Frame completions of conditional frames (report on a recent communication by B. Banaschewski)

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There are obvious dual notions of *meet-infinite distributive law* (MID), *coframe* and *conditional coframe*.

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$$L \to (a_1] \times [a_1, a_2] \times \cdots \times [a_{n-1}, a_n] \times [a_n),$$
$$x \mapsto \langle x \wedge a_1, a_1 \vee x \wedge a_2, \ldots, a_{n-1} \vee x \wedge a_n, a_n \vee x \rangle$$

where $a_1 < \cdots < a_n$ runs over all finite chains in L.

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For L complete, there were two more simple constructions.

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In particular, a frame acquires an embedding into a complete Boolean algebra with preservation of all joins.

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There is a frame embedding $L \rightarrowtail \mathbf{N}(L)$ sending $a \in L$ to the nucleus $a \vee \_$.

Each $a \vee \_$ has a complement $a \to \_$ in $\mathbf{N}(L)$; moreover the embedding $L \rightarrowtail \mathbf{N}(L)$ is universal among all frame maps $L \to F$ whose images consist of complemented elements of F.

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It follows that the embedding $L \rightarrowtail \mathbf{N}(L)_{\neg\neg}$ preserves arbitrary joins.

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For the first proof, we extend the Isbell embedding $L \rightarrowtail \mathbf{N}(L)_{\neg\neg}$ along $L \rightarrowtail \mathbf{B}(L)$ to obtain an embedding $\mathbf{B}(L) \rightarrowtail \mathbf{N}(L)_{\neg\neg}$.

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It turns out that the image of $\mathbf{B}(L)$ is join-dense in $\mathbf{N}(L)_{\neg\neg}$, hence by the universal property of the MacNeille completion, $\mathbf{N}(L)_{\neg\neg}$ must be isomorphic to $\mathbf{M}\,\mathbf{B}(L)$.

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For the second proof, we employed Esakia duality to give topological descriptions of the Grätzer and Isbell constructions, which yielded an isomorphism between $\mathbf{M}(\mathbf{B}(L))$ and $\mathbf{N}(L)_{\neg\neg}$ restricting to the identity of L with respect to the above embeddings.

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In a nutshell — both $\mathbf{M}(\mathbf{B}(L))$ and $\mathbf{N}(L)_{\neg\neg}$ are isomorphic to the Boolean algebra of **regular closed** subsets of the Esakia dual X_L of L.

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We did not know whether there is any analog of the Isbell construction in the non-complete case.

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Proposition. *A (bounded distributive) lattice is a conditional frame iff it embeds into a frame with preservation of all existing joins.*

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Proposition. *A (bounded distributive) lattice is a conditional frame iff it embeds into a frame with preservation of all existing joins.*

The 'if' part is clear; the main thing is the beautiful embedding of a conditional frame into a frame.

Embedding conditional frames into frames

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For a conditional frame L let $\mathscr{F}L$ be the set of join-closed ideals of L.

Then the embedding $\downarrow \_ : L \rightarrowtail \mathscr{F}L$ defined by $a \mapsto \downarrow a$ preserves all existing joins. As simple as that!

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One then has

Theorem (Banaschewski). *For any conditional frame morphism $f : L \to F$ to a frame F there is a unique frame homomorphism $f' : \mathscr{F}L \to F$ with $f' \circ \downarrow\_ = f$.*

Embedding conditional frames into Boolean frames

We now can apply the Isbell construction to the frame $\mathscr{F}L$ and obtain

$$L \rightarrowtail \mathscr{F}L \rightarrowtail \mathbf{N}\mathscr{F}L \twoheadrightarrow (\mathbf{N}\mathscr{F}L)_{\neg\neg}.$$

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Recall that

$$\mathscr{F}L \rightarrowtail \mathbf{N}\mathscr{F}L \twoheadrightarrow (\mathbf{N}\mathscr{F}L)_{\neg\neg}$$

is a join-preserving embedding, so we obtain a conditional frame morphism

$$L \rightarrowtail (\mathbf{N}\mathscr{F}L)_{\neg\neg}.$$

into a complete Boolean algebra.

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Proposition (Banaschewski). *For a conditional frame L, any join-preserving embedding $\mu : L \rightarrowtail B$ into a Boolean frame B which is frame-generated by the elements $\mu(a)$ and $\neg\mu(a)$, $a \in L$, is equivalent to $L \rightarrowtail (\mathbf{N}\mathscr{F}L)_{\neg\neg}$.*

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Here *equivalent* means there is an isomorphism between $(\mathbf{N}\mathscr{F}L)_{\neg\neg}$ and B which carries the embeddings of L to each other.

Uniqueness – proof sketch

Idea of the proof: consider the diagram

$$
\begin{array}{ccccc}
L & \rightarrowtail & \mathscr{F}L & \rightarrowtail & \mathbf{N}\mathscr{F}L \\
& \searrow^{\mu} & & & \\
& & B & &
\end{array}
$$

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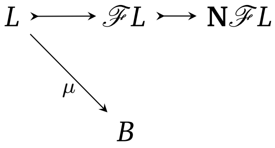

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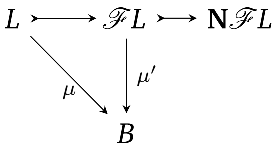

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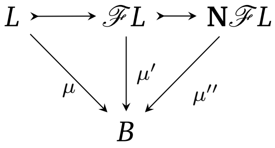

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Since B is generated by the elements $\mu(a)$ and $\neg\mu(a)$, $a \in L$, μ'' is surjective and *dense* — that is, ${\mu''}^{-1}(0) = \{0\}$.

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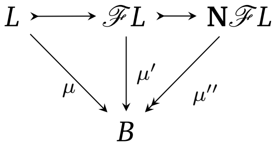

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Since B is generated by the elements $\mu(a)$ and $\neg\mu(a)$, $a \in L$, μ'' is surjective and *dense* — that is, ${\mu''}^{-1}(0) = \{0\}$.
It now follows from a result of Isbell that μ'' induces an isomorphism from $(\mathbf{N}\mathscr{F}L)_{\neg\neg}$ onto B.

Generalization of B-G-J to conditional frames

Corollary. *For a conditional frame L, the Grätzer embedding $L \rightarrowtail \mathbf{M}\,\mathbf{B}(L)$ is equivalent to $L \rightarrowtail (\mathbf{N}\mathscr{F}L)_{\neg\neg}$.*

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Corollary. *For a conditional frame L, the Grätzer embedding $L \rightarrowtail \mathbf{M}\,\mathbf{B}(L)$ is equivalent to $L \rightarrowtail (\mathbf{N}\mathscr{F}L)_{\neg\neg}$.*

Indeed, one sees easily that the Grätzer embedding is of the kind described in Banaschewski's proposition: $\mathbf{M}\,\mathbf{B}(L)$ is generated by elements of L and their complements.

Further generalization?

We thus have a transparent proof of Funayama's theorem that a distributive lattice L is a conditional frame if and only if there is a lattice embedding of L into a complete Boolean algebra which preserves all joins which exist in L.

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But what about distributive lattices in general?

Grätzer's construction can be carried out for any distributive lattice.

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In response to Banaschewski's note Guram sent him some relevant material, including his new joint paper with John Harding "Proximity Frames and Regularization" (2014).

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Among many other things they use the notions of *admissible subset* and *D-ideal* of Bruns and Lakser (1970).

Bruns and Lakser (1970)

A subset $S \subseteq L$ of a distributive lattice L is *admissible* if it has a join $\bigvee S$ in L and moreover $a \wedge \bigvee S$ is the join of $\{ a \wedge s \mid s \in S \}$ for any $a \in L$.

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The embedding $\downarrow \_ : L \rightarrowtail \mathscr{F}L$ *preserves joins of admissible subsets, and is universal among those lattice homomorphisms to frames* $L \to F$ *which preserve such joins.*

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Suppose that B is generated as a frame by elements of L and their complements. Then this embedding is equivalent to the composite

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It is easy to see that for any distributive lattice L the Grätzer construction $L \rightarrowtail \mathbf{B}(L) \rightarrowtail \mathbf{M}\,\mathbf{B}(L)$ satisfies these conditions, so the Isbell and Grätzer embeddings are equivalent in the general case too.

THANK YOU!