LOCAL HOMEOMORPHISMS AND ESAKIA DUALITY

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What do we want

There is known an algebraic description of those homomorphisms $f^{-1} : \mathrm{Op}(Y) \to \mathrm{Op}(X)$ between Heyting algebras of open sets of topological spaces which correspond to *local homeomorphisms* $f : X \to Y$ between these spaces.

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The question we want to investigate is this.

- What is the analog of local homeomorphisms for Esakia spaces?
- What kind of Heyting algebra homomorphisms do they correspond to?

- Local homeomorphisms in topological semantics.
- Esakia duality (reminder)
- Common ground – the finite case: strict p-morphisms.
- Infinite over finite – Stone presheaves.
- Duality between gluing local homeomorphisms and HSP – the variety $\mathscr{V}_H$ of étale H-algebras.
- Special properties of $\mathscr{V}_H$.
- Conclusion.

Open continuous maps

Definition 1.

A continuous map $f : Y \to X$ between topological spaces is *open* if $f(U) \in \mathrm{Op}(X)$ for any $U \in \mathrm{Op}(X)$.

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For a topological space X, let Op/X be the *category of open continuous maps over* X. Its objects are open continuous maps with codomain X and morphisms from $f : Y \to X$ to $f' : Y' \to X$ are open continuous maps $g : Y \to Y'$, such that $f' \circ g = f$.

Local homeomorphisms

Definition 2 (Local homeomorphism).

A continuous map $f : Y \to X$ between topological spaces is a *local homeomorphism* if for any point $y \in Y$ there exists an open U, such that $y \in U$ and $f|_U$ is a homeomorphism between U and $f(U)$.

In particular, any local homeomorphism is an open map.

Local homeomorphisms

Proposition 1 (Folklore).

If $f : Y \to X$ is a local homeomorphism then the Heyting algebra homomorphism $f^{-1} : \mathrm{Op}(X) \to \mathrm{Op}(Y)$ satisfies

$$\bigcup_{U \in \mathrm{Op}(X)} \left(V \leftrightarrow f^{-1}(U)\right) = Y$$

for any $V \in \mathrm{Op}(Y)$.

Local homeomorphisms

For a topological space X, let LH/X be the *category of local homeomorphisms over* X. Its objects are local homeomorphisms with codomain X and morphisms from $f : Y \to X$ to $f' : Y' \to X$ are local homeomorphisms $g : Y \to Y'$, such that $f' \circ g = f$.

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It follows, that LH/X is a subcategory of Op/X. In fact it turns out to be a *full* subcategory.

Some results

Our main intuitive starting point is

Proposition 2 (see “Elephant” 4.2.4(e)).

For every topological space X, the category LH/X of local homeomorphisms over X is a topos.

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Proposition 3 (Folklore?).

$f : Y \to X$ is a local homeomorphism if and only if f is an open map and $\Delta_f : Y \to Y \times_X Y$ is also an open map.

Some results

Theorem 2.1.

LH / X *is the smallest subcategory of* Op / X *which contains the object* $1_X : X \to X$ *(identity map of* X*) and is closed under coproducts, quotient objects, and subobjects.*

Some results

Theorem 2.1.

$\mathrm{LH}\,/\,\mathrm{X}$ *is the smallest subcategory of* $\mathrm{Op}\,/\,\mathrm{X}$ *which contains the object* $1_X : X \to X$ *(identity map of* X*) and is closed under coproducts, quotient objects, and subobjects.*

Remark.

Coprodruts in $\mathrm{Op}\,/\,\mathrm{X}$ coincide with coproducts in the category of topological spaces and are given by disjoint unions. A quotient object of $f : Y \to X$ is given by an object $f' : Y' \to X$ and a surjective open map $q : Y \twoheadrightarrow Y'$, such that $f' \circ q = f$. A subobject of $f : Y \to X$ is any open subset $U \subseteq Y$ with the map $f \circ i$, where i is the embedding of U into Y.

Esakia spaces

Definition 3.

A *Priestley space* is an ordered Stone space $\mathcal{X} = (X, \leqslant)$ which satisfies the *Priestley separation axiom*

- for every $x, y \in X$ with $x \nleqslant y$ there is a clopen upset U such that $x \in U$ and $y \notin U$.

Definition 4.

$\leqslant$ is called *point-closed* if $\uparrow(x)$ is closed for every $x \in X$.
$\leqslant$ is called *clopen* if $\downarrow(C)$ is clopen for every clopen set C.
A Priestley space $\mathcal{X} = (X, \leqslant)$ is an *Esakia space* if $\leqslant$ is a point-closed clopen relation.

Esakia morphisms

Definition 5.

A monotone map $f : (Y, \leqslant_Y) \to (X, \leqslant_X)$ between ordered sets is a *p-morphism* if for arbitrary $y \in Y$ and arbitrary $f(y) \leqslant_X x \in X$ there exists an $y \leqslant_Y y'$ such that $f(y') = x$.

Definition 6.

An *Esakia morphism* $(Y, \leqslant_Y) \to (X, \leqslant_X)$ between Esakia spaces is a monotone continuous map which is a p-morphism.

Esakia duality

Let $\mathcal{HA}$ denote the category of Heyting algebras and their homomorphisms and let $\mathcal{E}$ denote the category of Esakia spaces and Esakia morphisms.

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The following well known and important theorem holds:

Theorem 3.1 (Esakia).

There exists duality between category of Esakia spaces and the category of Heyting algebras.

$$\mathcal{HA}^{\mathrm{op}} \simeq \mathcal{E}$$

Strict p-morphisms

Definition 7.

A *strict p-morphism* $f : (Y, \leqslant_Y) \to (X, \leqslant_X)$ between ordered sets is a p-morphism such that for arbitrary $y \in Y$ and arbitrary $f(y) \leqslant_X x \in X$ there exists a **unique** $y \leqslant_Y y'$, such that $f(y') = x$.

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Proposition 4.

A p-morphism $f : Y \to X$ between ordered sets is strict if and only if the map $\Delta_f : Y \to Y \times_X Y$ is a p-morphism too.

Strict p-morphisms

Proposition 5.

A p-morphism $f : (Y, \leqslant_Y) \to (X, \leqslant_X)$ is strict **if and only if** *the Heyting algebra homomorphism $f^{-1} : \uparrow(X) \to \uparrow(Y)$ between Heyting algebras of up-sets satisfies*

$$\bigcup_{U \in \uparrow(X)} \left(V \leftrightarrow f^{-1}(U)\right) = Y$$

for any $V \in \uparrow(Y)$.

Slices $\mathcal{P}/(X,\leqslant)$, $\mathcal{SP}/(X,\leqslant)$.

Let $\mathcal{P}/(X,\leqslant)$ be the category of p-morphisms over $(X,\leqslant)$ and $\mathcal{SP}/(X,\leqslant)$– the category of strict p-morphisms over $(X,\leqslant)$.

Theorem 4.1.

$\mathcal{SP}/(X,\leqslant)$ is the smallest full subcategory of $\mathcal{P}/(X,\leqslant)$, which contains the object $1_X : X \to X$ and is closed under coproducts, quotient objects, and subobjects.

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Theorem 4.1.

$\mathcal{SP}/(X, \leqslant)$ is the smallest full subcategory of $\mathcal{P}/(X, \leqslant)$, which contains the object $1_X : X \to X$ and is closed under coproducts, quotient objects, and subobjects.

In fact it is easy to see that if we equip posets with their *Alexandroff topologies* (open sets are all up-sets) then a map $(Y, \leqslant) \to (X, \leqslant)$

- is continuous iff it is monotone
- is open iff it is a p-morphism and
- is a local homeomorphism iff it is a strict p-morphism.

Strict p-morphisms between Esakia spaces

Similarly, for an Esakia space $\mathcal{X}$ let $\mathcal{E}/\mathcal{X}$ be the category of Esakia morphisms over $\mathcal{X}$ and let $\mathcal{SE}/\mathcal{X}$ be the category of strict Esakia morphisms over $\mathcal{X}$.

Corollary 1.

For a finite Esakia space $\mathcal{X}$, the full subcategory of $\mathcal{SE}/\mathcal{X}$ consisting of $\mathcal{Y} \to \mathcal{X}$ with finite $\mathcal{Y}$, is the smallest subcategory of $\mathcal{E}/\mathcal{X}$ which contains the object $1_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ and is closed under **finite** *coproducts, quotient objects, and subobjects.*

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This is clear since finite Esakia spaces are essentially nothing else but finite posets.

Presheaves.

If $f : (Y, \leqslant) \to (X, \leqslant)$ is a strict p-morphism then for arbitrary $x_1 \leqslant x_2$ from X it determines a mapping from $f^{-1}(x_1)$ to $f^{-1}(x_2)$: $y_1 \in f^{-1}(x_1)$ goes to the unique $y_2 \in f^{-1}(x_2)$ with $y_1 \leqslant y_2$.
In this way we obtain a functor $F_f : (X, \leqslant) \to Set$.
Moreover a morphism from $f : (Y, \leqslant) \to (X, \leqslant)$ to $f' : (Y', \leqslant) \to (X, \leqslant)$ gives rise to a natural transformation $F_f \to F_{f'}$.

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Proposition 6 ("Elephant" A1.1.7).

The assignment $f \mapsto F_f$ determines an equivalence of categories

$$\mathcal{SP}/(X, \leqslant) \to Set^{(X,\leqslant)}$$

Stone presheaves.

If $f : \mathcal{Y} \to \mathcal{X}$ is a strict p-morphism, where $\mathcal{Y}$ is arbitrary Esakia space and $\mathcal{X}$ is a finite Esakia space, then for arbitrary $x \in \mathcal{X}$, $f^{-1}(x)$ is a clopen subset of $\mathcal{Y}$, so it is a Stone space.
Moreover the map $f^{-1}(x_1) \to f^{-1}(x_2)$ we defined before is continuous for any $x_1 \leqslant x_2$.
Thus in this case F_f may be viewed as a functor $F_f : \mathcal{X} \to \mathit{Stone}$ to the category of Stone spaces.

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Theorem 5.1.

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H-algebras

What is the dual algebraic counterpart of the above Theorem 4.1?

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Definition 8.

For a Heyting algebra H, let H-alg denote the category of *H-algebras* whose objects are Heyting algebra homomorphisms $f : H \to A$. A morphism from $f : H \to A$ to $f' : H \to A'$ is a Heyting algebra homomorphism $g : A \to A'$ with $g \circ f = f'$.

Étale H-algebras

Definition 9.

Define the variety $\mathscr{V}_H$ of *étale* H-algebras as the subvariety of the variety of H-algebras generated by the H-algebra H, i. e. by the identity map $1_H : H \to H$.

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By the Birkhoff theorem, $\mathscr{V}_H$ corresponds to the smallest subcategory of H-alg which contains H and is closed under products, subalgebras and homomorphic images.

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By the Birkhoff theorem, $\mathscr{V}_H$ corresponds to the smallest subcategory of H-alg which contains H and is closed under products, subalgebras and homomorphic images.
Note that by Esakia duality the category $H\text{-alg}^{\mathrm{op}}$ is dual to $\mathcal{E}/\mathcal{X}_H$ where $\mathcal{X}_H$ is the Esakia space dual to H. Thus the way $\mathscr{V}_H$ is obtained from H-alg is dual to the way $\mathcal{SE}/\mathcal{X}_H$ is obtained from $\mathcal{E}/\mathcal{X}_H$.

Identity of Étale H-algebras

Theorem 6.1.

For a finite Heyting algebra H,

- *Every H-algebra $i : H \to A$ in the variety $\mathscr{V}_H$ satisfies the identity*
$$\bigvee_{h \in H} (i(h) \leftrightarrow x) = 1.$$
- *If an algebra $H \to A$ in the variety H-alg satisfies the above identity then its dual Esakia morphism $\mathcal{X}_A \to \mathcal{X}_H$ is a strict p-morphism.*

Forgetful functor

Fact.

The forgetful functor $F : \mathcal{E}/\mathcal{X} \to Set/|\mathcal{X}|$ *does not preserve* inverse limits.

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For example $2 \leftarrow 2 \times 2 \to 2$ which is the limit of the diagram $2 \to 1 \leftarrow 2$ in $Set/1$, is not the limit of this diagram in $\mathcal{E}/1$.

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The forgetful functor $F : \mathcal{E}/\mathcal{X} \to Set/|\mathcal{X}|$ *does not preserve* inverse limits.

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Theorem 7.1.

For an arbitrary Esakia space $\mathcal{X}$ the forgetful functor $F : \mathcal{SE}/\mathcal{X} \to Set/|\mathcal{X}|$ preserves all inverse limits.

These facts signify that the categories of type $\mathscr{V}_H^{\mathrm{op}}$ or $\mathcal{SE}/\mathcal{X}$ although are not toposes but their categorical properties make them closer to toposes than the categories H-alg$^{\mathrm{op}} \simeq \mathcal{E}/\mathcal{X}_H$.

Thanks
for
attention