A topological duality for filter-distributive congruential logics

RAMON JANSANA

University of Barcelona

(join work with María Esteban)

TOLO IV

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- Extensions of the dualities to reducts of those classes of algebras, like distributive meet-semilattices and implicative meet-semilattices (S. Celani, G. Hansoul, G. Bezhanishvili, R. J.)

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- Extensions of the dualities to reducts of those classes of algebras, like distributive meet-semilattices and implicative meet-semilattices (S. Celani, G. Hansoul, G. Bezhanishvili, R. J.)
- Many of the varieties of algebras with a bounded distributive lattice reduct as well as the class of implicative meet-semilattices are the class of algebras of some logic with the following properties:

  a) it is congruential,
  b) the lattices of logical filters of its algebras are distributive lattices.

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- This is the goal of the work I will expound. I will present
  - a framework to develop Priestley-style dualities for the classes of algebras canonically associated with a congruential and filter-distributive logic (with theorems),
  - some correspondence results between logical properties of congruential and filter-distributive logics and properties of the duals spaces of their algebras. The properties I will discuss are:
    - having a conjunction, - having a disjunction,
    - having an implication for which the deduction theorem and modus ponens hold, - having an inconsistent formula.

Outline of the talk

- A Priestley style duality for distributive meet-semilattices
- Basic on logics: congruential and filter-distributive logics.
- The meet-semilattice of an algebra for a congruential and filter-distributive logic
- The strategy towards a duality
- The duality
- Dual correspondence of some logical properties

Duality for distributive meet-semilattices

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Distributive meet-semilattices with top

A meet-semilattice with top $M = \langle M, \wedge, 1 \rangle$ is *distributive* if its lattice of filters is distributive.

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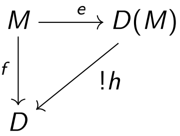
A homomorphism $h : M \to M'$ from a meet-semilattice M to a meet-semilattice M' is a *sup-homomorphism* if it preserves the existing finite joins (besides finite infimums).
In particular, if M has a lower bound, then M' should have a lower bound and h preserves the lower bounds.

The distributive lattice envelop of a distributive meet-semilattice

The *distributive lattice envelope* of a meet-semilattice M is a pair $(e, D(M))$ where:

- $D(M)$ is a bounded distributive lattice.
- e is a meet-embedding from M to $D(M)$
- for every bounded distributive lattice D and every (meet-semilattice) sup homomorphism $f : M \to D$ there exist a unique lattice homomorphism $h : D(M) \to D$ such that $h \circ e = f$.



If f is one-to-one, then h is also one-to-one.
The distributive lattice envelop exists and is unique (up to isomorphism).

It follows that $e[M]$ is (finitely) join-dense in $D(M)$, that is, every element of $D(M)$ is the join of a finite subset of $e[M]$.

The embedding e induces an isomorphism

$$\mathrm{Filters}(M) \cong \mathrm{Filters}(D(M))$$

defined by

$$F \mapsto \text{filter generated by } e[F] \text{ in } D(M)$$

with inverse given by

$$F \mapsto e^{-1}[F].$$

Definition

A filter of M is an optimal filter if it is the inverse image by e of a prime filter of $D(M)$. It is *irreducible* if it is a meet irreducible element of the lattice of filters of M.

Notation:

- $\mathrm{Op}(M)$: set of optimal filters of M
- $\mathrm{Irr}(M)$: set of irreducible filters of M

Every irreducible filter is optimal, but the converse is not always true.

There is an intrinsic characterization of optimal filters using Frink ideals .

POSETS

Let $P = \langle P, \leq \rangle$ be a poset. A *Frink ideal* of P (Frink, 1954) is any set $I \subseteq P$ such that for every finite $X \subseteq P$

$$X \subseteq I \;\Rightarrow\; X^{ul} \subseteq I$$

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Note that:

- $\emptyset$ is a Frink ideal of P if and only if P has no lower bound.
- The order ideals of P (up-directed down sets) are Frink ideals.
- The set of Frink ideals is a closure system (closed under intersections of arbitrary families)

A Frink ideal I of a distributive meet-semilattice M is *prime* if it is proper and for every finite and non-empty $U \subseteq M$, if $\bigwedge U \in I$, then $I \cap U \neq \emptyset$.

Proposition

1. A filter F of a meet-semilattice M is optimal if and only if its complement is a prime Frink ideal of $\leq_M$.

2. A filter F of a meet-semilattice M is irreducible if and only if its complement is an up-directed prime Frink ideal.

The dual space of a distributive meet-semilattice M has:

- points: the optimal filters
- order between points: inclusion
- topology: the topology inherited from the Priestley topology of the dual of $D(M)$; it has as subasis

$$\{\varphi(a) : a \in M\} \cup \{\varphi(a)^c : a \in M\}$$

where

$$\varphi(a) = \{F \in \mathrm{Op}(M) : a \in M\}$$

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The dual space of M is then: $\langle \mathrm{Op}(M), \tau, \subseteq, \mathrm{Irr}(M)\rangle$.

It satisfies:

- φ is an isomorphism between M and $\langle \varphi[M], \cap, \mathrm{Op}(M)\rangle$.
- $\varphi[M]$ is the set of clopen up-sets U such that $\max(U^c) \subseteq \mathrm{Irr}(M)$.

The abstract notion of dual space of a distributive meet-semilattice:

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A structure $\langle X, \tau, \leq X_0 \rangle$ is a pre-generalized Priestley space if

1. $\langle X, \tau, \leq \rangle$ is a Priestley space,
2. X_0 is a dense subset of X.

If $\langle X, \tau, \leq X_0 \rangle$ is a pre-generalized Priestley space we let

$$X^* = \{U \in \mathcal{C}\ell\mathcal{U}(X) : \max(U^c) \subseteq X_0\}.$$

The elements of X^* are called X_0-*admissible sets*.

Definition

A pre-generalized Priestley space $\langle X, \tau, \leq X_0 \rangle$ is a generalized Priestley space if

1. $x \leq y$ if and only if $(\forall U \in X^*)(x \in U \Rightarrow y \in U)$,
2. $X_0 = \{x \in X : \{U \in X^* : x \notin U\}$ is non-empty and up-directed$\}$.

The dual meet-semilattice of a generalized Priestley space $\langle X, \tau, \leq X_0 \rangle$ is the meet-semilattice $X^* = \langle X^*, \cap, X \rangle$.

Let $h : M \to M'$ be a homomorphism of distributive meet-semilattice with top element. The dual of h is the relation $R_h \subseteq \mathrm{Op}(M') \times \mathrm{Op}(M)$ defined by

$$(P, Q) \in R_h \;\; \Leftrightarrow \;\; h^{-1}[P] \subseteq Q$$

This relation satisfies:

1. if $(P, Q) \notin R_h$, then there is $a \in M$ such that $Q \notin \varphi(a)$ and $R_h[P] \subseteq \varphi(a)$,
2. the map $\Box_{R_h} : \mathcal{P}(\mathrm{Op}_{\mathcal{S}}(\mathbf{A})) \to \mathcal{P}(\mathrm{Op}_{\mathcal{S}}(\mathbf{B}))$ defined by

$$\Box_{R_h}(Y) = \{P \in \mathrm{Op}(M') : R_h[P] \subseteq Y\}$$

restricts to a homomorphism from $\varphi[M]$ to $\varphi[M']$.

Definition

Let $\langle X, \tau, \leq, X_0 \rangle$ and $\langle X', \tau', \leq', X_0' \rangle$ be two generalized Priestley spaces. A relation $R \subseteq X \times X'$ is an generalized Priestley morphism when:

- $\Box_R \in \mathrm{Hom}((X')^*, X^*)$,
- if $(x, y) \notin R$, then there is $U \in (X')^*$ such that $y \notin U$ and $R[x] \subseteq U$.

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Composition of generalized Priestley morphism is as follows:

For generalized Priestley spaces X_1, X_2 and X_3 and generalized Priestley morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, its composition as generalized Priestley morphisms is the relation $(S \star R) \subseteq X_1 \times X_3$ defined by:

$$
\begin{aligned}
(x, z) \in (S \star R) \quad &\text{iff} \quad \forall U \in B_3 \big(x \in \Box_R \circ \Box_S(U) \Rightarrow z \in U \big) \\
&\text{iff} \quad \forall U \in B_3 \big((S \circ R)[x] \subseteq U \Rightarrow z \in U \big).
\end{aligned}
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▶ A logic $\mathcal{S}$ has the congruence property if the relation (of mutual consequence) on **Fm** given by

$$\varphi \dashv\vdash_{\mathcal{S}} \psi$$

is a congruence.

Let $\mathcal{S}$ be a logic and $\mathbf{A}$ an algebra.

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- A set $F \subseteq A$ is an (logical) $\mathcal{S}$-filter if for every valuation v from **Fm** on **A**, and every $\Gamma \cup \{\varphi\} \subseteq Fm$

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The $\mathcal{S}$-filter of $\mathbf{A}$ generated by $X \subseteq A$ is denoted by $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X)$ and the one generated by $a \in A$, by $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a)$.

Similarly, $\mathrm{C}_{\mathcal{S}}(\Gamma) = \{\gamma \in Fm : \Gamma \vdash_{\mathcal{S}} \gamma\}$.

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Definition

A logic $\mathcal{S}$ is *filter-distributive* if for every $\mathbf{A}$ the the lattice $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ is distributive.

- The binary relation $\Lambda_{\mathcal{S}}^{\mathbf{A}}$ on $\mathbf{A}$ (know as the Frege $\mathcal{S}$-relation of $\mathbf{A}$) is defined by:

$$\langle a, b\rangle \in \Lambda_{\mathcal{S}}^{\mathbf{A}} \Leftrightarrow \forall F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}(a \in F \Leftrightarrow b \in F).$$

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A logic $\mathcal{S}$ is *congruential* if for every $\mathbf{A}$ the relation $\Lambda_{\mathcal{S}}^{\mathbf{A}}$ is a congruence.

The algebraic counterpart of a logic

Let $\mathcal{S}$ be a logic.

- The algebraic counterpart of $\mathcal{S}$ is the class $\mathbf{Alg}\mathcal{S}$ of algebras $\mathbf{A}$ such that the only congruence included in $\Lambda^{\mathbf{A}}_{\mathcal{S}}$ is the identity. The elements of $\mathbf{Alg}\mathcal{S}$ are called *$\mathcal{S}$-algebras*.

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A logic $\mathcal{S}$ is filter-distributive if and only for every algebra $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ the lattice $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ is distributive.

- Fix a congruential logic $\mathcal{S}$. Let $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$.

The *specialization partial order* on A is defined by:

$$a \leq_{\mathcal{S}}^{\mathbf{A}} b \iff (\forall F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A})\ (a \in F \Rightarrow b \in F)$$

In other words,

$$a \leq_{\mathcal{S}}^{\mathbf{A}} b \iff b \in \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(a) \iff \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(b) \subseteq \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(a)$$

Note that $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(a) = {\uparrow_{\leq_{\mathcal{S}}^{\mathbf{A}}}} a$.

The $\mathcal{S}$-semilattice of an $\mathcal{S}$-algebra

Let us fix a congruential and filter-distributive logic $\mathcal{S}$ with theorems.

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This semilattice is the $\mathcal{S}$-semilattice of $\mathbf{A}$. We denote it by $M(\mathbf{A})$.

[These $\mathcal{S}$-semilattices are used in (M.Gehrke, R.J., A. Palmigiano, APAL 2010) to obtain canonical extensions for the $\mathcal{S}$-algebras of $\mathcal{S}$ a congruential logic with the deduction theorem.]

$\mathbf{A}$ is $\leq_{\mathcal{S}}^{\mathbf{A}}$-embedded in $M(\mathbf{A})$ by the map j given by

$$a \mapsto \uparrow_{\leq_{\mathcal{S}}^{\mathbf{A}}} a$$

The embedding j induces an an isomorphism between the lattice of filters of $M(\mathbf{A})$ and the lattice $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$, given by $F \mapsto j^{-1}[F]$, whose inverse is the map that assigns to any $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ the filter of $M(\mathbf{A})$ generated by $\{\uparrow_{\leq_{\mathcal{S}}^{\mathbf{A}}} a : a \in F\}$.

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It follows that $M(\mathbf{A})$ is a distributive meet-semilattice with top.

Let **A** be an $\mathcal{S}$-algebra.

We have the embedding $\uparrow : \langle A, \leq_{\mathcal{S}}^{\mathbf{A}} \rangle \to M(\mathbf{A})$ and the distributive envelop $(e, D(M(\mathbf{A}))$ of $M(\mathbf{A})$:

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$$\begin{array}{ccc} A & \xrightarrow{\ \uparrow\ } & M(\mathbf{A}) \\ & {\scriptstyle e\circ\uparrow} \searrow & \downarrow e \\ & & D(M(\mathbf{A})) \end{array}$$

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Definition

An $\mathcal{S}$-filter $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ is *optimal* if it is the inverse image of an optimal filter of $M(\mathbf{A})$, thus if $F = (e \circ \uparrow)^{-1}[P]$ for some prime filter of $D(M(\mathbf{A}))$.

We denote by $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ the collection of all optimal $\mathcal{S}$-filters of $\mathbf{A}$.

An intrinsic characterization of the $\mathcal{S}$-optimal filters can be given.

Definition

A subset $I \subseteq A$ is an *$\mathcal{S}$-ideal* of an algebra $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ if it is a Frink ideal of the poset $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$.

$\mathrm{Id}_{\mathcal{S}}(\mathbf{A})$ denotes the collection of all $\mathcal{S}$-ideals of $\mathbf{A}$. Note that $\emptyset \in \mathrm{Id}_{\mathcal{S}}(\mathbf{A})$ if and only if the poset $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$ has no lower bound.
A certain subclass of $\mathcal{S}$-ideals is fundamental:

Definition

An $\mathcal{S}$-ideal $I \in \mathrm{Id}_{\mathcal{S}}(\mathbf{A})$ is *strong* when for any finite $I' \subseteq I$ and any finite $B \subseteq A$, if $\bigcap \{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b) : b \in I'\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$, then $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap I \neq \emptyset$.

We denote by $\mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ the collection of all strong $\mathcal{S}$-ideals. It is easy to check that when $\mathbf{A}$ is an $\mathcal{S}$-algebra all order ideals of $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$ are strong $\mathcal{S}$-ideals.

Proposition

An $\mathcal{S}$-filter $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ is optimal if and only if its complement is a strong $\mathcal{S}$-ideal.

Notice that from the definition it follows that $A \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ if and only if $\emptyset \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$. For any congruential and filter-distributive logic with theorems, we have the following two separation lemmas

Proposition

Let $\mathcal{S}$ be a congruential and filter-distributive logic with theorems, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. Then

1. *for every $I \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ such that $F \cap I = \emptyset$ there exists $Q \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ such that $F \subseteq Q$ and $Q \cap I = \emptyset$.*
2. *for every $I \in \mathrm{Id}(\mathbf{A})$ such that $F \cap I = \emptyset$ there exists $Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ such that $F \subseteq Q$ and $Q \cap I = \emptyset$.*

If follows:

Proposition

Let $\mathcal{S}$ be a congruential and filter-distributive logic with theorems, let **A** *be an $\mathcal{S}$-algebra and let $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. Then*

1. *F is the intersection of all optimal $\mathcal{S}$-filters that contain F*
2. *F is the intersection of all irreducible $\mathcal{S}$-filters that contain F.*

The dual space of an $\mathcal{S}$-algebra

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- points: the optimal $\mathcal{S}$-filters of $\mathbf{A}$
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- points: the optimal $\mathcal{S}$-filters of $\mathbf{A}$
- order: inclusion
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$$\{\varphi(a) : a \in A\} \cup \{\varphi(a)^c : a \in A\}$$

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- the set $\varphi[A]$ endowed whit the operations defined as follows. Let $\star$ be any n-ary function symbol of the language of $\mathcal{S}$. We define $\star^{\varphi[\mathbf{A}]}$ by

  $$\star^{\varphi[\mathbf{A}]}(\varphi(a_1), \ldots, \varphi(a_n)) = \varphi(\star^{\mathbf{A}}(a_1, \ldots, a_n))$$

  Then we have an algebra $\varphi[\mathbf{A}]$ isomorphic to $\mathbf{A}$.

Definition

A structure $\mathfrak{X} = \langle X, \tau, \mathbf{B} \rangle$ is an $\mathcal{S}$-Priestley space when:

- $\langle X, \mathbf{B} \rangle$ is a reduced $\mathcal{S}$-referential algebra, whose associated partial order is denoted by $\leq$,
- for any finite $\mathcal{V} \subseteq B$ and any $U \in B$, if $\bigcap \mathcal{V} \subseteq U$, then $U \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{V})$,
- $\langle X, \tau \rangle$ is a compact space,
- B is a family of clopen up-sets and $X \in B$,
- the set $X_B := \{x \in X : \{U \in B : x \notin U\}$ is non-empty and up-directed$\}$ is dense in $\langle X, \tau \rangle$.

$\langle X, \mathbf{B} \rangle$ is a reduced $\mathcal{S}$-referential algebra means that $\mathbf{B}$ is an algebra of the type of $\mathcal{S}$ with universe a set of subsets of X such that the relation between points of X given by

$$x \preceq y \iff (\forall U \in B)(x \in U \Rightarrow y \in U)$$

is a partial order. This order is the associated partial order of $\langle X, \mathbf{B} \rangle$.

From the definition follows:

- $\langle X, \tau, \leq \rangle$ is a Priestley space,
- $\langle X, \tau, \leq, X_B \rangle$ is a generalized Priestley space,
- $B \cup \{X \setminus U : U \in B\}$ is a subbasis for τ,
- $\mathbf{B} \in \mathbf{Alg}\mathcal{S}$,
- the meet-semilattice of the X_B-admissible sets of $\langle X, \tau, \leq, X_B \rangle$ is the $\mathcal{S}$-semilattice of $\mathbf{B}$.

$\mathbf{B}$ is the dual algebra of $\langle X, \tau, \mathbf{B} \rangle$.

If $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$, then $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau : \mathbf{A}, \varphi[\mathbf{A}] \rangle$ is an $\mathcal{S}$-Priestley space, the dual of $\mathbf{A}$. The dual $\mathcal{S}$-Priestley space of $\mathbf{B}$ is then $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{B}), \tau_{\mathbf{B}}, \varphi_{\mathbf{B}}[\mathbf{B}] \rangle$.

Duality for homomorphisms

Let $\mathbf{A}, \mathbf{B} \in \mathbf{Alg}\mathcal{S}$ and $h : \mathbf{A} \to \mathbf{B}$ a homomorphism.
We define $R_h \subseteq \mathrm{Op}_{\mathcal{S}}(\mathbf{B}) \times \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ by

$$(P, Q) \in R_h \iff h^{-1}[P] \subseteq Q$$

This relation satisfies:

1. if $(P, Q) \notin R_h$, then there is $a \in A$ such that $Q \notin \varphi(a)$ and $R_h[P] \subseteq \varphi(a)$,
2. the map $\Box_{R_h} : \mathcal{P}(\mathrm{Op}_{\mathcal{S}}(\mathbf{A})) \to \mathcal{P}(\mathrm{Op}_{\mathcal{S}}(\mathbf{B}))$ defined by

$$\Box_{R_h}(Y) = \{P \in \mathrm{Op}_{\mathcal{S}}(\mathbf{B}) : R_h[P] \subseteq Y\}$$

restricts to a homomorphism from $\varphi[\mathbf{A}]$ to $\varphi[\mathbf{B}]$.

Definition

Let $\mathfrak{X}_1 = \langle X_1, \tau_1, \mathbf{B}_1 \rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_2, \mathbf{B}_2 \rangle$ be two $\mathcal{S}$-Priestley spaces. A relation $R \subseteq X_1 \times X_2$ is an $\mathcal{S}$-Priestley morphism when:

- $\Box_R \in \mathrm{Hom}(\mathbf{B}_2, \mathbf{B}_1)$,
- if $(x, y) \notin R$, then there is $U \in B_2$ such that $y \notin U$ and $R[x] \subseteq U$.

The dual of R is $\Box_R : \mathbf{B}_2 \to \mathbf{B}_1$.
Composition of $\mathcal{S}$-Priestley morphisms is like for distributive meet-semilattices.

Dual correspondence of some logical properties

A logic $\mathcal{S}$ has the property of conjunction (PC) for a binary formula $x \wedge y$ if for all formulas δ, γ,

$$\delta \wedge \gamma \vdash_{\mathcal{S}} \delta, \qquad \delta \wedge \gamma \vdash_{\mathcal{S}} \gamma, \qquad \delta, \gamma \vdash_{\mathcal{S}} \delta \wedge \gamma.$$

This property transfers to every $\mathcal{S}$-algebra $\mathbf{A}$, because for every $a, b \in A$

$$\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a \wedge^{\mathbf{A}} b) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a, b).$$

We say that a logic $\mathcal{S}$ has (PC) if it has (PC) for some binary formula.

Theorem

Let $\mathcal{S}$ be a congruential and filter-distributive logic with theorems. Then $\mathcal{S}$ has (PC) if and only if for every $\mathcal{S}$-Priestely space $\langle X, \tau, \mathbf{B}\rangle$ the B is the collection of the X_B-admissible clopen up-sets.

A logic $\mathcal{S}$ satisfies the property of weak disjunction (PWDI) for a set of formulas in two variables $\nabla(x, y)$ if

$$\delta \vdash_{\mathcal{S}} \nabla(\delta, \gamma), \qquad \delta \vdash_{\mathcal{S}} \nabla(\gamma, \delta),$$

$$\text{if} \quad \delta \vdash_{\mathcal{S}} \mu \quad \& \quad \gamma \vdash_{\mathcal{S}} \mu, \quad \text{then} \quad \nabla(\delta, \gamma) \vdash_{\mathcal{S}} \mu.$$

If for every algebra $\mathbf{A}$, and every $a, b \in A$:

$$\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\nabla^{\mathbf{A}}(a, b)) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) \cap \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b)$$

we say that (PWDI) for $\nabla(x, y)$ *transfers to every algebra*.

A logic $\mathcal{S}$ satisfies the property of disjunction (PDI) for a set of formulas in two variables $\nabla(x, y)$ if for every $\Gamma \subseteq Fm$ and all $\delta, \gamma, \mu \in Fm$:

$$\delta \vdash_{\mathcal{S}} \nabla(\delta, \gamma), \qquad \delta \vdash_{\mathcal{S}} \nabla(\gamma, \delta),$$

$$\text{if} \quad \Gamma, \delta \vdash_{\mathcal{S}} \mu \quad \& \quad \Gamma, \gamma \vdash_{\mathcal{S}} \mu, \quad \text{then} \quad \Gamma, \nabla(\delta, \gamma) \vdash_{\mathcal{S}} \mu.$$

If $\mathcal{S}$ satisfies (PDI) for a one-element set $\nabla(x, y)$, we say that $\mathcal{S}$ has (PDI) for a single formula.

If for every algebra $\mathbf{A}$, and every $\{a, b\} \cup X \subseteq A$:

$$\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, \nabla^{\mathbf{A}}(a, b)) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, a) \cap \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, b)$$

we say that (PDI) for $\nabla(x, y)$ *transfers to every algebra*.
It is well known that:

- (PDI) transfers to every algebra,
- if a logic has (PDI), then it is filter-distributive,
- for any filter-distributive logic $\mathcal{S}$, $\mathcal{S}$ has (PWDI) for a set of formulas $\nabla(x, y)$ if and only if it has (PDI) for the same set ∇.

Theorem

Let $\mathcal{S}$ be a congruential and filter-distributive logic with theorems. Then $\mathcal{S}$ has (PDI) if and only if

- *for every $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$, $X = X_B$,*
- *every $\mathcal{S}$-Priestley morphism $R \subseteq X_1 \times X_2$ from an $\mathcal{S}$-Priestley space $\langle X_1, \tau_1, \mathbf{B}_1\rangle$ to an $\mathcal{S}$-Priestley space $\langle X_2, \tau_2, \mathbf{B}_2\rangle$ is functional, that is, it holds:*

$$(\forall x \in X_1)(\exists y \in X_2)(\forall U \in B_2)R(x) = \uparrow y \quad (1)$$

Theorem

Let $\mathcal{S}$ be a congruential and filter-distributive logic with theorems. Then $\mathcal{S}$ has (PDI) **for a single formula** *if and only if*

- *for every $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B} \rangle$ the set B is closed under non-empty finite unions,*
- *for every $\mathcal{S}$-Priestley morphism $R \subseteq X_1 \times X_2$ from an $\mathcal{S}$-Priestley space $\langle X_1, \tau_1, \mathbf{B}_1 \rangle$ to an $\mathcal{S}$-Priestley space $\langle X_2, \tau_2, \mathbf{B}_2 \rangle$ it holds that for every $x \in X_1$ and every $U, V \in B_2$,*

$$\text{if } R(x) \subseteq U \cup V, \text{ then } R(x) \subseteq U \text{ or } R(x) \subseteq V.$$

A logic $\mathcal{S}$ satisfies the deduction-detachment property (DDT) for a non-empty finite set of formulas in two variables $x \Rightarrow y$ if for every $\Gamma \subseteq Fm$ and all $\delta, \gamma \in Fm$:

$$\Gamma, \delta \vdash_{\mathcal{S}} \gamma \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \delta \Rightarrow \gamma.$$

When in addition for every algebra $\mathbf{A}$, and every $\{a, b\} \cup X \subseteq A$:

$$b \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, a) \quad \text{iff} \quad a \Rightarrow^{\mathbf{A}} b \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X).$$

it is said that (DDT) *transfers to every algebra*.

A logic $\mathcal{S}$ satisfies (DDT) for a term $x \rightarrow y$ if it satisfies (DDT) for the set $\{x \rightarrow y\}$. In this case we say that $\mathcal{S}$ satisfies (DDT) for a single formula.

It is well known that:

- If $\mathcal{S}$ satisfies (DDT), then (DDT) transfers to every algebra,
- If $\mathcal{S}$ satisfies (DDT), then $\mathcal{S}$ is filter-distributive.

Theorem

Let $\mathcal{S}$ be a congruential and filter-distributive logic with theorems. If $\mathcal{S}$ is protoalgebraic, then $\mathcal{S}$ has (DDT) **for a single formula** *if and only if for every $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B} \rangle$, $(\downarrow(U \cap V^c))^c \in B$ for all $U, V \in B$, i.e. B is closed under the implication of the Heyting algebra of the up-sets of $\langle X, \tau, \mathbf{B} \rangle$.*

A logic $\mathcal{S}$ satisfies the property of the inconsistent formula (PIF) for a formula ψ if for every formula δ:

$$\psi \vdash_{\mathcal{S}} \delta.$$

Such a formula is known as an *inconsistent formula*.

If $\mathcal{S}$ is congruential and satisfies (PIF), then every $\mathcal{S}$-algebra **A** has a lower bound in the order $\leq_{\mathcal{S}}^{\mathbf{A}}$ and every inconsistent formula is interpreted by every homomorphism to the lower bound.

Theorem

Let $\mathcal{S}$ be a congruential and filter-distributive logic with theorems. Then $\mathcal{S}$ satisfies (PIF) if and only if for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B} \rangle$, $\emptyset \in B$.

Thank you

A logic $\mathcal{S}$ is protoalgebraic if there is a set of formulas in two variables $x \Rightarrow y$ such that

$$\vdash_{\mathcal{S}} x \Rightarrow x \qquad x, x \Rightarrow y \vdash_{\mathcal{S}} y$$