [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}} \text{ iff } x \in [\![\varphi]\!] \text{ and } x \in [\![\psi]\!]^{\mathcal{M}_\varphi}.$$

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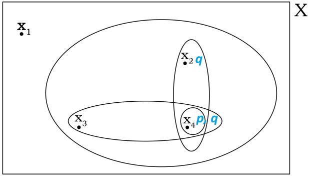


Figure: e.d.

(X, τ)

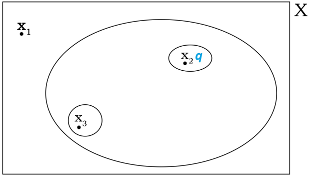
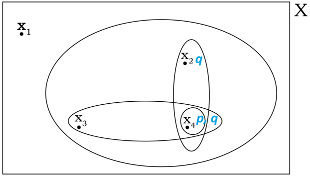


Figure: not e.d.

$([\![\neg p]\!]^{\mathcal{M}}, \tau_{\neg p})$

Updates on E.D. spaces: *inconsistent beliefs!*



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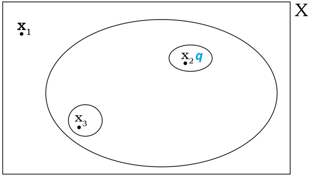

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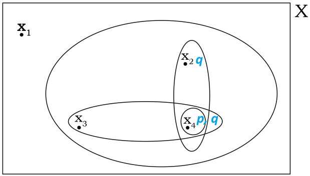

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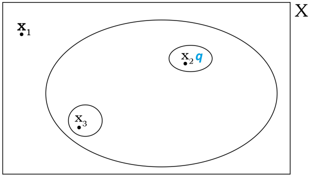

Figure: not e.d.

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- $x_1 \models Bq \wedge B\neg q$ (against *Consistency of Beliefs*)
- $x_1 \not\models Bq \wedge B\neg q \rightarrow B(q \wedge \neg q)$ (against *Normality*)

Solutions

1. *Hereditarily Extremally Disconnected Space (h.e.d.)*
2. *All Topological Spaces*

Updates on H.E.D. spaces

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Axioms of **KD45**
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Axioms of **S4.3**
$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
$K\varphi \to \varphi$
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$K(K\varphi \to \psi) \vee K(K\psi \to \varphi)$

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- **S4.3** is the logic of h.e.d. spaces (under the interior semantics).

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$B\varphi \to \langle B \rangle \varphi$
$B\varphi \to BB\varphi$
$B\langle B \rangle B\varphi \to B\varphi$

- **S4** is the logic of all topological spaces.

Future Work

- Multi-agent case
- Action models
- Different Dynamic attitudes
- Relation with topo-logic and *effort modality*

Thank you!