Modal Logics of Metric Spaces

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Joint work with:
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ToLo 4 Tbilisi, Georgia

25 June 2014

Modal Language and Interpretation

A Topological Interpretation (in space X)

Variable	:	p	'is'	A	: Subset
Negation	:	$\neg$	'is'	$X-$	: Complement
Disjunction	:	$\vee$	'is'	$\cup$	: Union
Diamond	:	$\Diamond$	'is'	$\mathbf{c}$	: Closure

S4 and Kuratowski Closure

$\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$	'is'	$\mathbf{c}(A \cup B) = \mathbf{c}A \cup \mathbf{c}B$
$\Diamond\bot \leftrightarrow \bot$	'is'	$\mathbf{c}\varnothing = \varnothing$
$p \rightarrow \Diamond p$	'is'	$A \subseteq \mathbf{c}A$
$\Diamond\Diamond p \rightarrow \Diamond p$	'is'	$\mathbf{cc}A \subseteq \mathbf{c}A$

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Historical Development

Theorem (McKinsey and Tarski 1944)

The logic of all topological spaces is **S4**.

Theorem (McKinsey and Tarski 1944)

The logic of any separable dense-in-itself metric space is **S4**.

Corollaries

$\text{Log}(\mathbb{R}) = \mathbf{S4}$,
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Strengthening M&T's Results

Theorem (Rasiowa and Sikorski 1963)

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Special Case (Bezhanishvili and Harding 2011)

Characterized the logics of metric Stone spaces.

Goal

Characterize the logic of each metric space.

Approach

X → Weakly Scattered

X → Not Weakly Scattered

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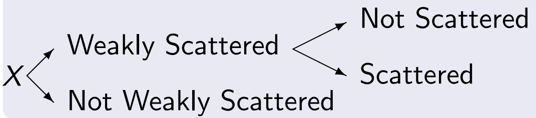

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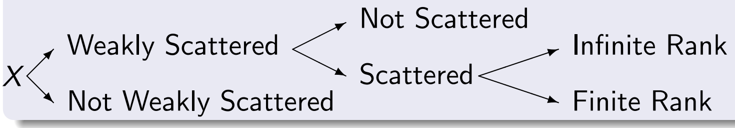

Some Definitions

Recall

1. $x \in X$ isolated: $\{x\}$ is open
   $\mathrm{iso}(X)$: set of isolated points in X
2. X dense-in-itself (dii): $\mathrm{iso}(X) = \varnothing$
3. X is scattered: $\mathrm{iso}(Y) \neq \varnothing$ for every subspace $Y (\neq \varnothing)$ of X
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Main Tools: Frames

S4-Frame

1. Frame $\mathfrak{F} = (W, R)$: $W \neq \varnothing$ and $R \subseteq W \times W$
2. S4-Frame $\mathfrak{F}$: R is reflexive and transitive
3. Rooted $\mathfrak{F}$: $\exists r \in W, \forall w \in W, rRw$

Finite Model Property (FMP)

Logic **L** has FMP: for any nontheorem φ of **L**, there is a finite frame $\mathfrak{F}$ for **L** refuting φ.

All the logics at play herein have the FMP!

Example: S4 and $\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q)$

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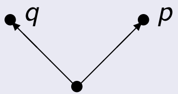

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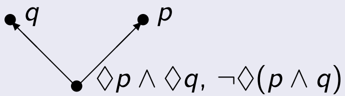

Main Tools: Truth Preserving Operations

Truth Preserving Operations

Open Subspace: $Y \subseteq X$ and Y open in X

Interior Image: $f : X \to Y$ interior (continuous and open) and onto

$\mathrm{Log}(X) \subseteq \mathrm{Log}(Y)$, equivalently $Y \not\models \varphi$ implies $X \not\models \varphi$.

Viewing Frames as Spaces

S4-frame $\mathfrak{F} = (W, R)$, Alexandroff topology τ_R: R-upsets in $\mathfrak{F}$

Via τ_R, Kripke semantics is a special case of topological semantics.

Move freely: (W, R) vs. (W, τ_R)

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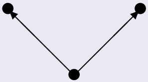

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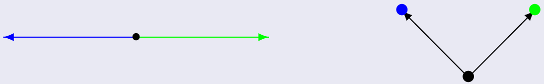

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Any finite rooted **S4**-frame is an interior image of any dii metric space.

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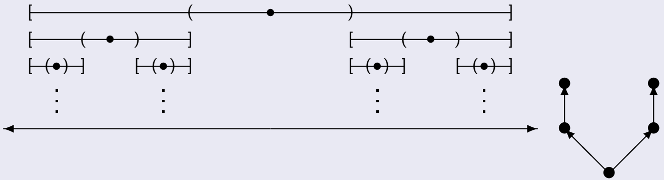

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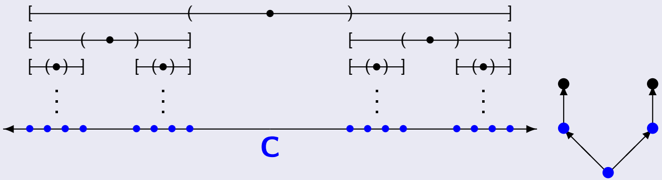

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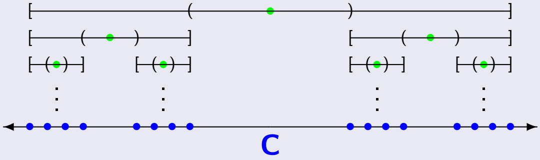

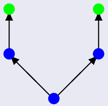

Cantor-Bendixon Decomposition

The CB Theorem

Let X be a space.
There are subspaces S and D of X such that
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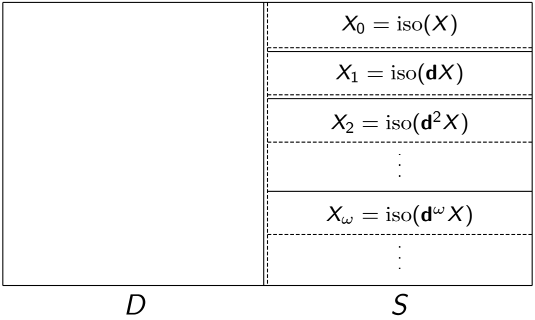


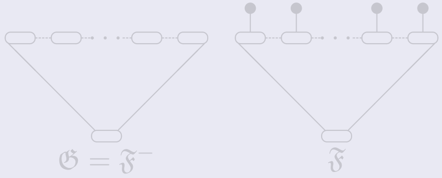
Quasitrees, Trees and Top Thin Quasitrees

Quasitrees (qtree)

1. qtree $\mathfrak{F} = (W, R)$: rooted **S4**-frame satisfying $\forall u, v \in R^{-1}(w)$ either uRv or vRu
2. Tree $\mathfrak{T} = (W, R)$: antisymmetric qtree
3. Height of finite tree $\mathfrak{T}$: greatest cardinality of a chain in $\mathfrak{T}$

Top Thin Quasitrees (tt-qtrees)

tt-qtree $\mathfrak{F}$: Built from finite qtree $\mathfrak{G}$ by adding a ‘new top’ to each maximal cluster; denote $\mathfrak{G}$ by $\mathfrak{F}^-$



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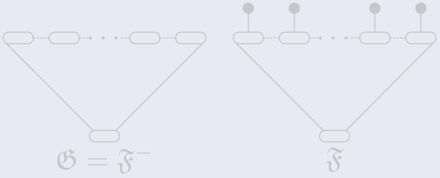

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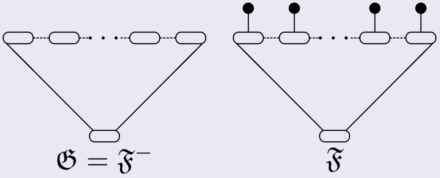

The Logics of Interest

Logics

$$\begin{aligned}
&\mathbf{S4} \\
\mathbf{S4.1} &= \mathbf{S4} + \Box\Diamond p \to \Diamond\Box p \\
\mathbf{S4.Grz} &= \mathbf{S4} + \mathbf{grz} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to p \\
\mathbf{S4.Grz}_n &= \mathbf{S4.Grz} + \mathbf{bd}_n
\end{aligned}$$

Formulas

$$\begin{aligned}
\mathbf{bd}_1 &= \Diamond\Box p_1 \to p_1 \\
\mathbf{bd}_{n+1} &= \Diamond(\Box p_{n+1} \wedge \neg\mathbf{bd}_n) \to p_{n+1}
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Qtrees for Logics of Interest and Main Result

(Known) Lemma

1. **S4** is the logic of finite rooted qtrees.
2. **S4.1** is the logic of tt-qtrees.
3. **S4.Grz** is the modal logic of finite trees.
4. $\mathbf{S4.Grz}_n$ is the modal logic of finite trees of height $\leq n$.

Theorem: Main Result (Brief Version)

The modal logics of metric spaces form the chain

$$\mathbf{S4.Grz}_1 \supset \mathbf{S4.Grz}_2 \supset \mathbf{S4.Grz}_3 \supset \cdots \ \mathbf{S4.Grz} \supset \mathbf{S4.1} \supset \mathbf{S4}.$$

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Case I

Theorem

S4 is the logic of any non-weakly scattered metric space X.

Proof Sketch

$Y = X - \mathbf{c}(\mathrm{iso}(X))$ is nonempty open dii subspace of X.
φ: nontheorem of **S4**
Finite rooted qtree $\mathfrak{F}$: $\mathfrak{F} \not\models \varphi$
$\mathfrak{F}$ is interior image of Y
$Y \not\models \varphi$
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Case I

Theorem

S4 is the logic of any non-weakly scattered metric space X.

Proof Sketch

$Y = X - \mathbf{c}(\mathrm{iso}(X))$ is nonempty open dii subspace of X.

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Finite rooted qtree $\mathfrak{F}$: $\mathfrak{F} \not\models \varphi$

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Case II: Strongly Zero-Dimensional

Telgarski's Theorem

Each Scattered metric space is strongly zero-dimensional.

Two Useful Lemmas

1. If $F_1, \ldots, F_n$ are nonempty pairwise disjoint closed subsets of a strongly zero-dimensional normal space X, then there are pairwise disjoint clopen subsets $U_1, \ldots, U_n$ of X such that $F_i \subseteq U_i$ and $X = U_1 \cup \cdots \cup U_n$.
2. For any discrete subset A of a metric space X, there is a disjoint family of balls $\{B_{\varepsilon_a} : a \in A\}$.

Definition

Strongly zero-dimensional X: completely regular space such that every finite cover of X consisting of cozero-sets has a finite pairwise disjoint open refinement

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Case II: The Mapping

A Mapping Lemma

$n \in \omega$,
$\mathfrak{T}$: finite tree, height at most $n+1$,
X: scattered metric space;
If $X_n \neq \varnothing$ then there is an onto interior map $f : X_0 \cup \cdots \cup X_n \to \mathfrak{T}$ such that $f(x)$ is the root of $\mathfrak{T}$ for each $x \in X_n$.

Proof

By induction on $n \in \omega$

Illustrated by pictures

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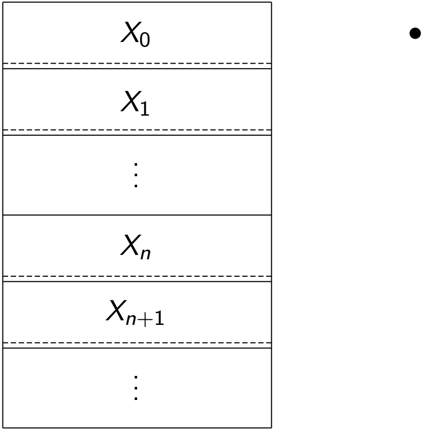

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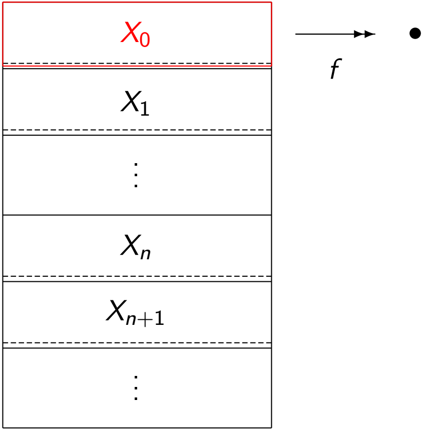

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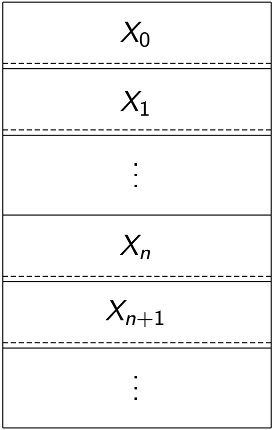

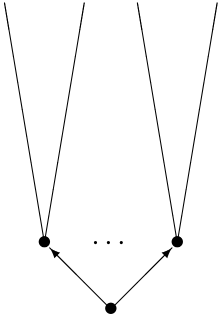

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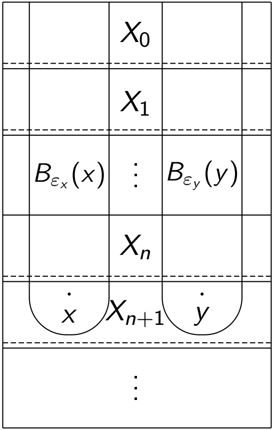

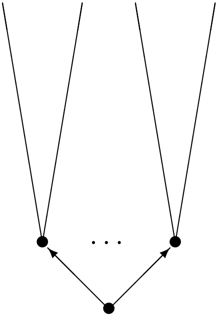

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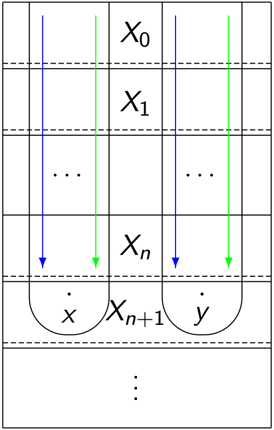

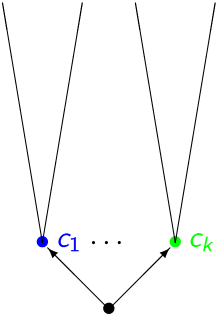

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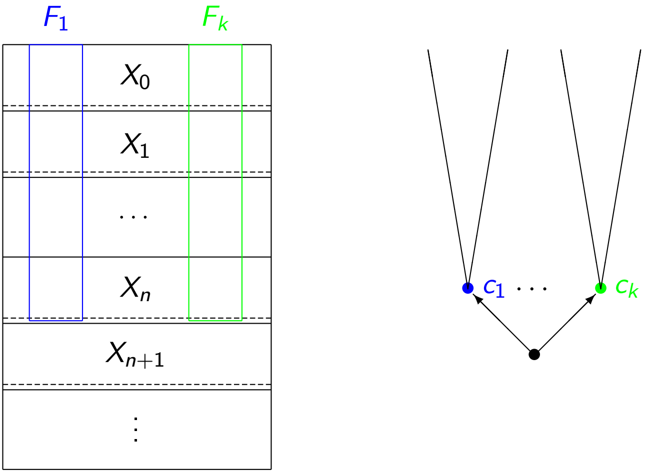

$X_{n+1} \subseteq \mathbf{c}F_i$

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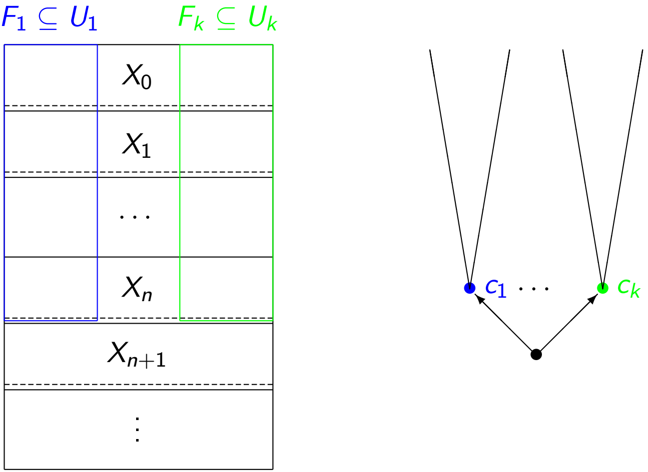

$$X_0 \cup \cdots \cup X_n = U_1 \cup \cdots \cup U_k$$

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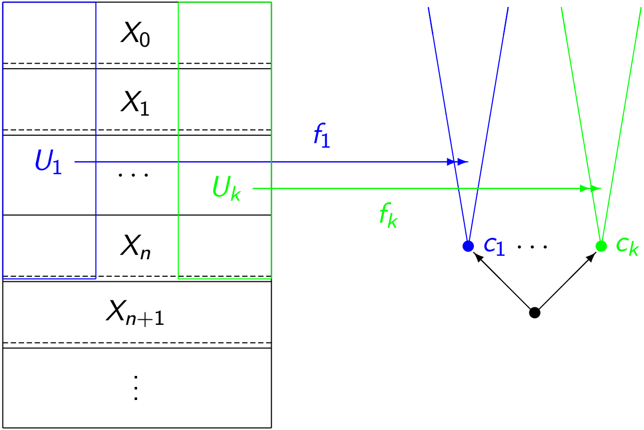

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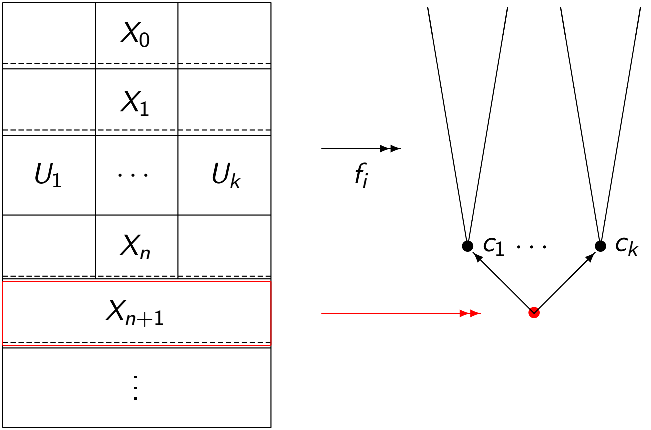

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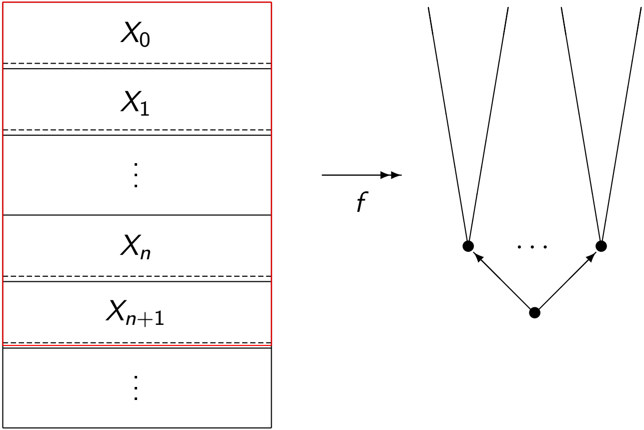

Case II: Infinite Rank

Theorem

S4.Grz is the logic of ...

1. all scattered spaces. (Esakia 1981)
2. any ordinal $\alpha \geq \omega^\omega$. (Abashidze 1987/Blass 1990)

Theorem

S4.Grz is the logic of any scattered metric space X of infinite rank.

Proof Sketch

$X \models$ **grz**: by above Lemma

φ: nontheorem of **S4.Grz**

Finite tree $\mathfrak{T}$ of height $n (\geq 1)$: $\mathfrak{T} \not\models \varphi$

$\mathfrak{T}$ is interior image of $X_0 \cup \cdots \cup X_{n-1}$: $X_0 \cup \cdots \cup X_{n-1} \not\models \varphi$

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Lemma

In any scattered space X and any $n \in \omega$, the interpretation of $\mathbf{bd}_{n+1}$ contains $X_0 \cup \cdots \cup X_n$.

Theorem

$\mathbf{S4.Grz}_n$ is the logic of any scattered metric space X of rank $n \in \omega$.

Proof Sketch

$X = X_0 \cup \cdots \cup X_{n-1} \models \mathbf{bd}_n$: by above Lemma
$X \models \mathbf{grz}$: by Lemma on previous slide
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Case III: The Mapping

A Mapping Lemma

Any tt-qtree $\mathfrak{F}$ is an interior image of any weakly scattered non-scattered metric space X.

Key Idea of Proof

$X = S \cup D$;
$\mathfrak{F}^- = \mathfrak{F} - \max(\mathfrak{F})$;
$\exists g : D \to \mathfrak{F}^-$;
Extend g to $f : X \to \mathfrak{F}$ such that $f(S) = \max(\mathfrak{F})$.
Proceed by induction on the height of $\mathfrak{F}$.

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Proceed by induction on the height of $\mathfrak{F}$.

Case III: The Mapping

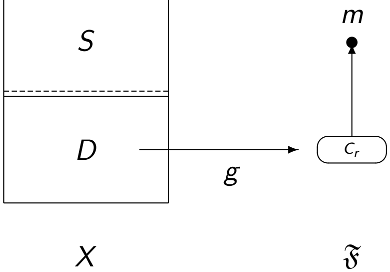
A Mapping Lemma

Any tt-qtree $\mathfrak{F}$ is an interior image of any weakly scattered non-scattered metric space X.

Key Idea of Proof

$X = S \cup D$;
$\mathfrak{F}^- = \mathfrak{F} - \max(\mathfrak{F})$;
$\exists g : D \to \mathfrak{F}^-$;
Extend g to $f : X \to \mathfrak{F}$ such that $f(S) = \max(\mathfrak{F})$.
Proceed by induction on the height of $\mathfrak{F}$.

Case III: The Mapping–Base Case



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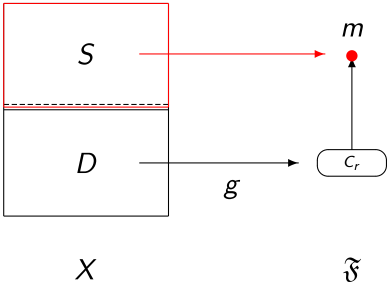

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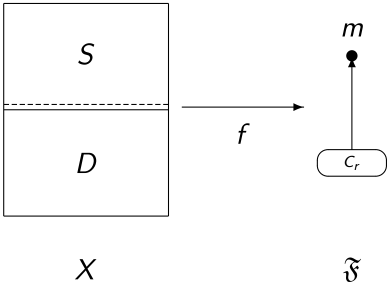
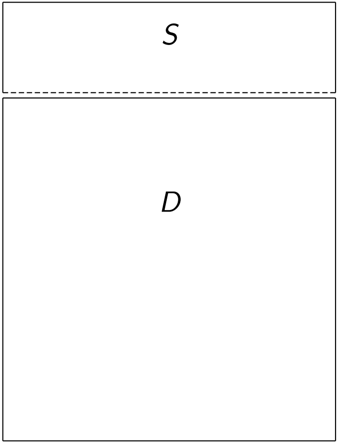
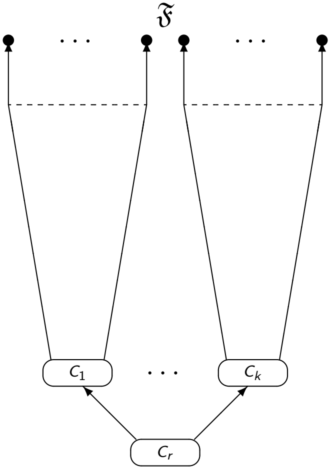


Case III: The Mapping–Inductive Case





$$\mathfrak{F}^- = \mathfrak{F} - \max(\mathfrak{F})$$

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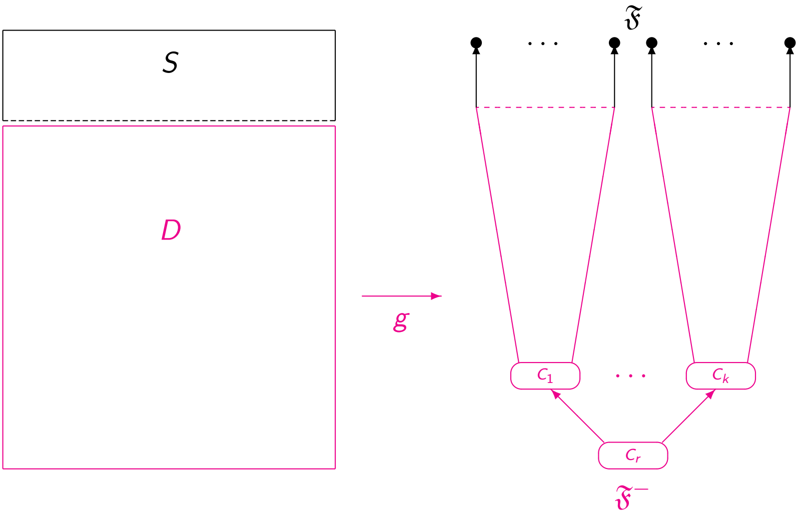

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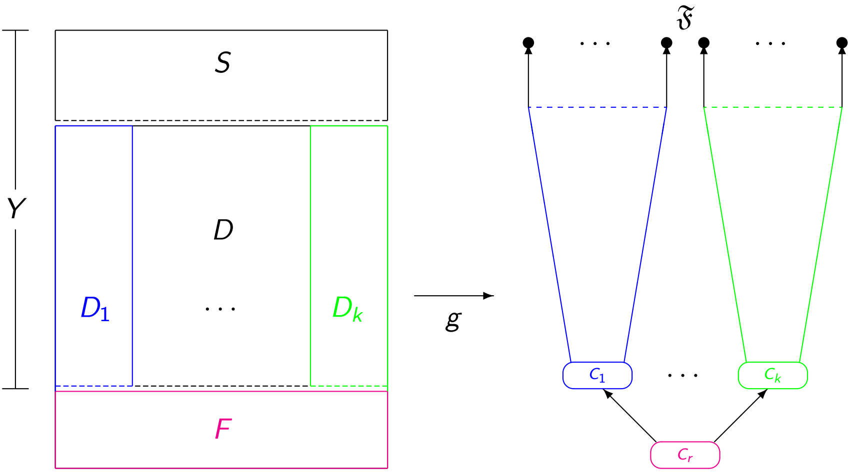
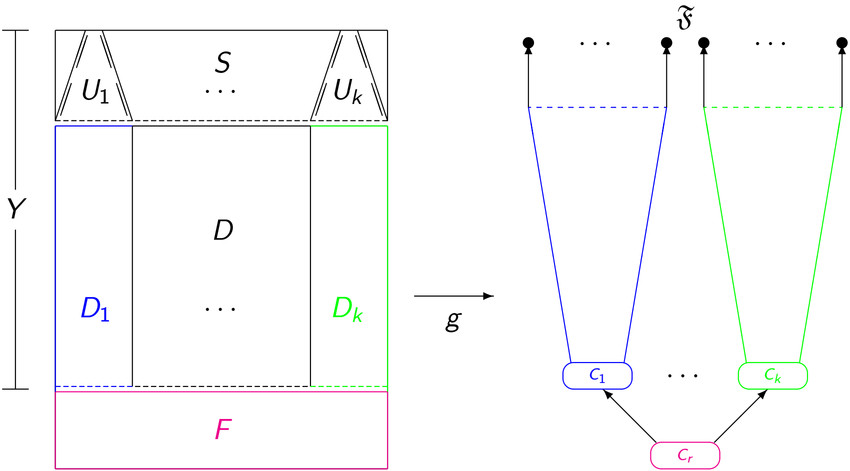


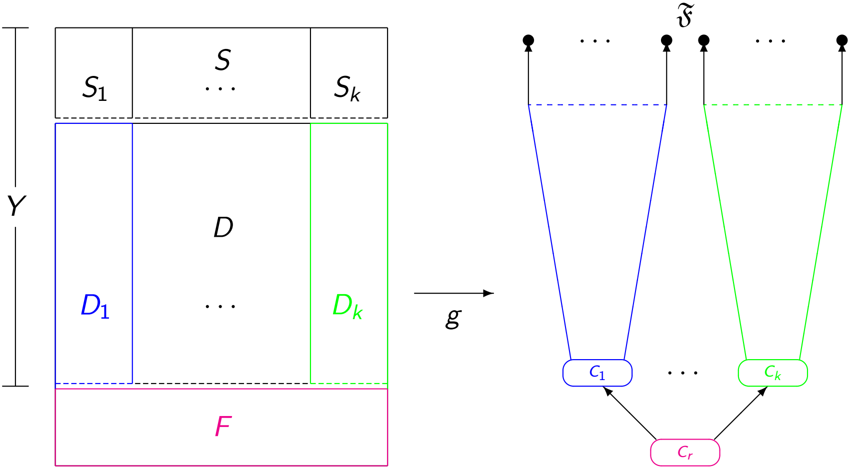
D_i closed in $Y = X - F$

Case III: The Mapping–Inductive Case



$D_i \subseteq U_i$ open in Y
$\mathbf{c}_Y U_i$ pairwise disjoint
$S \cap \mathbf{c}_Y U_i$ closed in S

Case III: The Mapping–Inductive Case



S_i clopen partition of S
$\mathbf{c}_Y S_i = S_i \cup D_i$

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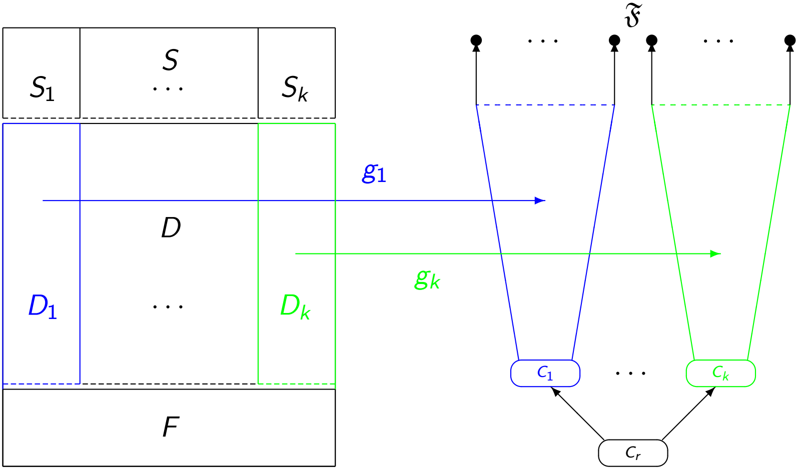
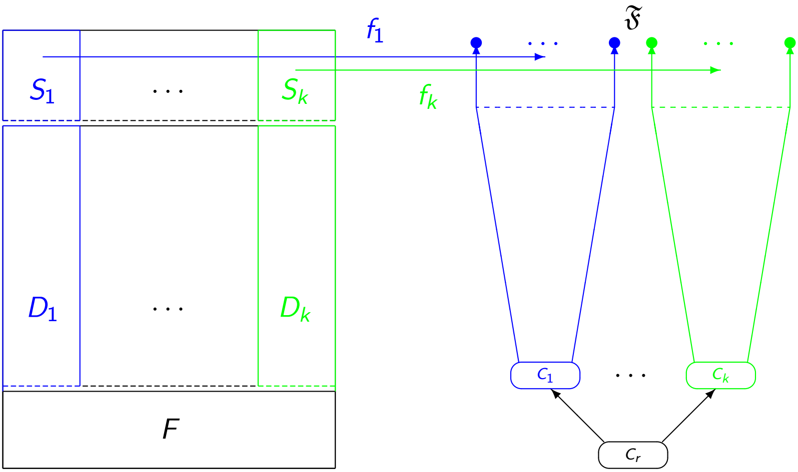


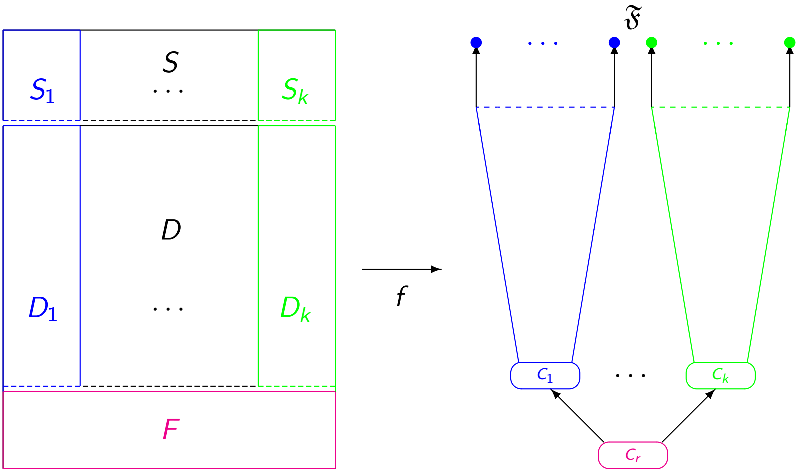
g_i: the restriction of g to D_i

Case III: The Mapping–Inductive Case



f_i the extension of g_i given by inductive hypothesis

Case III: The Mapping–Inductive Case



f is defined by 'the colors'

Case III

(Known) Lemma

S4.1 is the logic of all weakly scattered spaces.

Theorem

S4.1 is the logic any weakly scattered non-scattered metric space X.

Proof Sketch

$X \models$ **S4.1**: by above Lemma
φ: nontheorem of **S4.1**
tt-qtree $\mathfrak{F}$: $\mathfrak{F} \not\models \varphi$
$\mathfrak{F}$ is interior image of X
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$X \not\models \varphi$

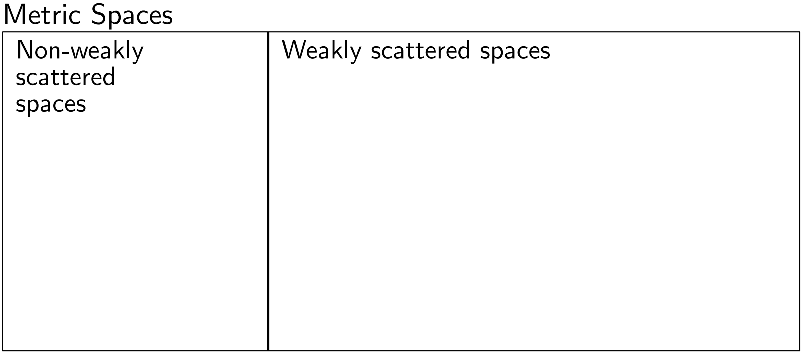
Modal Logics of Metric Spaces

Theorem: Main Result (Full Version)

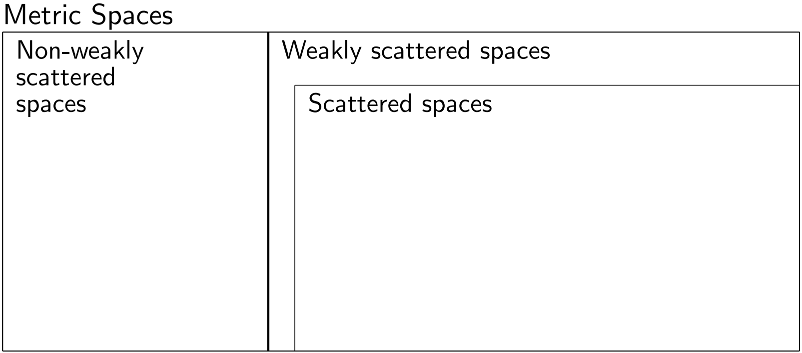
Let X be a (nonempty) metric space.

1. If X is not weakly scattered, then $\mathrm{Log}(X) = \mathbf{S4}$.
2. If X is weakly scattered but not scattered, then $\mathrm{Log}(X) = \mathbf{S4.1}$.
3. If X is scattered and has infinite rank, then $\mathrm{Log}(X) = \mathbf{S4.Grz}$.
4. If X is scattered and has rank $n \in \omega - \{0\}$, then $\mathrm{Log}(X) = \mathbf{S4.Grz}_n$.

Main Result: Picture

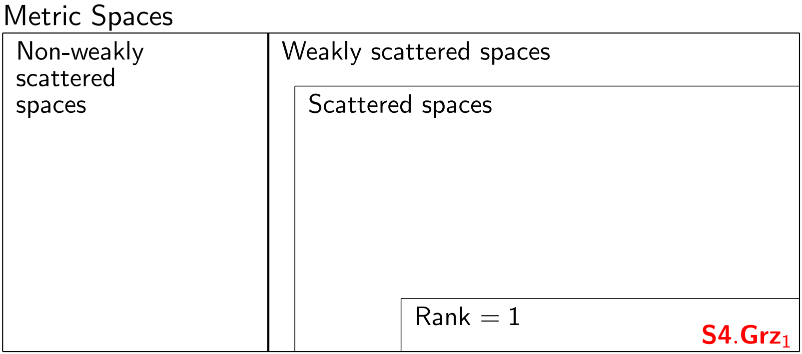


Main Result: Picture



Main Result: Picture

$$\mathbf{S4.Grz_1} \supset \mathbf{S4.Grz_2} \supset \mathbf{S4.Grz_3} \supset \cdots \ \mathbf{S4.Grz} \supset \mathbf{S4.1} \supset \mathbf{S4}$$



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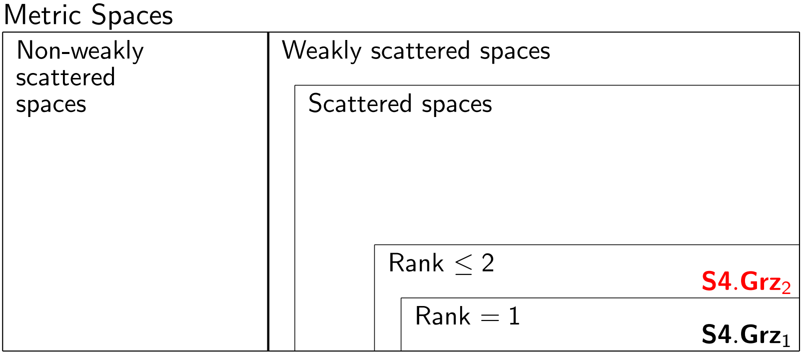
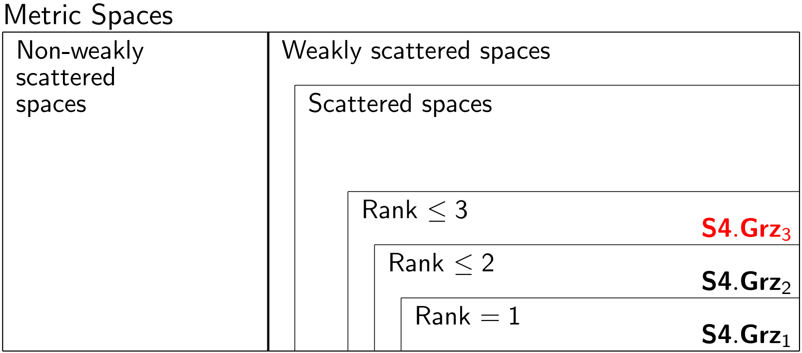


Main Result: Picture

$$\mathbf{S4.Grz_1} \supset \mathbf{S4.Grz_2} \supset \mathbf{S4.Grz_3} \supset \cdots \quad \mathbf{S4.Grz} \supset \mathbf{S4.1} \supset \mathbf{S4}$$



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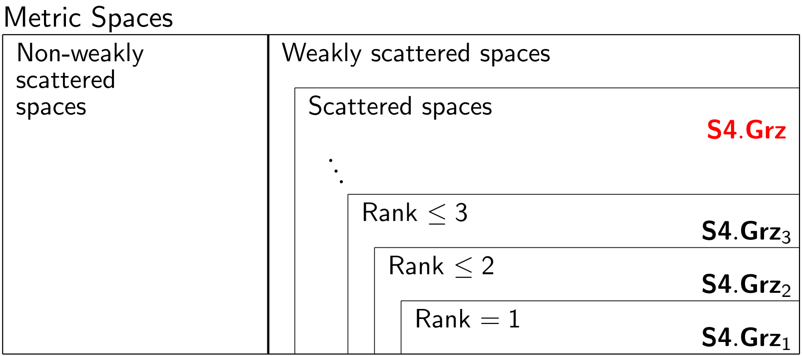

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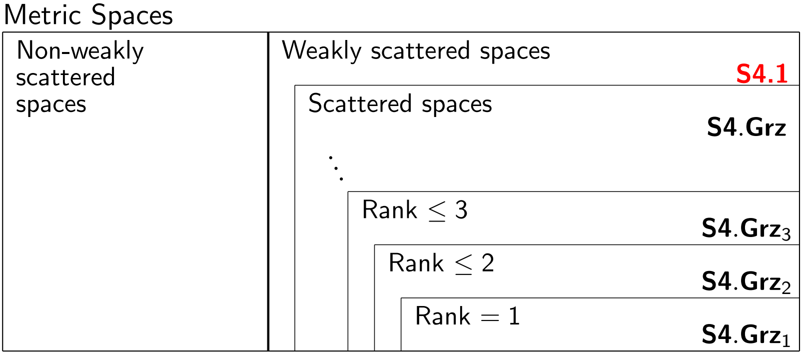
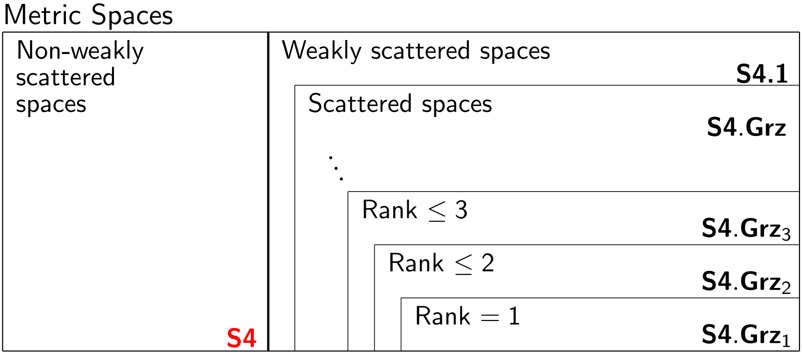


Main Result: Picture

$$\mathbf{S4.Grz}_1 \supset \mathbf{S4.Grz}_2 \supset \mathbf{S4.Grz}_3 \supset \cdots \; \mathbf{S4.Grz} \supset \mathbf{S4.1} \supset \mathbf{S4}$$



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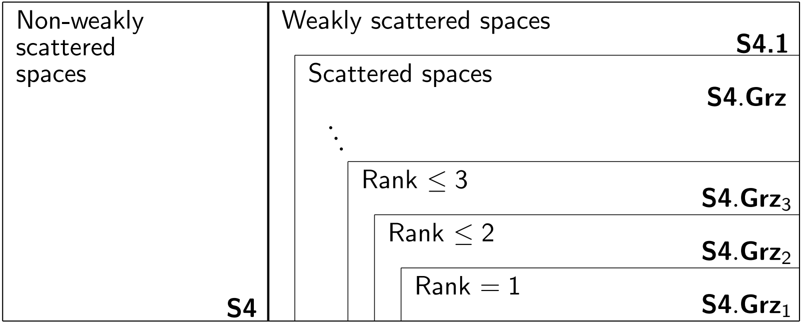
Metric Spaces



Open Question

Generalize to paracompact spaces?

The End

We are happy to distribute a pre-published version of the paper containing complete details on the results presented today, please inquire!

Thanks...

... for your attention!
And
... to the organizers!

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