Axiomatizations via the pseudo-complemented bounded lattice reduct of Heyting algebras

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Background and outline

- Zakharyaschev's canonical formulas axiomatize all intermediate logics.
- Algebra-based versions of canonical formulas via the $(\wedge, \to)$-reduct[1] and the $(\wedge, \vee)$-reduct[2] of Heyting algebras axiomatize all intermediate logics.
- We will define algebra based canonical formulas using the $(\wedge, \vee, \neg)$-reduct of Heyting algebras.
- We investigate intuitionistic multi-conclusion rules via the $(\wedge, \vee, \neg)$-reduct of Heyting algebras.[3].

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[1] [1] G. Bezhanishvili, N. Bezhanishvili. "An algebraic approach to canonical formulas: Intuitionistic case." In: Review of Symbolic Logic 2.3 (2009).

[2] [2] G. Bezhanishvili, N. Bezhanishvili. "Locally finite reducts of Heyting algebras and canonical formulas". To appear in Notre Dame Journal of Formal Logic. 2014.

[3] [3] G. Bezhanishvili, N. Bezhanishvili, R. Iemhoff. "Stable canonical rules." 2014

$(\wedge, \vee, \neg)$-homomorphisms

- We are interested in classes of Heyting algebras that are (partially) closed under bounded pseudo-complemented sublattices.

Definition

Let $\mathbf{HA}_{(\wedge,\vee,\neg)}$ be the category of

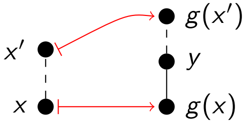
- Heyting algebras and
- homomorphism of bounded pseudo-complemented lattices.

We call these homomorphims $(\wedge, \vee, \neg)$-homomorphism.

$(\wedge, \vee, \neg)$-homomorphisms and quasi p-morphisms

Definition

Let $g : X \to Y$ be a Priestley-morphism between Esakia spaces. We call g a *quasi p-morphism* if for all $x \in X$ such that $g(x) \leq y$ for some $y \in Y$ there is $x' \in X$ with $x \leq x'$ such that $y \leq g(x')$.



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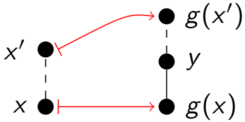


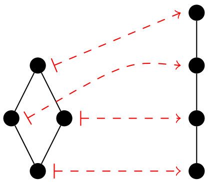
- Let $\mathbf{Esakia}_{\neg}$ be the category of Esakia spaces and quasi p-morphisms.

Theorem

The categories $\mathbf{HA}_{(\wedge,\vee,\neg)}$ *and* $\mathbf{Esakia}_{\neg}$ *are dually equivalent.*

p-morphisms vs. quasi p-morphisms

Example of a quasi p-morphism that is not a p-morphism.



$(\wedge, \vee, \neg)$-canonical rules

Definition

Let A be a finite Heyting algebra and let $D \subseteq A^2$. For every $a \in A$ let p_a be a propositional letter. The $(\wedge, \vee, \neg)$-*canonical rule* associated to A and D is $\rho(A, D, \neg) = \Gamma/\Delta$ where

$$\begin{aligned}\Gamma = &\{p_0 \leftrightarrow 0\} \cup \\ &\{p_{a \vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\ &\{p_{a \wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\} \cup \\ &\{p_{a \to b} \leftrightarrow p_a \to p_b \mid (a, b) \in D\}\end{aligned}$$

and

$$\Delta = \{p_a \leftrightarrow p_b \mid a, b \in A \text{ with } a \neq b\}$$

Characterization of $(\wedge, \vee, \neg)$-canonical rules

Proposition (as Thm. 5.3 of [3])

Let A be a finite Heyting algebra, $D \subseteq A^2$. Then for every Heyting algebra B the following are equivalent.

1. $B \not\models \rho(A, D, \neg)$
2. *There is a $(\wedge, \vee, \neg)$-embedding*

$$A \overset{h}{\rightarrowtail} B$$

such that $h(a \to b) = h(a) \to h(b)$ for all $(a, b) \in D$.

Axiomatizations via $(\wedge, \vee, \neg)$-canonical rules

Proposition (as Thm. 5.6 of [3])

1. *Every intermediate logic is axiomatizable by $(\wedge, \vee, \neg)$-canonical rules.*
2. *If an intermediate logic is finitely axiomatizable then it can be axiomatized by finitely many $(\wedge, \vee, \neg)$-canonical rules.*

Stable and cofinal stable rules

- *Cofinal stable rules* are $(\wedge, \vee, \neg)$-canonical rules of the form $\rho(A, \emptyset, \neg)$; notation: $\rho(A, \neg)$.
- *Stable rules* are $(\wedge, \vee)$-canonical rules of the form $\rho(A, \emptyset)$; notation: $\rho(A)$.

Corollary

Let B be a Heyting algbebra, $\mathfrak{F}$ an Esakia space.

1. $B \not\models \rho(A, \neg)$ *iff there is a* $(\wedge, \vee, \neg)$*-embedding* $h : A \to B$.

   Dually, the Esakia space $\mathfrak{F} \not\models \rho(A, \neg)$ *iff there is an onto quasi p-morphism* $g : \mathfrak{F} \to A_*$

2. $B \not\models \rho(A)$ *iff there is a* $(\wedge, \vee)$*-embedding* $h : A \to B$.

   Dually, the Esakia space $\mathfrak{F} \not\models \rho(A)$ *iff there is an onto Priestly morphism* $g : \mathfrak{F} \to A_*$

Stable and cofinal stable universal classes

Definition

1. A universal class of Heyting algebras that is closed under $(\wedge, \vee)$-subalgebras is called a *stable universal class*.
2. A universal class of Heyting algebras that is closed under $(\wedge, \vee, \neg)$-subalgebras is called a *cofinal stable universal class*.

Proposition (as Thm. 7.3 of [3])

1. *A universal class is stable if and only if it is axiomatizable by stable rules.*
2. *A universal class is cofinal stable if and only if it is axiomatizable by cofinal stable rules.*

Stable universal classes vs. cofinal stable classes

Clearly, every stable class is cofinal stable.

Theorem (as 6.13 of [2])

There are continuum many stable universal classes.

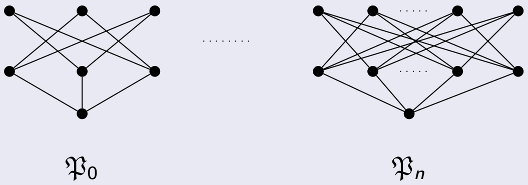
Theorem

There are continuum many cofinal stable universal classes that are not stable.

Continuum many cofinal stable classes that are not stable.

Proof.

The following sequence $\{\mathfrak{P}_i\}_{i\in\mathbb{N}}$ of frames forms an anti-chain with respect to onto quasi p-morphisms.



,

- For every $j \in \mathbb{N}$, $\mathfrak{P}_j \not\models \rho(\mathfrak{P}_j, \neg)$, and $\mathfrak{P}_j \models \rho(\mathfrak{P}_i, \neg)$ for all $i \neq j$.
- For every $J \subseteq \mathbb{N}$, $\mathcal{S}_J = \mathcal{S}_{\mathbf{IPC}} + \{\rho(\mathfrak{P}_i^*, \neg) \mid i \in J\}$ is cofinal stable.
- If $J \neq J'$, then $\mathcal{S}_J \neq \mathcal{S}'_J$.
- Let $\Delta := \{J \subseteq \mathbb{N} \mid J \text{ infinite}, \mathbb{N} \setminus J \text{ infinite}\}$. For $J \in \Delta$, $\mathcal{S}_J$ axiomatizes a cofinal stable class that is not stable.

From universal classes to logics

Recall, with an intuitionistic rule system $\mathcal{S}$ we associate the logic $\Lambda(\mathcal{S}) := \{\varphi \mid /\varphi \in \mathcal{S}\}$. Then $V_{\Lambda(\mathcal{S})} = V(\mathcal{U}_{\mathcal{S}})$

Corollary

Let L be an intermediate logic.

1. *V_L is generated by a stable universal class iff it is axiomatizable by stable rules.*
2. *V_L is generated by a cofinal stable universal class iff it is axiomatizable by cofinal stable rules.*

Moreover, if one of the above is satisfied, L has the finite model property.

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- **LC**: $V_{\mathbf{LC}}$ is generated by the linear Heyting algbras. These form a stable universal class.
- **KC** : $V_{\mathbf{KC}}$ is generated by the stable universal class $\mathcal{U}(\mathcal{K})$, where $\mathcal{K}$ is the class of finite rooted frames with a maximal element.

Intermediate logics via the $(\wedge, \vee, \neg)$ and the $(\wedge, \vee)$-reduct.

We suggest two ways to get classes of intermediate logics via the reducts:

- Logics generated by stable or cofinal stable rule systems.
- Logics via algebra based $(\wedge, \vee)$- or $(\wedge, \vee, \neg)$-canonical formulas.
- The two approaches may lead to different classes of intermediate logics.

$(\wedge, \vee)$- canonical formulas [2]

- Let A be a finite subdirectly irreducible Heyting algebra and $D \subseteq A^2$. For every element $a \in A$ let p_a be a propositional letter. The *$(\wedge, \vee)$-canonical formula associated to A and D* is defined as:

$$
\begin{aligned}
\gamma(A, D) := \quad & p_0 \leftrightarrow 0 \quad \wedge p_1 \leftrightarrow 1 \\
& \bigwedge\{p_{a \wedge b} \leftrightarrow p_a \wedge p_b \mid a, b \in A\} \wedge \\
& \bigwedge\{p_{a \vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \wedge \\
& \bigwedge\{p_{a \to b} \leftrightarrow p_a \to p_b \mid (a, b) \in D\} \\
& \longrightarrow \bigvee\{p_a \to p_b \mid a, b \in A, a \not\leq b\}.
\end{aligned}
$$

- The valuation $p_a \mapsto a$ witnesses that $A \not\models \gamma(A, D)$.
- Similarly, we can define the *$(\wedge, \vee, \neg)$-canonical formula associated to A and D*.

Refutation patterns for restricted formulas [2]

- To every finite subdirecty irreducible Heyting algebra A let $\gamma(A)$ be the $(\wedge, \vee)$-canonical formula associated to A and $\emptyset$.
- Similary, define for every s.i Heyting algebra A its $(\wedge, \vee, \neg)$-canonical formula $\gamma(A, \neg)$.

Theorem (3.4 of [2])

For every s.i. Heyting algebra B,

1. $B \not\models \gamma(A)$ *iff there is a s.i Heyting algebra C, an onto homomorphism of Heyting algebras f and a $(\wedge, \vee)$-embedding h as in*

$$A \overset{h}{\rightarrowtail} C \overset{f}{\twoheadleftarrow} B \ .$$

2. $B \not\models \gamma(A, \neg)$ *iff there is a s.i Heyting algebra C, an onto homomorphism of Heyting algebras f and a $(\wedge, \vee, \neg)$-embedding h as in*

$$A \overset{h}{\rightarrowtail} C \overset{f}{\twoheadleftarrow} B \ .$$

Stable logics and cofinal stable logics

Definition

Let L be an intermediate logic and let V_L its corresponding variety.

1. L is called a *stable intermediate logic* iff for all B, A s.i. Heyting algebras such that A is a bounded sublattice of B then

$$B \in V_L \Rightarrow A \in V_L.$$

2. L is called a *cofinal stable intermediate logic* iff for all B, A s.i. Heyting algebras such that A is a $(\wedge, \vee, \neg)$-sublattice of B then

$$B \in V_L \Rightarrow A \in V_L.$$

Stable logics vs. cofinal stable logics

- Every stable logic is cofinal stable, and there are cofinal stable logics that are not stable (cf. $\mathbf{BD_2}$).

- **Theorem (6.13 of [2])**

  There are continuum many stable logics.

- **Theorem (6.8 of [2])**

  All cofinal stable logics (and therefore also all stable logics) have the fmp.

- **Theorem (6.11 of [2])**

  An intermediate logic is stable iff it is axiomatizable by $\gamma(A)$ formulas.

- **Proposition**

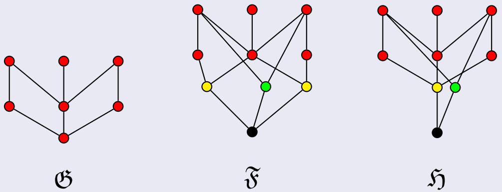
  Every cofinal stable logic is axiomatizable by $\gamma(A, \neg)$-formulas.

  - The converse of the last proposition does not hold.

Proposition

There is a logic axiomatized by $\gamma(A, \neg)$-formulas that is not cofinal stabe.

Proof.



- Let $L = \mathbf{IPC} + \gamma(\mathfrak{G}^*, \neg)$.
- $\mathfrak{G}$ is not a quasi p-morphic image of any rooted upset of $\mathfrak{F}$, so $\mathfrak{F} \models L$.
- $\mathfrak{H} \not\models L$ since $\mathfrak{G}$ is isomorphic to a rooted upset of $\mathfrak{H}$.
- However, $\mathfrak{H}$ is a quasi p-morphic image of $\mathfrak{F}$. It follows that L is not cofinal stable.

Connection to logics axiomatized by rules

- What is the connection to the logics generated by stable and cofinal stable universal classes?

Proposition

Let L be a logic such that V_L is finitely generated.

1. *L is a stable intermediate logic iff V_L is generated by a stable universal class.*
2. *If L is a cofinal stable intermediate logic then V_L is generated by a cofinal stable universal class.*

The converse in (2) does not hold, i.e there is a finite cofinal stable universal $\mathcal{U}$ class such that $V(\mathcal{U})$ is not cofinal stable.

- Question: What if V_L is not finitely generated?

Summary

- Axiomatizations of intuitionistic rule systems and intermediate logics using the $(\wedge, \vee, \neg)$-reduct are obtained analogously to the $(\wedge, \vee)$ case.
- Subtle issues that distinguish stable and cofinal stable logics.
- In particular, considering intermediate logics generated by canonical rule systems the two reducts show different behavior.

Future work

- Further investigate the usage of canonical rules and formulas in the intuitionistic case.
- Develop a notion of cofinal stable logics in the modal case.
- Apply the method of algebra based canonical formulas to other non-classical logics.