Modal logic of products of neighborhood frames

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PLAN

1. Language
2. History of the topic
3. Neighborhood frames
4. Product of n-frames
5. Weak product
6. Completeness theorem
7. Future work

Language and logics

$$\phi ::= p \mid \bot \mid \phi \to \phi \mid \Box_i \phi,\ i = 1, 2.$$

Normal modal logic.
$\mathsf{K_n}$ denotes the minimal normal modal logic with n modalities and $\mathsf{K} = \mathsf{K_1}$.
$\mathsf{L_1}$ and $\mathsf{L_2}$ — two modal logics with one modality $\Box$ then the fusion of these logics is defined as

$$\mathsf{L_1} * \mathsf{L_2} = \mathsf{K_2} + \mathsf{L_1'} + \mathsf{L_2}';$$

where $\mathsf{L_i'}$ is the set of all formulas from $\mathsf{L_i}$ where all $\Box$ replaced by $\Box_i$.

The product of Kripke frames

For two frames $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$

$$F_1 \times F_2 = (W_1 \times W_2, R_1^*, R_2^*), \text{ where } \begin{array}{r} (a_1, a_2)R_1^*(b_1, b_2) \Leftrightarrow a_1 R_1 b_1 \ \& \ a_2 = b_2 \\ (a_1, a_2)R_2^*(b_1, b_2) \Leftrightarrow a_1 = b_1 \ \& \ a_2 R_2 b_2 \end{array}$$

For two logics L_1 and L_2

$$\mathsf{L}_1 \times \mathsf{L}_2 = Log(\{F_1 \times F_2 \mid F_1 \models \mathsf{L}_1 \ \& \ F_2 \models \mathsf{L}_2\})$$

(Shehtman, 1978)
For two classes of frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$
$Log(\{F_1 \times F_2 \mid F_1 \in \mathfrak{F}_1 \ \& \ F_2 \in \mathfrak{F}_2\}) \supseteq Log(\mathfrak{F}_1) * Log(\mathfrak{F}_2) +$
$+ \Box_1\Box_2 p \leftrightarrow \Box_1\Box_2 p + \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p.$

$$\mathsf{K} \times \mathsf{K} = \mathsf{K} * \mathsf{K} + \Box_1\Box_2 p \leftrightarrow \Box_1\Box_2 p + \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$$

$$\mathsf{S4} \times \mathsf{S4} = \mathsf{S4} * \mathsf{S4} + \Box_1\Box_2 p \leftrightarrow \Box_1\Box_2 p + \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$$

$$\vdots$$

The product of topological spaces

(van Benthem et al, 2005)
For two topological space $\mathfrak{X}_1 = (X_1, \tau_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2)$

$\mathfrak{X}_1 \times \mathfrak{X}_2 = (X_1 \times X_2, \tau_1^*, \tau_2^*)$, where τ_1^* has base $\{U_1 \times \{x_2\} \mid U_1 \in \tau_1 \;\&\; x_2 \in X_2\}$

τ_2^* has base $\{\{x_1\} \times U_2 \mid x_1 \in X_1 \;\&\; U_2 \in \tau_2\}$

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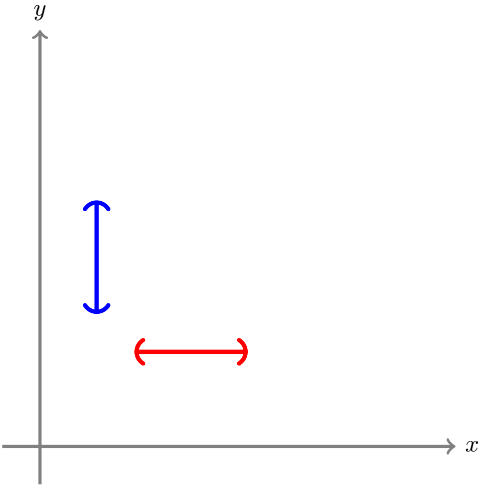

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For two logics L_1 and L_2

$$\mathsf{L}_1 \times_t \mathsf{L}_2 = Log(\{\mathfrak{X}_1 \times \mathfrak{X}_2 \,|\, \mathfrak{X}_1 \models \mathsf{L}_1 \;\&\; \mathfrak{X}_2 \models \mathsf{L}_2\}$$
$$\mathsf{S4} \times_t \mathsf{S4} = Log(\mathbb{Q} \times \mathbb{Q}) = \mathsf{S4} * \mathsf{S4} \quad \text{(van Benthem et al, 2005)}$$
$$Log(\mathbb{R} \times \mathbb{R}) \neq \mathsf{S4} * \mathsf{S4} \quad \text{(Kremer, 2010?)}$$
$$Log(\mathbb{C} \times \mathbb{C}) \neq \mathsf{S4} * \mathsf{S4}$$

d-logic of product of topological spaces was considered by L. Uridia (2011). He proved

$$Log_{\,d}(\mathbb{Q} \times \mathbb{Q}) = \mathsf{D4} * \mathsf{D4}$$

Generalization to neighborhood frames was done by K. Sano (2011).

Neighborhood frames

A (normal) neighborhood frame (or an n-frame) is a pair $\mathfrak{X} = (X, \tau)$, where

- $X \neq \varnothing$;
- $\tau : X \to 2^{2^X}$, such that $\tau(x)$ is a filter on X;

τ — neighborhood function of $\mathfrak{X}$,
$\tau(x)$ — neighborhoods of x.
Filter on X: nonempty $\mathcal{F} \subseteq 2^X$ such that
1) $U \in \mathcal{F} \;\&\; U \subseteq V \Rightarrow V \in \mathcal{F}$
2) $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$ (filter base)

The neighborhood model (n-model) is a pair $(\mathfrak{X}, V)$, where $\mathfrak{X} = (X, \tau)$ is a n-frame and $V : PV \to 2^X$ is a valuation. Similar: neighborhood 2-frame (n-2-frame) is (X, τ_1, τ_2) such that τ_i is a neighborhood function on X for each i.
Validity in model:

$$M, x \models \Box_i \psi \iff \exists V \in \tau_i(x) \forall y \in V (M, y \models \psi).$$

$M \models \varphi \quad \mathfrak{X} \models \varphi \quad \mathfrak{X} \models L \quad Log(\mathcal{C}) = \{\varphi \mid \mathfrak{X} \models \varphi \text{ for some } \mathfrak{X} \in \mathcal{C}\}$

$$nV(L) = \{\mathfrak{X} \mid \mathfrak{X} \text{ is an n-frame and } \mathfrak{X} \models L\}$$

Connection with Kripke frames

Definition

Let $F = (W, R)$ be a Kripke frame. We define neighborhood frame $\mathcal{N}(F) = (W, \tau)$ as follows. For any $w \in W$

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}.$$

Lemma

Let $F = (W, R)$ be a Kripke frame. Then

$$Log(\mathcal{N}(F)) = Log(F).$$

Bounded morphism for n-frames

Definition

Let $\mathfrak{X} = (X, \tau_1, \ldots)$ and $\mathcal{Y} = (Y, \sigma_1, \ldots)$ be n-frames. Then function $f : X \to Y$ is a bounded morphism if

1. f is surjective;
2. for any $x \in X$ and $U \in \tau_i(x)$ $f(U) \in \sigma_i(f(x))$;
3. for any $x \in X$ and $V \in \sigma_i(f(x))$ there exists $U \in \tau_i(x)$, such that $f(U) \subseteq V$.

In notation $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$.

Lemma

If $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$ then $Log(\mathcal{Y}) \subseteq Log(\mathfrak{X})$.

Product of n-frames

Definition

Let $\mathfrak{X}_1 = (X_1, \tau_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2)$ be two n-frames. Then the product of these n-frames is an n-2-frame defined as follows

$$
\begin{aligned}
&\mathfrak{X}_1 \times \mathfrak{X}_2 = (X_1 \times X_2, \tau_1', \tau_2'),\\
&\tau_1'(x_1, x_2) = \{U \subseteq X_1 \times X_2 \,|\, \exists V (V \in \tau_1(x_1) \;\&\; V \times \{x_2\} \subseteq U)\},\\
&\tau_2'(x_1, x_2) = \{U \subseteq X_1 \times X_2 \,|\, \exists V (V \in \tau_2(x_2) \;\&\; \{x_1\} \times V \subseteq U)\}.
\end{aligned}
$$

Definition

For two unimodal logics L_1 and L_2, such that $nV(\mathsf{L}_i) \neq \varnothing$. We define n-product of them as follows

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = Log(\{\mathfrak{X}_1 \times \mathfrak{X}_2 \,|\, \mathfrak{X}_1 \in nV(\mathsf{L}_1) \;\&\; \mathfrak{X}_2 \in nV(\mathsf{L}_2)\})$$

Lemma

$\mathsf{L}_1 * \mathsf{L}_2 \subseteq \mathsf{L}_1 \times_n \mathsf{L}_2$ *for any two unimodal logics* L_1 *and* L_2.

Theorem (AK, 2012)

Let L_1 *and* L_2 *be from the set* $\{\mathsf{D}, \mathsf{T}, \mathsf{D4}, \mathsf{S4}\}$ *then*

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = \mathsf{L}_1 * \mathsf{L}_2.$$

n-product of logics

It is not the case for logic K!

Lemma

For any two n-frames $\mathfrak{X}_1$ and $\mathfrak{X}_2$

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models \Box_1\bot \to \Box_2\Box_1\bot.$$

And even more, for any closed $\Box_1$-free formula ϕ and any closed $\Box_2$-free formula ψ

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models \phi \to \Box_1\phi, \qquad \mathfrak{X}_1 \times \mathfrak{X}_2 \models \psi \to \Box_2\psi.$$

Proof.

$$\begin{aligned} \mathfrak{X}_1 \times \mathfrak{X}_2, (x, y) \models \Box_1\bot &\iff \varnothing \in \tau_1'(x, y) \iff \\ \varnothing \in \tau_1(x) &\iff \forall y' \in X_2 \ (\varnothing \in \tau_1'(x, y')) \iff \\ \forall y' \in X_2 \ (\mathfrak{X}_1 \times \mathfrak{X}_2, (x, y') \models \Box_1\bot) &\implies \mathfrak{X}_1 \times \mathfrak{X}_2, (x, y) \models \Box_2\Box_1\bot. \end{aligned}$$

Hence, $\mathfrak{X}_1 \times \mathfrak{X}_2 \models \Box_1\bot \to \Box_2\Box_1\bot$. $\Box$

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Proof.

Since ψ does not contain neither $\Box_2$, nor variables, its value does not depend on the second coordinate. Let $F = \mathfrak{X}_1 \times \mathfrak{X}_2$. So $F, (x, y) \models \psi$, then $\forall y' (F, (x, y') \models \psi)$, hence, $F, (x, y) \models \Box_2 \psi$. □

n-product of logics

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Definition

For two unimodal logics L_1 and L_2, we define

$$\langle L_1, L_2 \rangle = L_1 * L_2 + \Delta, \text{ where}$$

$$\Delta = \{\phi \to \Box_2 \phi \mid \phi \text{ is closed and } \Box_2\text{-free}\} \cup \{\psi \to \Box_1 \psi \mid \psi \text{ is closed and } \Box_1\text{-free}\}.$$

Lemma

For any two normal modal logics L_1 and L_2 $\langle L_1, L_2 \rangle \subseteq L_1 \times_n L_2$.

Note that if $\Diamond \top \in L_1 \cap L_2$ then $L_1 * L_2 \models \Delta$.

Goal

Theorem

$K \times_n K = \langle K, K \rangle$.

Plan:

1. Find proper Kripke frames for $\langle K, K \rangle$.
2. Construct n-frames for which there is a bounded morphism to the proper frames.

Weak product of frames

$F_1 = F_1^{x_0}$ and $F_2 = F_2^{y_0}$ — Kripke frames with roots x_0 and y_0. A path in the product $F_1 \times F_2$ is a sequence of the following type

$$(x_0, y_0)S_1(x_1, y_1)S_2 \ldots S_n(x_n, y_n),$$

where $S_i \in \{R_1^h, R_2^v\}$ and for any $i \leq n$ $(x_{i-1}, y_{i-1})S_i(x_i, y_i)$ holds.
$\mathcal{P}(F_1 \times F_2)$ — the set of all paths in $F_1 \times F_2$.
for any two paths $\alpha, \beta \in \mathcal{P}(F_1 \times F_2)$

$$\alpha R_1' \beta \iff \beta = \alpha R_1^h(a, b)$$
$$\alpha R_2' \beta \iff \beta = \alpha R_2^v(a, b)$$

The following Kripke frame is the weak product of F_1 and F_2

$$\langle F_1, F_2 \rangle = (\mathcal{P}(F_1 \times F_2), R_1', R_2').$$

Weak product of frames

Lemma

For any two Kripke frames F_1 *and* F_2 $\langle F_1, F_2 \rangle \models \Delta$.

Theorem

Logic $\langle \mathsf{K}, \mathsf{K} \rangle$ *is complete w.r.t. weak products of Kripke frames, and even more, w.r.t. weak products of trees.*

Paths with stops

Definition

$F = (W, R)$ — frame with root a_0, $0 \notin W$ we define a path with stops as a tuple $a_0 a_1 \dots a_n$, so that $a_i \in W \cup \{0\}$ and after eliminating zeros each point is related to the next one by relation R. We also consider infinite paths with stops that end with infinitely many zeros. We call these sequences pseudo-infinite paths (with stops). Let W_ω be the set of all pseudo-infinite paths in W.

Define $f_F : W_\omega \to W$ in the following way: for $\alpha = a_0 a_1 \dots a_n 0^\omega$, $a_n \neq 0$, we put

$$f_F(\alpha) = a_n.$$

$$st(\alpha) = \min\{N \mid \forall k \geq N (a_k = 0)\};$$

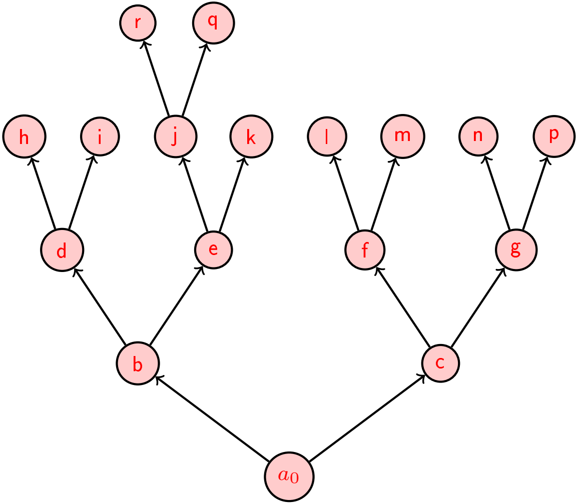
$$\alpha|_k = a_1 \dots a_k;$$

$$U_i^k(\alpha) = \{\beta \in W_\omega \mid \alpha|_m = \beta|_m \ \&\ f_F(\alpha) R_i f_F(\beta),\ \text{where } m = \max(k, st(\alpha))\}.$$

Lemma

$U_i^k(\alpha) \subseteq U_i^m(\alpha)$ *whenever* $k \geq m$ *for any* $i \in \{1, 2\}$.

Example



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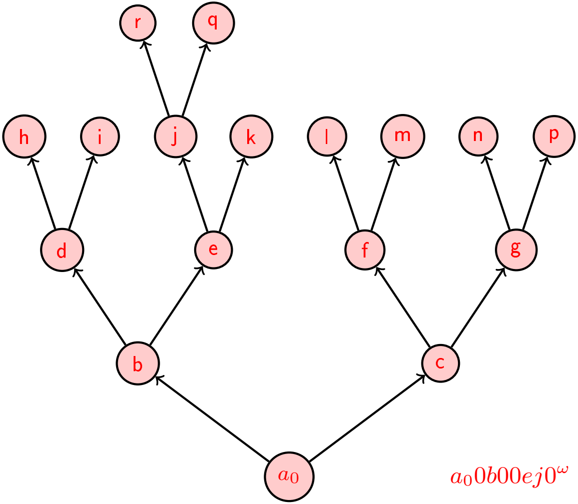

$a_0 0 b 0 0 e j 0^\omega$

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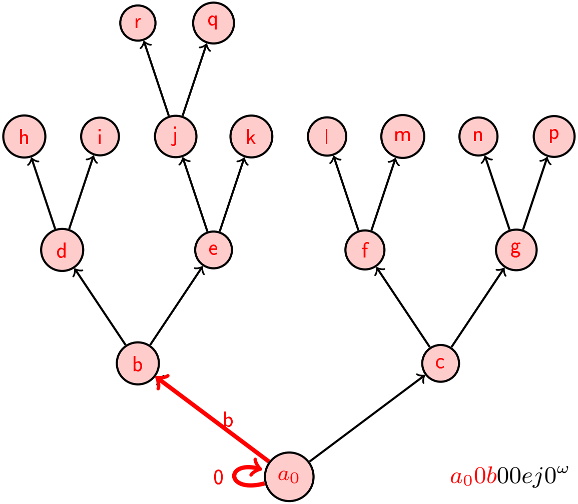

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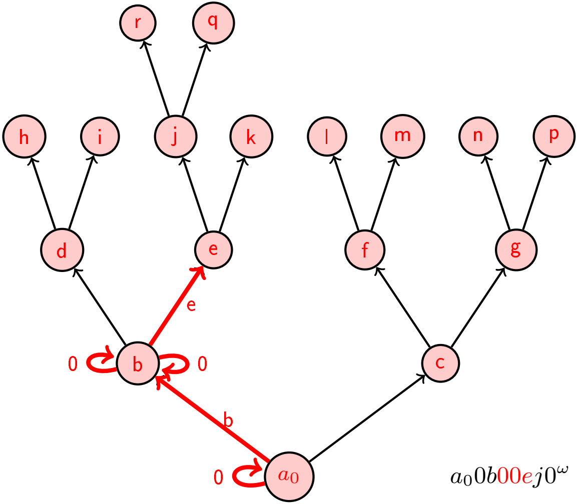

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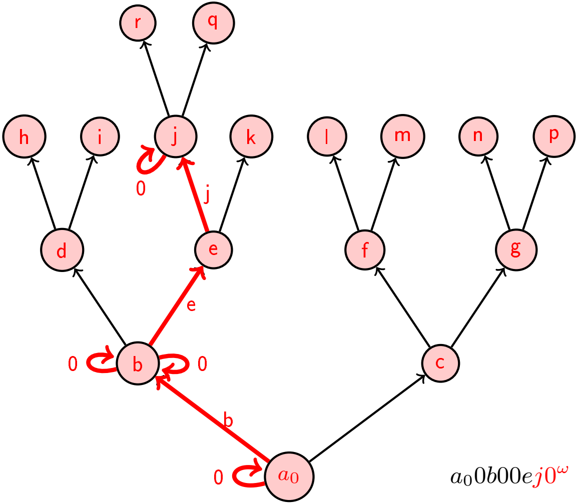

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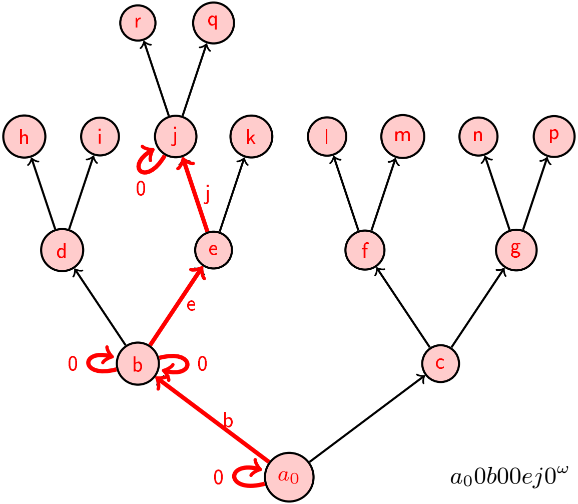

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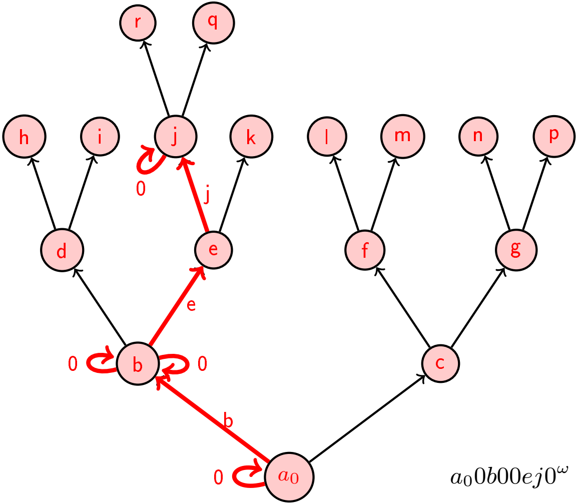

$a_0 0 b 0 0 e j 0^\omega$

$a_0 \;\; b \;\;\; ej$

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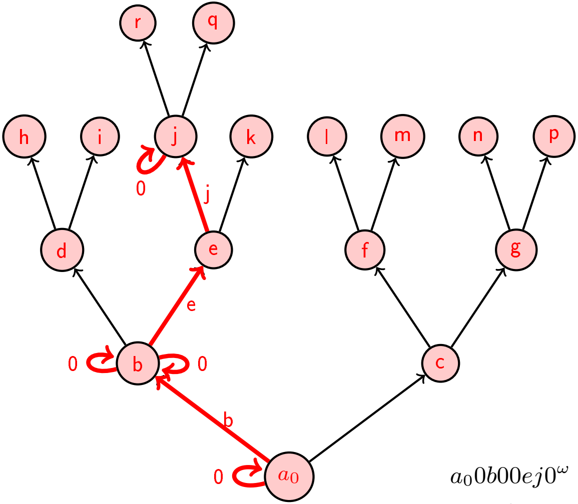

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$$a_0 b e j$$

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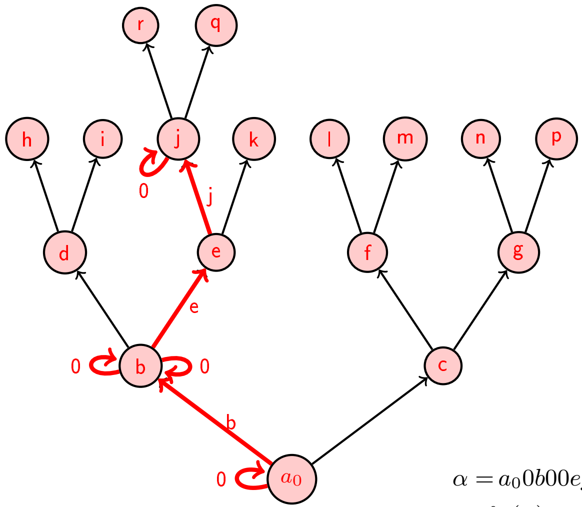

$$\alpha = a_0 0 b 0 0 e j 0^\omega$$

$$f_F(\alpha) = j$$

$N_\omega(F)$

Definition

Sets $U_n(\alpha)$ form a filter base. So we can define

$$\tau(\alpha) - \text{the filter with base } \{U_n(\alpha) \mid n \in \mathbb{N}\};$$
$$\mathcal{N}_\omega(F) = (W_\omega, \tau) \text{ — is a dense n-frame based on } F.$$

Frame $\mathcal{N}_\omega(F)$ is dense in a sense that the intersection of all neighborhoods of a point is empty. So, there are no minimal neighborhoods unlike $\mathcal{N}(F)$.

Lemma

Let $F = (W, R)$ be a Kripke frame with root a_0, then

$$f_F : \mathcal{N}_\omega(F) \twoheadrightarrow \mathcal{N}(F).$$

Corollary

For any frame F $Log(\mathcal{N}_\omega(F)) \subseteq Log(\mathcal{N}(F)) = Log(F)$.

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Completeness theorem

Logic K is complete w.r.t. trees.

Lemma

For any two trees F_1 and F_2

$$\mathcal{N}_\omega(F_1) \times \mathcal{N}_\omega(F_2) \twoheadrightarrow \mathcal{N}(\langle F_1, F_2 \rangle).$$

$$\begin{aligned}
\mathsf{K} \times_n \mathsf{K} = & \bigcap_{\mathfrak{X}_1, \mathfrak{X}_2 \in nV(\mathsf{K})} Log(\mathfrak{X}_1 \times \mathfrak{X}_2) \subseteq \\
\subseteq & \bigcap_{F_1, F_2 - \text{trees}} Log(\mathcal{N}_\omega(F_1) \times \mathcal{N}_\omega(F_2)) \subseteq \\
\subseteq & \bigcap_{F_1, F_2 - \text{trees}} Log(\langle F_1, F_2 \rangle) \subseteq \langle \mathsf{K}, \mathsf{K} \rangle \subseteq \mathsf{K} \times_n \mathsf{K}.
\end{aligned}$$

Theorem

$\mathsf{K} \times_n \mathsf{K} = \langle \mathsf{K}, \mathsf{K} \rangle$.

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\end{aligned}$$

Theorem

$\mathsf{K} \times_n \mathsf{K} = \langle \mathsf{K}, \mathsf{K} \rangle$.

Future work

Conjecture

$K4 \times_n K4 = \langle K4, K4 \rangle$.

Question: What conditions of logics L_1 and L_2 are sufficient for

$$L_1 \times_n L_2 = \langle L_1, L_2 \rangle.$$

Future work

Conjecture

$\mathsf{K4} \times_n \mathsf{K4} = \langle \mathsf{K4}, \mathsf{K4} \rangle$.

Question: What conditions of logics L_1 and L_2 are sufficient for

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = \langle \mathsf{L}_1, \mathsf{L}_2 \rangle.$$

Thank you!