Two functors induced by z-ideals and d-ideals of pointfree function rings

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Definition

- An element a of L is rather below an element b, written $a \prec b$, in case there is an element s, called a separating element, such that $a \wedge s = 0$ and $s \vee b = 1$.
- The frame L is regular if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$.
- An element a is completely below b, written $a \prec\prec b$, if there are elements (x_r) indexed by rational numbers $\mathbb{Q} \cap [0, 1]$ such that $a = x_0$, $x_1 = b$ and $x_r \prec x_s$ for $r < s$.
- The frame L is completely regular if $a = \bigvee\{x \in L \mid x \prec\prec a\}$ for each $a \in L$.
- Coz L is the cozero part of L, and is the regular sub-σ-frame consisting of all the cozero elements of L.

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- An ideal I of a ring A is a radical ideal if for any $a \in A$, $a^2 \in I$ implies $a \in I$.
- An element c of a frame L is compact if for any $S \subseteq L$, $c \leq \bigvee S$ implies $c \leq \bigvee T$, for some finite $T \subseteq S$.
- A nucleus on a frame L is a closure operator $\ell : L \to L$ such that $\ell(a \wedge b) = \ell(a) \wedge \ell(b)$ for all $a, b \in L$.
- $\mathrm{Did}(\mathcal{R}L)$ is the lattice of d-ideals of $\mathcal{R}L$.
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Throughout, by “ring" we mean a commutative ring with identity. For a ring A and $a \in A$, we let

$$\mathfrak{M}(a) = \{M \in \operatorname{Max} A \mid a \in M\}.$$

In the article

G. Mason, *z-Ideals and Prime Ideals*, J. Alg. **26** (1973), 280-297.

Mason calls an ideal I of A a *z-ideal* if for any a and b in A,

$$a \in I \quad \text{and} \quad \mathfrak{M}(a) = \mathfrak{M}(b) \quad \Rightarrow \quad b \in I.$$

Let L be a completely regular frame, and $\mathcal{R}L$ be the ring of real-valued continuous functions on L.

An ideal Q of $\mathcal{R}L$ is a z-ideal iff $Q = \bigcup\{\mathbf{M}_{\operatorname{coz}\alpha} \mid \alpha \in Q\}$, where, for any $a \in L$,

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Recall that an ideal I of a ring A is singular if it consists entirely of zero-divisors. For any $a \in A$, let P_a denote the intersection of all minimal prime ideals of A containing a. We will denote the annihilator of a set $S \subseteq A$ by $S^{\perp}$, and the annihilator of the singleton $\{a\}$ will be abbreviated as $a^{\perp}$.

It is shown in

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that $P_a = a^{\perp\perp}$.

An ideal I of the ring A is called a *d*-ideal if $a^{\perp\perp} \subseteq I$, for every $a \in I$.

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Proposition

Let L be a completely regular frame

1. *$Zid(\mathcal{R}L) = Fix(z)$, for the z-nucleus on $Rad(\mathcal{R}L)$.*
2. *$Did(\mathcal{R}L) = Fix(d)$, for the d-nucleus on $Zid(\mathcal{R}L)$.*
3. *$\mathfrak{K}(Zid(\mathcal{R}L)) = \{\mathbf{M}_{coz\,\alpha} \mid \alpha \in \mathcal{R}L\} = \{\mathbf{M}_{coz\,\alpha} \mid \mathbf{0} \leq \alpha \in \mathcal{R}L\}$.*
4. *$\mathfrak{K}(Did(\mathcal{R}L)) = \{\mathbf{M}_{c^{**}} \mid c \in Coz\,L\}$.*
5. *$Zid(\mathcal{R}L)$ is a coherent frame.*
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We recall from

P. T. Johnstone, Stone Spaces
Cambridge Studies in Advanced Math. No. 3, Camb. Univ. Press 1982.

that if A and B are coherent frames, then any lattice homomorphism $\mathfrak{K}(A) \to \mathfrak{K}(B)$ extends to a frame homomorphism $A \to B$.

Proposition

The map $\bar{h}\colon \mathfrak{K}(Zid(\mathcal{R}L)) \to \mathfrak{K}(Zid(\mathcal{R}M))$ defined by

$$\bar{h}(\mathbf{M}_{\mathrm{coz}\,\alpha}) = \mathbf{M}_{\mathrm{coz}(h\cdot\alpha)}$$

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Proposition

*For any morphism $h\colon L \to M$ in **CRegFrm**, the map*

$$Zid(h)\colon Zid(\mathcal{R}L) \to Zid(\mathcal{R}M)$$

given by

$$Zid(h)(Q) = \bigvee \{\boldsymbol{M}_{coz\,(h\cdot\alpha)} \mid \alpha \in Q\}$$

is a coherent frame homomorphism.

We define Z: **CRegFrm** $\to$ **CohFrm** by

$$\mathrm{Z}(L) = Zid(\mathcal{R}L) \quad \text{and} \quad Z(h) = \bar{h}.$$

It must send an object $L \in$ **CRegFrm** to $\mathrm{Zid}(\mathcal{R}L)$, and a morphism $h \in$ **CRegFrm** to the morphism $\bar{h}$ in **CohFrm**.

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Z is a functor.

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Z is a functor.

In order to define a functor induced by Did along the lines of the functor Z, we first show that given any frame homomorphism $h\colon L \to M$, there is a coherent map

$$\mathrm{Did}(h)\colon \mathrm{Did}(\mathcal{R}L) \to \mathrm{Did}(\mathcal{R}M).$$

We write $\sqcup$ for the join in $\mathrm{Did}(\mathcal{R}L)$ and $\mathfrak{B}L$, where $\mathfrak{B}L$ is the Booleanization of L whose underlying set is $\{a^{**} \mid a \in L\}$ with the meet calculated as in L, and join $\bigsqcup$ given by

$$\bigsqcup S = \left(\bigvee\nolimits_L S\right)^{**}$$

for any $S \subseteq \mathfrak{B}L$.

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$$\mathrm{Did}(h)\colon \mathrm{Did}(\mathcal{R}L) \to \mathrm{Did}(\mathcal{R}M).$$

We write $\sqcup$ for the join in $\mathrm{Did}(\mathcal{R}L)$ and $\mathfrak{B}L$, where $\mathfrak{B}L$ is the Booleanization of L whose underlying set is $\{a^{**} \mid a \in L\}$ with the meet calculated as in L, and join $\bigsqcup$ given by

$$\bigsqcup S = \left(\bigvee\nolimits_L S\right)^{**}$$

for any $S \subseteq \mathfrak{B}L$.

Lemma

For any $a, b \in coz\,L$, we have:

1. $\boldsymbol{M}_{a^{**}} \wedge \boldsymbol{M}_{b^{**}} = \boldsymbol{M}_{(a\wedge b)^{**}}$
2. $\boldsymbol{M}_{a^{**}} \sqcup \boldsymbol{M}_{b^{**}} = \boldsymbol{M}_{(a\vee b)^{**}}$.

Lemma

Given a frame homomorphism $h\colon L \to M$, the map

$$\phi\colon \Re(Did(\mathcal{R}L)) \to \Re(Did(\mathcal{R}M))$$

given by

$$\phi(\boldsymbol{M}_{c^{**}}) = \boldsymbol{M}_{h(c)^{**}}$$

is a lattice homomorphism.

Lemma

For any $a, b \in \mathrm{coz}\, L$*, we have:*

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Lemma

Given a frame homomorphism $h\colon L \to M$*, the map*

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given by

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By the result recalled earlier from Johnstone's book, we have the following:

Corollary

For any frame homomorphism $h\colon L \to M$, the map $\tilde{h}\colon \mathrm{Did}(\mathcal{R}L) \to \mathrm{Did}(\mathcal{R}M)$ defined by

$$\tilde{h}(Q) = \bigvee \{ M_{(h(\operatorname{coz} \alpha))^{**}} \mid \alpha \in Q \}$$

is the unique coherent map extending the map ϕ defined above.

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Definition

We define $\mathsf{D}\colon \textbf{CRegFrm} \to \textbf{CohFrm}$ by setting $\mathsf{D}(L) = \mathrm{Did}(\mathcal{R}L)$ and $\mathsf{D}(h) = \tilde{h}$.

Proposition

D is a functor.

Proposition

Both D and Z are faithful.

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Both D and Z are faithful.

Proof.

We prove the faithfulness of D only because that of Z is similar; and, in fact, more straightforward. We will use the fact that if $x \lll a$, then $x^{**} \leq a$. Let $h \colon L \to M$ and $g \colon L \to M$ be two morphisms in **CRegFrm** such that $\mathrm{D}(h) = \mathrm{D}(g)$. Then, for any $c \in \mathrm{Coz}\, L$, $\mathrm{D}(h)(\boldsymbol{M}_{c^{**}}) = \mathrm{D}(g)(\boldsymbol{M}_{c^{**}})$, which implies $\boldsymbol{M}_{h(c)^{**}} = \boldsymbol{M}_{g(c)^{**}}$, and consequently, $h(c)^{**} = g(c)^{**}$. □

Proof.

Let $a \in L$. Then, by complete regularity,

$$a = \bigvee\{c \in \operatorname{Coz} L \mid c \lll a\},$$

and hence

$$\begin{aligned} h(a) &= \bigvee\{h(c) \mid c \in \operatorname{Coz} L \text{ and } c \lll a\} \\ &\leq \bigvee\{h(c)^{**} \mid c \in \operatorname{Coz} L \text{ and } c \lll a\} \\ &= \bigvee\{g(c)^{**} \mid c \in \operatorname{Coz} L \text{ and } c \lll a\} \\ &\leq g(a) \quad \text{since } g(c) \lll g(a) \text{ whenever } c \lll a. \end{aligned}$$

By symmetry, we conclude that $h(a) = g(a)$, so that $h = g$. Therefore D is faithful. □

For any frame L we write δ_L for the frame homomorphism

$$\delta_L\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Did}(\mathcal{R}L)$$

induced by the d-nucleus on $\mathrm{Zid}(\mathcal{R}L)$. Now we have the following result.

Proposition

The correspondence $L \mapsto \delta_L$ is a natural transformation $Z \to D$.

Proof.

(OUTLINE) We need to check that the diagram

$$\begin{array}{ccc} \mathrm{Zid}(\mathcal{R}L) & \xrightarrow{\;Z(h)\;} & \mathrm{Zid}(\mathcal{R}M) \\ \Big\downarrow{\scriptstyle \delta_L} & & \Big\downarrow{\scriptstyle \delta_M} \\ \mathrm{Did}(\mathcal{R}L) & \xrightarrow{\;D(h)\;} & \mathrm{Did}(\mathcal{R}M) \end{array}$$

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commutes. □

Follow the compact element $\boldsymbol{M}_c$:

$$\begin{array}{ccc} \boldsymbol{M}_c & \xrightarrow{\mathrm{Z}(h)} & \boldsymbol{M}_{h(c)} \\ \downarrow{\scriptstyle \delta_L} & & \downarrow{\scriptstyle \delta_M} \\ \boldsymbol{M}_{c^{**}} & \xrightarrow{\mathrm{D}(h)} & \boldsymbol{M}_{h(c)^{**}} \end{array}$$

Recall that a frame homomorphism $h\colon L \to M$ is said to be skeletal if it sends dense elements to dense elements. By a result of Banaschewski and Pultr, in the article

Variants of Openness
Applied Categorical Structures. No. 2, 331–350, 1994.

h is skeletal precisely if $h(a^{**}) \leq h(a)^{**}$ for every $a \in L$. Weakening this, we introduce the following definition.

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A frame homomorphism $h\colon L \to M$ is coz-skeletal if $h(c^{**}) \leq h(c)^{**}$ for every $c \in \mathrm{Coz}\, L$.

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A frame homomorphism $h\colon L \to M$ is coz-skeletal if $h(c^{**}) \leq h(c)^{**}$ for every $c \in \mathrm{Coz}\, L$.

Lemma

For any $L \in$ **CRegFrm**, *the map* $\sigma_L \colon \mathrm{Zid}(\mathcal{R}L) \to L$ *given by*

$$\sigma_L(Q) = \bigvee \{\mathrm{coz}\,\alpha \mid \alpha \in Q\}$$

is a dense onto frame homomorphism.

Proposition

Let $h\colon L \to M$ be a skeletal frame homomorphism between completely regular frames. Then in the diagram,

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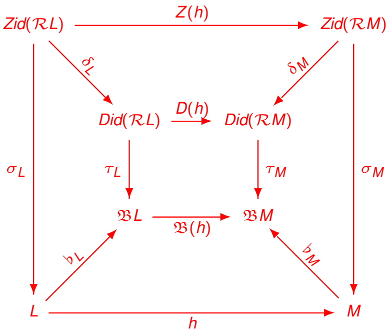

every quadrilateral is commutative.

Proposition

The following are equivalent for a frame homomorphism $h\colon L \to M$.

1. h *is skeletal*.
2. $\mathrm{Z}(h)$ *is skeletal*.
3. $\mathrm{D}(h)$ *is skeletal*.

Proposition

The following are equivalent for any frame homomorphism $h\colon L \to M$.

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Definition

A frame homomorphism $h\colon L \to M$ is open if h has a left adjoint $h_! : M \to L$ which satisfies the Frobenius identity

$$h_!(h(a) \wedge b) = a \wedge h_!(b),$$

for all $a \in L$ and $b \in M$.

We recall from

T. Dube and I. Naidoo, On openness and surjectivity of lifted frame homomorphisms,
Top. Appl., ***157*** *(2010), 2159–2171.*

that a homomorphism $h\colon L \to M$ is a λ-map if the diagram

$$\begin{array}{ccc} \lambda L & \xrightarrow{\ h^{\lambda}\ } & \lambda M \\ {\scriptstyle (\lambda_L)_*}\uparrow & & \uparrow{\scriptstyle (\lambda_M)_*} \\ L & \xrightarrow[\ h\ ]{} & M \end{array}$$

is round; that is, if $(\lambda_M)_* \cdot h = h^{\lambda} \cdot (\lambda_L)_*$.

Lemma

$Z(h)(\boldsymbol{M}_a) = \boldsymbol{M}_{h(a)}$ for every $a \in L$ if and only if h is a λ-map.

Proposition

A λ-map $h\colon L \to M$ has a left adjoint if and only if $Z(h)$ has a left adjoint.

Corollary

A λ-map $h\colon L \to M$ is open if and only if $Z(h)$ is open.

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Example

Let X be a pseudocompact Tychonoff space which is not locally compact. Let $L = \mathfrak{O}X$. The map $j_L \colon \beta L \to L$ is coz-surjective and coz-faithful since L is pseudocompact. Thus, by the result quoted above, $\mathrm{Zid}(j_L)$ is an isomorphism, and hence open. However, j_L is not open. Indeed, because X is not locally compact, the inclusion map $X \hookrightarrow \beta X$ is not open, and hence the induced frame homomorphism $\mathfrak{O}(\beta X) \to \mathfrak{O}X$ is not open. But $\mathfrak{O}(\beta X) \cong \beta(\mathfrak{O}X)$, so the claim is established.

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