A dichotomy for some elementarily generated modal logics

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Tbilisi, June 2014

Language $\mathcal{M}l_{\Lambda}$ (propositional modal formulas)

$$PV = \{p_1, p_2, p_3, \dots\}$$

$$\phi ::= p_i \mid \bot \mid (\phi \vee \phi) \mid (\phi \wedge \phi) \mid \neg\phi \mid \Diamond_{\lambda}\phi \mid \Box_{\lambda}\phi,$$

where $\lambda \in \Lambda$.

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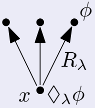
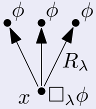
Kripke semantics

- Kripke frame: $F = (W, (R_\lambda : \lambda \in \Lambda))$ where $R_\lambda \in W \times W$,
- Kripke model: $M = (F, \theta)$ where $\theta : PV \to 2^W$,
- $M, x \models \phi$ $\quad$ ϕ is true at x in M,
- $F, x \models \phi$ $\quad$ ϕ is valid at x in F,
- $F \models \phi$ $\quad$ ϕ is valid in F,





Language $\mathcal{L}f_\wedge$

The corresponding first-order language $\mathcal{L}f_\wedge$ consists of:

- object variables $x, y, z, \ldots$
- binary relational symbols $R_\lambda(x, y)$ and equality
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$$F \models A(x_0) \Leftrightarrow F, x_0 \models \phi$$

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Global question

Given C, determine what properties hold for $\mathrm{Log}(C)$?
How are they related to the properties of the first-order formula defining C ?

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What if C is given by some very simple first-order formula?

Definition

A set of modal formulas L is a normal modal logic if

- L contains $\Box_\lambda(p \wedge q) \equiv \Box_\lambda p \wedge \Box_\lambda q$ and $\Box_\lambda p \equiv \neg\Diamond_\lambda\neg p$
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Properties of logics we are interested in

- finite axiomatisability (or, axiomatisability by a single formula)
- (generalised) Sahlqvist axiomatisability
- using finitely many variables
- using a single non-canonical formula and arbitrary many canonical f-las
- elementarity (when $V(L) = \{F \mid F \models L\}$ is elementary)

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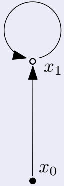
Example: "see itself in two steps" vs "have a reflexive successor"

$\forall x_0 \exists x_1 (x_0 R x_1 \wedge x_1 R x_0)$

modally definable
by a Sahlqvist formula

$L = K + p \to \Diamond\Diamond p$

L is elementary



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modally undefinable

L is not axiomatisable
using finitely many variables

L is not Δ-elementary

(Hughes, 1990)

any axiomatisation of L requieres
infinitely many
non-canonical formulas
(I. Hodkinson, Y. Venema, 2003)

Both are of the form

$$\forall x_0 \exists x_1 \dots \exists x_n \bigwedge R_\lambda(x_i, x_j)$$

Problem statement

Classify $\mathcal{C}$ given by $\forall x_0 E(x_0)$, where $E(x_0) = \exists x_1 \ldots \exists x_n \bigwedge x_i R x_j$ w.r.t.

(i) $E(x_0)$ is locally modally definable by a generalised Sahlqvist formula;

(ii) $E(x_0)$ is locally modally definable;

(iii) $\forall x_0 E(x_0)$ is globally modally definable;

(iv) $\mathrm{Log}(\mathcal{C})$ is axiomatisable by a single generalised Sahlqvist formula;

(v) $\mathrm{Log}(\mathcal{C})$ is finitely axiomatisable;

(vi) $\mathrm{Log}(\mathcal{C})$ is axiomatisable using finitely many variables ;

(vii) $\mathrm{Log}(\mathcal{C})$ is axiomatisable by canonical formulas;

(viii) $\mathrm{Log}(\mathcal{C})$ is axiomatisable using finitely many non-canonical formulas;

(ix) $\{F \mid F \models \mathrm{Log}(\mathcal{C})\} = \mathcal{C}$;

(x) $\mathrm{Log}(\mathcal{C})$ is elementary;

(xi) $\mathrm{Log}(\mathcal{C})$ Δ-elementary (?).

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Let $W^D = \{x_0, x_1, \ldots, x_n\}$, $x^D_0 = x_0$, $\quad K^D(x_0, x_1, \ldots, x_n) = \bigwedge_{\substack{z_1,\ z_2 \in W^D, \\ z_1 R^D_\lambda z_2}} z_1 R_\lambda z_2,$

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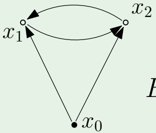

$$K_D(x_0, x_1, x_2) = x_0Rx_1 \wedge x_0Rx_2 \wedge x_1Rx_2 \wedge x_2Rx_1$$

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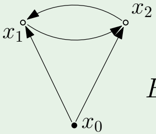

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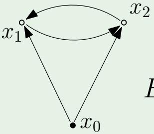

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A diagram D is called

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for any diagram D' which is obtained from D by deleting an edge.

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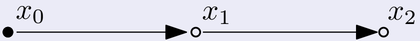

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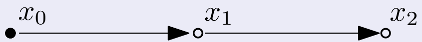

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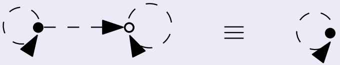
Minimality is important



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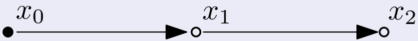

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Claim 1. If all undirected cycles in a rooted D pass through its root, then it is locally definable by a generalised Sahqvist formula.

This was more or less known:

- 2005[a] — definability was noticed and proved;
- 2013[b] — definability for many "root" variables by generalised Sahqvist formulas;
- 2013[c] — an easy-to-implement translating algorithm is presented.

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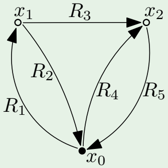
[a] E. Zolin. Query answering based on modal correspondence theory. In *Proceedings of the 4th "Methods for modalities" Workshop (M4M-4)*, pages 21–37, 2005.

[b] S. Kikot and E. Zolin. Modal definability of first-order formulas with free variables and query answering. In *Journal of Applied Logic*, 11:190–216, 2013.

[c] S. Kikot, D. Tsarkov, M. Zakharyaschev and E. Zolin. Query Answering via Modal Definability with FaCT++: First Blood. In Informal Proceedings of *DL 2013*: 26th International Workshop on Description Logics (Ulm, 22–27 July), pp. 328–340, CEUR Workshop Proceedings, vol. 1014, 2013.

Example



f.o. formula $\quad E(x_0) = \exists x_1 \exists x_2 (x_0 R_1 x_1 \wedge x_1 R_2 x_0 \wedge x_1 R_3 x_2 \wedge x_0 R_4 x_2 \wedge x_2 R_5 x_0)$

$$\Updownarrow$$

generalised Kracht formula $\quad x \in R_4^{-1}(R_5^{-1}(x) \cap R_3(R_1(x) \cap R_2^{-1}(x)))$

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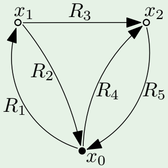

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Indeed, $\theta_{\min}(p) = \{x\}$ and $\theta_{\min}(q) = R_3(R_1(x) \cap R_2^{-1}(x))$.

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E.g., negations of (i) and (ii) follow from

Theorem (2013[b])

If a rooted D is locally minimal and contains the mentioned cycle, then $E_D(x_0)$ is not locally definable.

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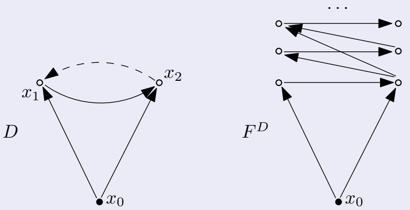

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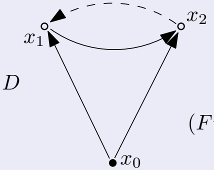

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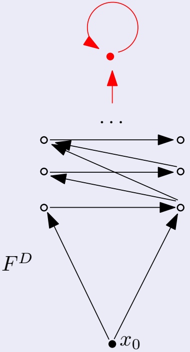
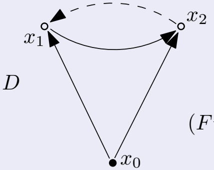


Can we use this idea to show global undefinability?



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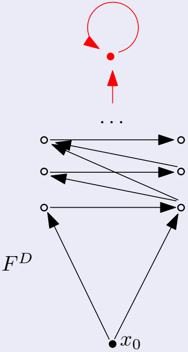


From this example we learn:

- for proving global properties it is convenient to "repair" the destroyed diagram to make all points except root satisfy E^D;
- this can often be done by adding a reflexive point on top of the diagram.

Lemma (about repairing the diagram)

For any rooted globally minimal D with an interior cycle there exist two pointed Kripke frames $F^+ = (W, (R^+_\lambda : \lambda \in \Lambda), x^0_0)$ and $F^- = (W, (R^-_\lambda : \lambda \in \Lambda), x^0_0)$, points $x_d, x_{d'} \in W^D$, index $\lambda_d \in \Lambda$, and embedding $g : D \to F^+$ sending x_0 to x^0_0, such that:

(i) $R^+_{\lambda_d} = R^-_{\lambda_d} \cup \{(g(x_d), g(x'_d))\}$ и $R^+_\lambda = R^-_\lambda$ для $\lambda \neq \lambda_d$;

(ii) $F^- \not\models E_D(x^0_0)$

(iii) $F^+ \models E_D(x^0_0)$

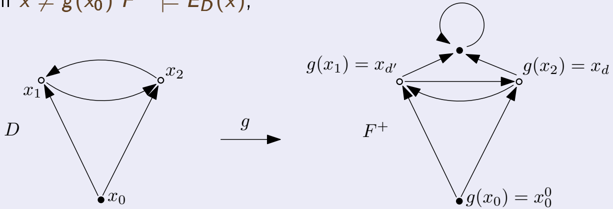
(iv) the points $g(x_d)$ and $g(x_{d'})$ can be connected in F^- by an indirected path, not passing through x_0

(v) if for some $x^0_1, \ldots, x^0_n \in W$ $F^+ \models K_D(x^0_0, x^0_1, \ldots, x^0_n)$, then

$$\{x^0_0, \ldots, x^0_n\} = \{g(x_0), \ldots, g(x_n)\},$$

and for all $0 \leq i, j \leq n$ $x^0_i R^+_\lambda x^0_j$ implies $x_i R^D_\lambda x_j$;

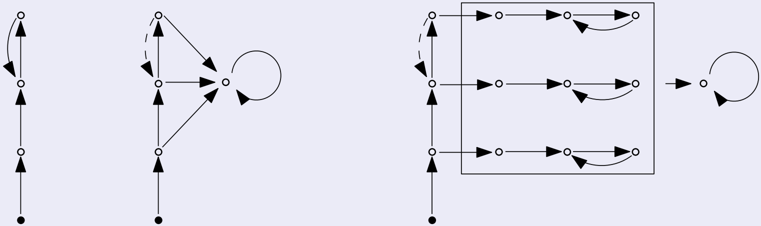
(vi) for all $x \neq g(x_0)$ $F^- \models E_D(x)$;



Can we always make do with adding a reflexive point?

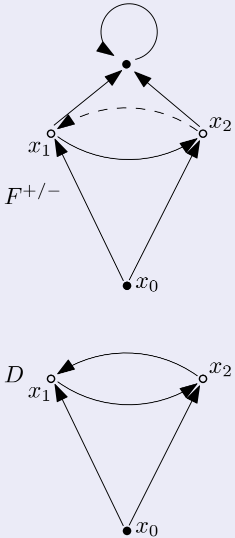
Can we always make do with adding a reflexive point?

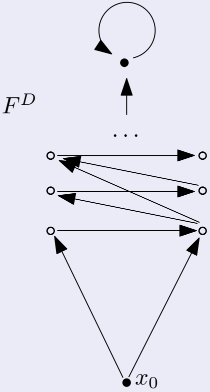
No !

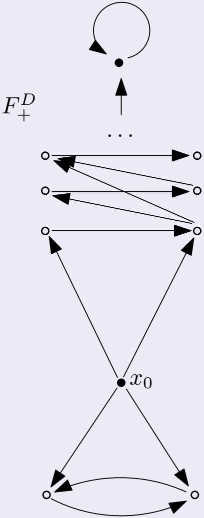


Non-axiomatisability using finitely many variables ((iv) – (vi) are false)

Proof idea: use F^+ and F^- while constructing F^D.



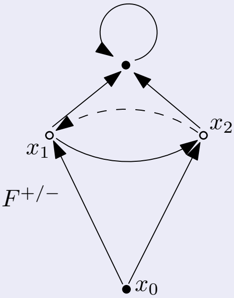




$$F^D \not\models E_D(x_0)$$

$$F^D_+ \models E_D(x_0)$$

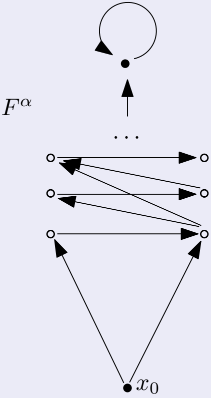
Pseudo-products with order

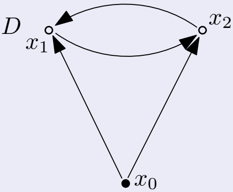


$\times$

α

$=$





α is a linear discrete order with the first element

$F^{\alpha} \in V(L)$ for infinite α

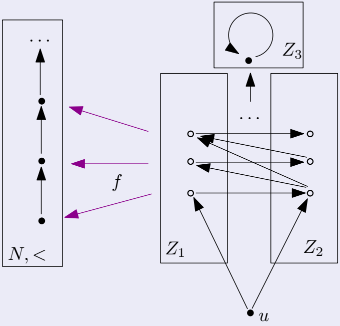
$F^{\alpha} \notin V(L)$, for finite α

Non-elementarity

In the signature Σ which consists of:

- binary relational symbols R_λ, $<$, f and $=$
- unary predicate symbols $N, Z_1, \ldots, Z_m$ and
- a constant u (which goes instead of Z_0)

one can write a formula ζ_k which says:
"The subframe, generated by u and R_λ, is isomorphic to F^α for some linear order α, while $|\alpha| >= k$".



Let $F \in V(L) \Leftrightarrow F \models \eta$.

$T = \{\neg\eta, \zeta_1, \zeta_2, \zeta_3, \ldots\}$

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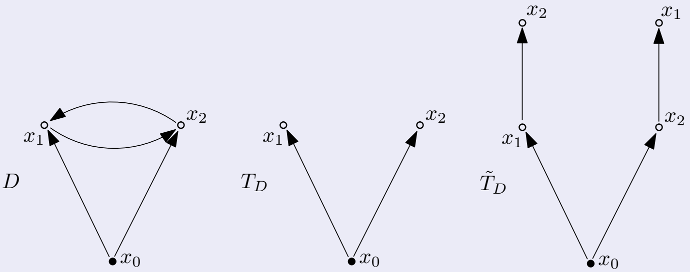
Axiomatisation

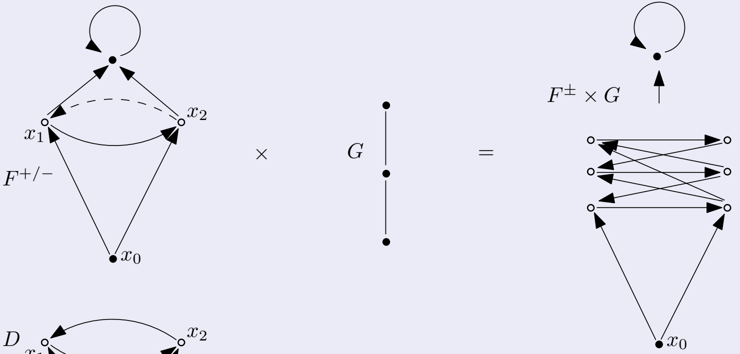


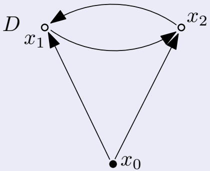
$$\exists j_1 \exists j_2 (\Diamond(j_1 \wedge \Diamond j_2) \wedge \Diamond(j_2 \wedge \Diamond j_1))$$

$$\Box(p_1 \vee \cdots \vee p_n) \to \bigvee_{i,j=1}^{n} \Diamond(p_i \wedge \Diamond p_j) \wedge \Diamond(p_j \wedge \Diamond p_i)$$

$F, x \models \gamma_n \iff$ If the d-neighborhood of x in F is painted in n colours, then there is a homomorphism from $\tilde{T}_D$ to F such that the nodes with same labels are mapped to the points with the same colours.

Pseudo-products with graphs





G is an arbitrary graph; b is the number of points in F^+.

If G cannot be painted in 2^{bk} colours, then $F^{\pm} \times G \models \gamma_k$.

If G can be painted in N colours, then $F^{\pm} \times G \not\models \gamma_{Nb}$.

Inverse limits of descriptive general frames

Let $F_i = (W_i, (R_{\lambda,i} : \lambda \in \Lambda), P_i)$ (for $i \in \mathbb{N}$) be descriptive general frames and $f_i : F_{i+1} \to F_i$ be p-morphisms.
We define the inverse limit of a system of general frames

$$\dots \to F_{i+1} \xrightarrow{f_i} F_i \to \dots$$

as $\mathcal{F} = (W, (R_\lambda : \lambda \in \Lambda), P)$, where

$$W = \{x \in \prod_{i \in \mathbb{N}} W_i : f_i(x_{i+1}) = x_i \text{ for all } i \in \mathbb{N}\},$$

$$R_\lambda = \{(x, y) \in W : R_{\lambda,i}(x_i, y_i) \text{ for all } i \in \mathbb{N}\},$$

$$P = \{pr_i^{-1}[S] : i \in \mathbb{N}, S \in P_i\},$$

where for each $i \in \mathbb{N}$ $pr_i : W \to W_i$ is the projection $pr_i(x) = x_i$.

The inverse limit of Kripke frames considered as general frames is not necessarily a Kripke frame!

Theorem (R. Goldblatt)

- The inverse limit of a system of descriptive frames is a descriptive frame.
- If a modal formula ϕ is valid on all F_i, then it is valid on F

Theorem (I. Hodkinson, Y. Venema)

Let $s \geq 2$. Then there is a sequence of graphs $G_0, G_1, \ldots$ and p-morphisms $\rho_i : G_{i+1} \to G_i$ such that

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Lemma

Let γ_i be a sequence of modal formulas such that γ_{i_1} implies γ_{i_2} if $i_2 < i_1$. Suppose that for all l there exists m such that for all k there exists an inverse system of finite Kripke frames $\{F_i\}$ such that:

(1) for all i $F_i \models \gamma_k$,

(2) $\varprojlim F_i \models \gamma_l$,

(3) $\varprojlim F_i \not\models \gamma_m$.

Then any axiomatisation of $L = \mathbf{K} + \{\gamma_n : n \in \omega\}$ has infinitely many non-canonical axioms.

Discussion

- We classified $\forall x_0 \exists x_1 \ldots \exists x_n \bigwedge x_i R x_j$ with respect to the properties of its modal logic.

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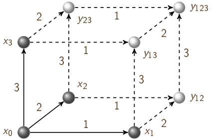

- What's next ?
  - AE-diagrams;
  - Sahlqvist successor property.

Thank you !