Proximity biframes and compactifications of completely regular ordered spaces

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Schauerte in 1993 extended Banaschewski's work to the biframe setting, defining bi-compactifications of a biframe and describing them in terms of strong inclusions on the biframe.

We will generalize Schauerte's work by defining a proximity on a biframe. This will capture the concept of an order-compactification in the spatial case.

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If (X, τ_1, τ_2) is a bispace, let $\tau = \tau_1 \vee \tau_2$ be the **patch** topology and $\leq_1$ the **specialization** order of τ_1. Then $(X, \tau, \leq_1)$ is an ordered space.

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We assume throughout that the topology of an ordered space is **strongly order convex**; that is, $\tau = \tau_u \vee \tau_d$.

Let $(Y, \tau, \leq)$ be an ordered space. We call it a **Nachbin space** if Y is compact and $\leq$ is closed in the product topology. This latter condition is equivalent to an order-theoretic separation axiom called order-Hausdorff.

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There is an isomorphism between the category **Nach** of Nachbin spaces and order-preserving continuous maps and the category **KRBSp** of compact regular bispaces and bi-continuous maps. Under this isomorphism $(X, \tau, \leq)$ goes to (X, τ_u, τ_d) and (X, τ_1, τ_2) goes to $(X, \tau_1 \vee \tau_2, \leq_1)$.

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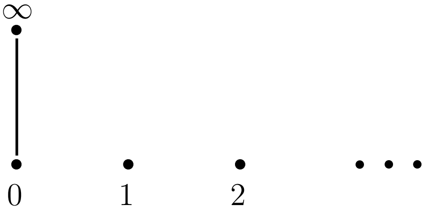
If (Y, e) is a bi-compactification of (X, τ_1, τ_2), then it is an order-compactification of $(X, \tau_1 \vee \tau_2, \leq_1)$.

However, an order-compactification of $(X, \tau, \leq)$ may not be a bi-compactification of (X, τ_u, τ_d), as the following example shows.

Example. Let $X = \mathbb{N}$ with the discrete topology and trivial order. Let Y be the one-point compactification of X, with order given by $0 \leq \infty$ as the only nontrivial relation. Then Y is an order-compactification of X.

However, since $\{0\}$ is an open upset of X but there is no open upset of Y contracting to it, Y is not a bi-compactification of X.



Distinction between compactifications

Let (Y, e) be an order-compactification of $(X, \tau, \leq)$. Then each open set U of X has the form $e^{-1}(V)$ for some open set V of Y, and each upset U of X has the form $e^{-1}(V)$ for some upset V of Y. However, as the previous example shows, an open upset U of X may not have the form $e^{-1}(V)$ for some open upset V of Y. This distinguishes order-compactifications and bi-compactifications.

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If (Y, e) is an order-compactification of X, then X has two bispace structures; one is (X, τ_u, τ_d), and the other is (X, τ'_u, τ'_d), where τ'_u is the set of open upsets of the form $e^{-1}(V)$ for an open upset V of Y. In general the second is smaller.

Ordered spaces and biframes

A **biframe** is a triple $L = (L_0, L_1, L_2)$ with L_i subframes of a frame L_0 such that L_0 is generated by $L_1 \cup L_2$.

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If (X, τ_1, τ_2) is a bispace and $\tau = \tau_1 \vee \tau_2$, then (τ, τ_1, τ_2) is a biframe.

In particular, if $(X, \tau, \leq)$ is strongly order convex, then $\Omega(X) = (\tau, \tau_u, \tau_d)$ is a biframe.

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If we change Schauerte's definition of bi-compactification to assume that $f(M_0) = L_0$ but only $f(M_i) \subseteq L_i$, we may not recover Nachbin's order-compactifications, as we see in the next example.

Example. Let $X = \{x, y\}$ be a two-point set with discrete topology τ and trivial order. Let $L = \Omega(X)$. Then $L_0 = L_1 = L_2$ is the four-element Boolean algebra. Let $(Y, \pi) = (X, \tau)$ and define order on Y by letting $x \leq y$ as the only nontrivial inequality. Let $i : X \to Y$ be the identity and let $M = \Omega(Y)$. Then M_1 and M_2 are the three-element chain, and $i^{-1} = f : M_0 \to L_0$ is an onto dense biframe homomorphism. Moreover, $f(M_i)$ is properly contained in L_i. But (Y, i) is not an order-compactification of X because $x \nleq y$ but $i(x) \leq i(y)$.

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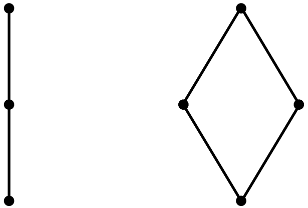

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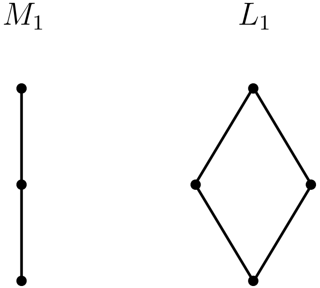

Thus, simply dropping the condition that $f(M_i) = L_i$ for $i = 1, 2$ does not capture the concept of order-compactification.

Points and Filters

Recall that if L is a frame, then a point of L is a frame homomorphism from L to $\mathbf{2}$. Points are in 1-1 correspondence with **completely prime filters** F, which are characterized by $\bigvee S \in F$ implies there is $s \in S$ with $s \in F$.

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A filter F is **Scott open** if $\bigvee S \in F$ implies there is a finite subset T of S with $\bigvee T \in F$. A completely prime filter is then Scott open.

Let $e : X \to Y$ be an order-preserving topological embedding onto a dense subspace of a Nachbin space Y, let $f = e^{-1}$, and let L (resp. M) be the biframe corresponding to X (resp. Y). Then $f : M \to L$ is a dense, onto biframe homomorphism.

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3. For $i = 1, 2$, let F_i be an open filter in L_i and let $F_1 \vee F_2 = L_0$. Then $f^{-1}(F_1) \vee f^{-1}(F_2) = M_0$.

The lemma is the motivation for Condition 3 below.

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Thus, if (Y, e) is an order-compactification of an ordered space X, then e^{-1} is a compactification of the corresponding biframe.

Proximities on a biframe

Banaschewski characterized compactifications of a frame in terms of **strong inclusions**, that is, binary relations $\triangleleft$ on L satisfying

(B1) $0 \triangleleft 0$ and $1 \triangleleft 1$.

(B2) If $a \triangleleft b$, then $a \prec b$.

(B3) If $a \leq b \triangleleft c \leq d$, then $a \triangleleft d$.

(B4) If $a, b \triangleleft c$, then $a \vee b \triangleleft c$.

(B5) If $a \triangleleft b, c$, then $a \triangleleft b \wedge c$.

(B6) If $a \triangleleft c$, then there is $b \in L$ with $a \triangleleft b \triangleleft c$.

(B7) If $a \triangleleft b$, then $\neg b \triangleleft \neg a$.

(B8) If $b \in L$, then $b = \bigvee \{a \in L : a \triangleleft b\}$.

Banaschewski's work has been generalized in two ways. Schaeuerte extended the notion of strong inclusion to the biframe setting and used it to characterize bi-compactifications. Together with Harding we generalized the notion of strong inclusion to that of a proximity on a frame and used it to characterize stable compactifications.

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Our notion of proximity on a biframe is a modification of Picado-Pultr's ideas and provides a common ground for the two cases above.

Associated subframes

If $(\triangleleft_1, \triangleleft_2)$ is a pair of relations on a biframe L, set

$$L_i' = \{b \in L_i : b = \bigvee\{a \in L_i : a \triangleleft_i b\}.$$

Then L_i' is a subframe of L_i (assuming appropriate properties of the relations).

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If $(\lhd_1, \lhd_2)$ is a pair of relations on a biframe L, set

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Then L_i' is a subframe of L_i (assuming appropriate properties of the relations).

Define $a \lhd_0 b$ if there are $u_i \in L_0$ with $a \lhd_i u_i$ and $u_1 \wedge u_2 \leq b$.

If F is a filter on L_i, then $F' := F \cap L_i'$ is a filter on F_i'.

Definition. A **proximity** on a biframe L is a pair $\triangleleft = (\triangleleft_1, \triangleleft_2)$ of relations on L_0 satisfying

(P1) $0 \triangleleft_i 0$ and $1 \triangleleft_i 1$.

(P2) If $a \triangleleft_i b$, then $a \prec_i b$.

(P3) If $a \leq b \triangleleft_i c \leq d$, then $a \triangleleft_i d$.

(P4) If $a, b \triangleleft_i c$, then $a \vee b \triangleleft_i c$.

(P5) If $a \triangleleft_i b, c$, then $a \triangleleft_i b \wedge c$.

(P6) If $a \triangleleft_i c$, then there is $b \in L_i'$ with $a \triangleleft_i b \triangleleft_i c$.

(P7) If $a \triangleleft_i b$, then $\neg_k b \triangleleft_k \neg_k a$.

(P8) $b = \bigvee \{a \in L_0 : a \triangleleft_0 b\}$.

(P9) For $i = 1, 2$, let F_i be an open filter in L_i and let $F_1 \vee F_2 = L_0$. Then $F_1' \vee F_2' = L_0$.

If $\triangleleft$ is a proxmity on L we call $(L, \triangleleft)$ a **proximity biframe**.

The difference between this definition and Picado-Pultr's is that we start with a biframe while they start with a frame. That the biframe (L_0, L_1', L_2') they (and we) get is different than (L_0, L_1, L_2) turns out to be the difference between compactifications and bi-compactifications.

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If $L'_i = L_i$ for $i = 1, 2$, then our definition is essentially the same as Schauerte's.

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If Y is an order-compactification of X and $\Delta = Y \setminus X$, then the induced proximity on the biframe of X is given by $U \lhd_1 V$ if $\uparrow \mathrm{cl}(U) \subseteq \mathrm{int}(V \cup \Delta)$, where closure and interior is computed in Y.

Compactifications from proximities

Definition. Let $(L, \triangleleft)$ be a proximity biframe. For $i = 1, 2$, we call an ideal I of L_0 an **i-round ideal** if for each $a \in I$ there is $b \in I$ with $a \triangleleft_i b$. Let $\mathcal{R}_i$ be the set of all i-round ideals of L_0.

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An ideal I is i-round iff $I = \bigvee\{\twoheaddownarrow_i b : b \in I\}$, and I is round iff $I = \bigvee\{\twoheaddownarrow_0 b : b \in I\}$.

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If I is a round ideal, it need not be the case that $a \in I$ implies there is $b \in I$ with $a \lhd_0 b$. However, I is generated by $\{a \in I : \exists b \in I, a \lhd_0 b\}$.

Let $\mathcal{R}_0$ be the frame of round ideals of L and let $\mathcal{R}_i$ be the subframe of i-round ideals. Then $\mathcal{R} = (\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2)$ is a compact regular biframe.

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Proposition. Let $\triangleleft$ be a proximity on a biframe L. Define $f : \mathcal{R}_0 \to L_0$ by $f(I) = \bigvee I$. Then $(\mathcal{R}, f)$ is a compactification of L. Moreover, the right adjoint $r : L_0 \to \mathcal{R}_0$ of $f : \mathcal{R}_0 \to L_0$ is given by $r(b) = \twoheaddownarrow_0 b$ for all $b \in L_0$.

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Corollary. A biframe has a compactification iff it has a proximity.

Equivalent compatifications

Two compactifications (M, f) and (M', f') of a biframe L are **equivalent** if there is a biframe isomorphism $k : M_0 \to M_0'$ with $f = f' \circ k$.

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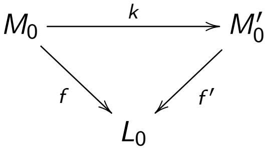
We denote by $[M, f]$ the equivalence class of a compactification (M, f), and define a partial order by $[M, f] \leq [M', f']$ if there is a biframe homomorphism $k : M_0 \to M_0'$ with $f = f' \circ k$.



Main theorem

The set of all proximities on L is a poset by setting $\lhd \leq \lhd'$ if $\lhd_1 \subseteq \lhd_1'$ and $\lhd_2 \subseteq \lhd_2'$.

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If (M, f) is a compactification of L, we saw how to get a proximity from it, via $a \triangleleft_i b$ iff $r(a) \prec_i r(b)$.

Going the other direction, if $\triangleleft = (\triangleleft_1, \triangleleft_2)$ is a proximity on L, then $\mathcal{R}(L, \triangleleft)$ is a compactification of L, and is equivalent to any compactification of L inducing $\triangleleft$.

Schauerte's theorem

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3. The poset of order-compactifications of X is isomorphic to the poset of proximities on the biframe $\Omega(X)$.

Blatter-Seever theorem

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Conversely, given a proximity $\lhd$ on $\Omega(X)$, we obtain a quasi-proximity on $\mathcal{P}(X)$ by defining $A \lhd_1 B$ if there are $U, V \in \tau_1$ with $A \subseteq U \lhd_1 V \subseteq B$.

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Blatter and Seever's arguments are analytic in nature. Our results yield an alternate point-free proof of their result.

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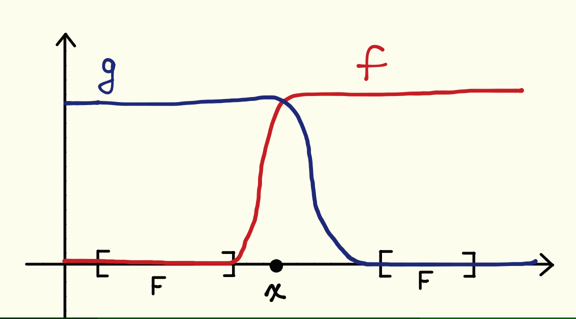

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Künzi (1990) gave an example of a completely regular ordered space that is not strictly completely regular. Thus, strictly completely regular is stronger than completely regular.

The rather inside relation on biframes

Let L be a biframe. For $i = 1, 2$, we define $\lll_i$ on L_0 by $a \lll_i b$ if there is a family $\{c_p\} \subseteq L_i$ for $p \in \mathbb{Q} \cap [0, 1]$ such that $a \leq c_0$, $c_1 \leq b$, and $c_p \prec_i c_q$ whenever $p < q$.

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$$L_i' = \{b \in L_i : b = \bigvee \{a \in L_i : a \lll_i b\}\}.$$

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For each filter F_i of L_i, let $F_i' = F_i \cap L_i'$. Then F_i' is a filter of L_i'.

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Künzi's example shows that there exist completely regular biframes that are not strictly completely regular.

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Thus, if a biframe L is completely regular, then $\triangleleft = (\lll_1, \lll_2)$ is a proximity on L. In fact, $\triangleleft$ is the largest proximity on L. Therefore, the compactification of L corresponding to $\triangleleft = (\lll_1, \lll_2)$ is the largest compactification of L.

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If L corresponds to a completely regular ordered space $(X, \tau, \leq)$, then the largest compactification of L corresponds to the Nachbin order-compactification of $(X, \tau, \leq)$, which is the largest order-compactification of $(X, \tau, \leq)$.

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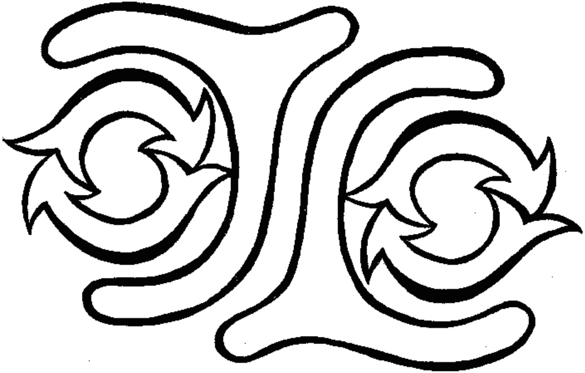

ToLo picture drawn by Marcel Erné