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Gödel Spaces and perfect *MV*–algebras

R. Grigolia

Tbilisi State University

(joint work with A. Di Nola and G. Lenzi)

University of Salerno

content

- Introduction
- Preliminaries
- *MV(C)*-algebras
- Duality

Introduction

MV-algebras are the algebraic counterpart of the infinite valued Lukasiewciz sentential calculus, as Boolean algebras are with respect to the classical propositional logic. In contrast with what happens for Boolean algebras, there are *MV -algebras which are not semisimple, i.e. the intersection of* their maximal ideals (the radical of A) is different from {0}.

MV -algebras form a category which is equivalent to the category of abelian lattice ordered groups (*ℓ-groups, for short) with* strong unit.

Perfect *MV-algebras do not form a variety and contains non-simple subdirectly* irreducible *MV-algebras.*

An important example of a perfect MV-algebra is the subalgebra *S of the Lindenbaum algebra L of the first ord*er Lukasiewicz logic generated by the classes of formulas which are valid when interpreted in [0,1] *but* non-provable.

Hence perfect MV-algebras are directly connected with the very important phenomenon of incompleteness in Lukasiewicz first order logic [P. Belluce, C. C. Chang, 1963].

Each perfect *MV -algebra is associated with an abelian ℓ-group with a strong unit. Moreover, the category of perfect MV–alge-bras* is equivalent to the category of abelian ℓ-*groups.*

The variety generated by all perfect *MV -algebras, denoted by* ***MV(C), is also generated*** by a single *MV -chain, actually the MV -algebra C, defined by Chang.*


1
¬c
(¬c)²
C
2c
c
0

$C_0 = \Gamma(Z, 1)$,

$C_1 = C = \Gamma(Z \times_{lex} Z, (1, 0))$

with generator $(0, 1) = c_1 (= c)$, ... ,

$C_m = \Gamma(Z \times_{lex} \cdots \times_{lex} Z, (1, 0, \ldots, 0))$

with generators

$c_1(= (0, 0, \ldots, 1)), \ldots, c_m(= (0, 1, \ldots, 0))$

Let us denote $Rad(A) \cup \neg Rad(A)$ through $R^*(A)$.

We are interested by the class

LSP$\{C_i : i \in \omega\}$ *of MV (C)-algebras*

which is generated from the set $\{C_i : i \in \omega\}$ by the operators of direct products, subalgebras and direct limits, where C_0 is two-element Boolean algebra, $C_1 = C$ *and* C_n $(n > 1)$ is *n-generated perfect MV -chain*

We give

1) the description of *m*-generated free *MV*-algebras in the variety **MV(C)** generated by perfect *MV*-algebras;

2) that there is a duality between the full subcategory $\mathbf{MV(C)}^{\mathbf{G}}$ (= $\mathbf{LSP}\{C_i : i \in \omega\}$) of the category **MV(C)** and the category of Gödel spaces *GS* , where any Gödel spaces are special case of Priestley spaces.

More precisely, we construct the functors

P : $\mathbf{MV(C)}^{\mathbf{G}} \to GS$, which is full, and

H : $GS \to \mathbf{MV(C)}^{\mathbf{G}}$ which is faithfull.

In the category theory,

a functor F: ***E*** $\longrightarrow$ ***D*** *is dense (or essentially surjective*)

if each object *D of* ***D*** is isomorphic to an object of the form F*(E)* for some object *E of* ***E.***

Preliminaries

Gödel Algebras and Gödel Spaces

A *Boolean space* is zero-dimensional, compact and Hausdorff topological space.

A Priestley space is a triple *(X,R, Ω), where* *(X, Ω)* is a Boolean space and *R* is an order relation on X such that, for all $x, y \in X$ with $\neg(xRy)$, there exists a clopen up-set V with $x \in V$ and $y \notin V$.

A morphism between Priestley spaces is a continuous order-preserving map.

Priestley duality relates the category of bounded distributive lattices to the category $\mathcal{PS}$ of Priestley spaces by mapping each bounded distributive lattice *L to its ordered space* $\mathcal{F}$*(L) of prime filters, and mapping each Priestley* space X to the bounded distributive lattice $\mathcal{L}$*(X) of clopen up-sets of X.*

When restricted to Heyting algebras and Heyting spaces respectively, these mappings give the restricted Priestley duality for Heyting algebras.

Gödel Algebras and Gödel Spaces

A *Heyting algebra* is an algebra $(A,\vee,\wedge,\rightarrow, 0, 1)$ *of type* (2, 2, 2, 0, 0), where $(A,\vee,\wedge, 0, 1)$ *is a bounded distributive lattice and the binary operation* $\rightarrow$, which is called *implication,* satisfies

$$(\forall a,b,x \in A) \quad x \wedge a \leq b \Leftrightarrow x \leq a \rightarrow b.$$

A *Heyting space* X is a Priestley space such that $R^{-1}(U)$ is open for every open subset U *of* X.

(Recall that $R^{-1}(U) = \{y \in X : (\exists u \in U) yRu\}$ and that $R^{-1}\{x\}$ is abbreviated to $R^{-1}(x)$). The sets $R(U)$ *and* $R(x)$ are defined dually.)

A morphism between Heyting spaces, called a *strongly isotone map(or Heyting morphism in other terminology),* is a continuous map $\varphi : X \to Y$ such that $\varphi(R(x)) = R(\varphi(x))$ for all $x \in X$.

The variety **G** , corresponding to Gödel logic *G* is generated by finite chain Heyting algebras.

Gödel Algebras and Gödel Spaces

For any Priestley space (X,R) we define $\mathrm{P}(X)$ as the set of all clopen up-sets of X.

For any U, $V \in \mathrm{P}(X)$ define:

$$U \vee V = U \cup V \text{ and } U \wedge V = U \cap V.$$

Then the algebra

$$\mathrm{P}((X,R)) = (P(X), \cup, \cap, \emptyset, X)$$

is a bounded distributive lattice.

On the other hand, for each bounded distributive lattice L, the set $\mathrm{F}(L)$ of all prime filters of L with the binary relation R on it, which is the inclusion between prime filters, and topologised by taking the family of

$$supp(a) = \{F \in \mathrm{F}(L) : a \in F\},$$

for $a \in L$, and their complements as a subbase, is an object of PS.

For any Heyting space (X,R) and $U,V \in \mathcal{H}(X)$ (= the set of all clopen up-sets of X) define:

$$U \to V = X - (R^{-1}(U - V))$$

Then the algebra

$$\mathrm{H}((X,R)) = (\mathrm{H}(X), \cup, \cap, \to, \varnothing, X)$$

is a Heyting algebra.

We have two contravariant functors

$$\mathrm{F} : \mathbf{HA} \rightarrow \mathit{HS} \text{ and } \mathrm{H} : \mathit{HS} \rightarrow \mathbf{HA}.$$

These functors establish a dual equivalence between the categories **HA** and HS.

A Heyting algebra A is said to be *Göodel algebra* if it satisfies the linearity condition:

$$(a \to b) \vee (b \to a) = 1$$

all $a, b \in A$.

It is well known that the Heyting spaces for Göodel algebras form root systems.

So we can define Gödel space X as such kind Heyting space that $R(x)$ is a chain for any $x \in X$.

The category of Gödel spaces and strongly isotone maps denote by *GS*.

MV(C)-algebras

An ***MV-algebra*** is an algebra

$$\mathrm{A} = (A, \oplus, \otimes, \neg, 0, 1)$$

where $(A, \oplus, 0)$ is an abelian monoid, and for all $x,y \in A$ the following identities hold:

$x \oplus 1 = 1$, $\neg\neg x = x$,

$\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$,

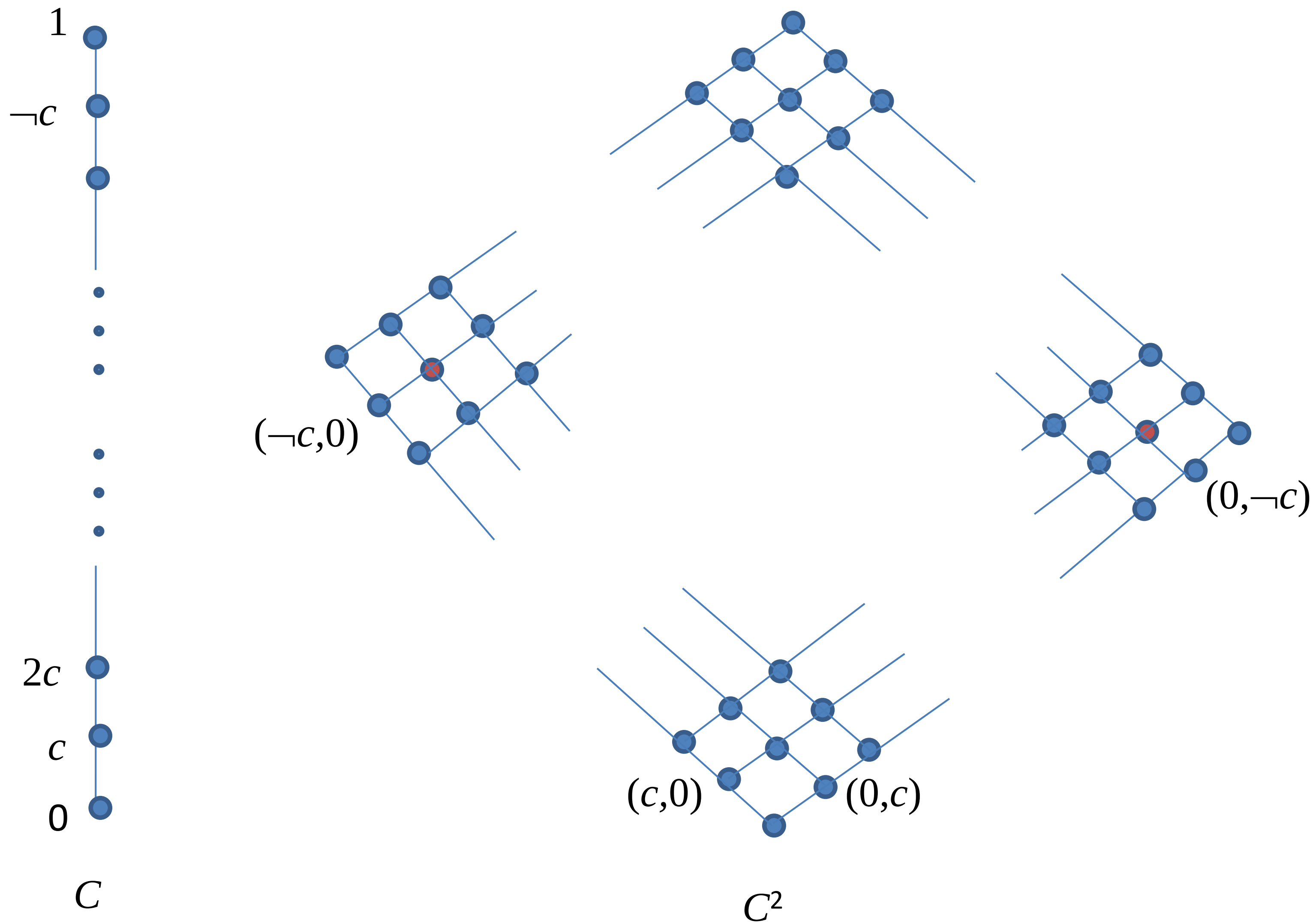
$x \otimes y = \neg(\neg x \oplus \neg y)$.

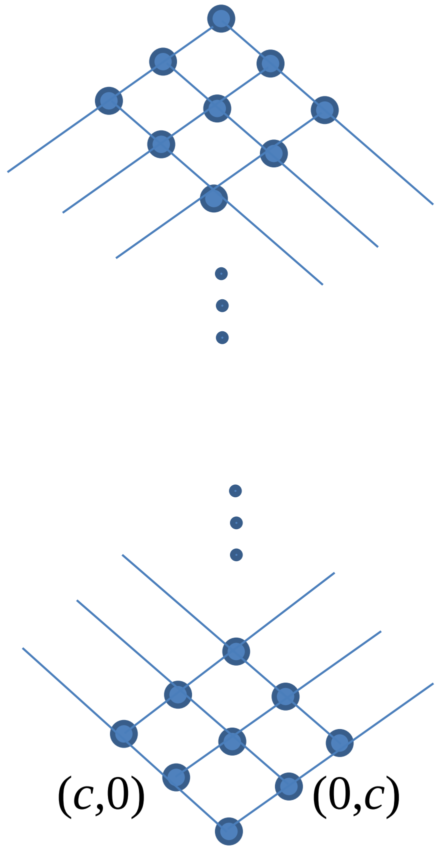
An ***MV-algebra*** is *MV(C)*- algebra if in addition holds $(2x)^2 = 2x^2$.

Let *A be* any *MV* -algebra. The least integer for which $nx = 1$ is called *the order of* $x \in A$. When such an integer exists, we denote it by $ord(x)$ and say that x has finite order, otherwise we say that x has infinite order and write $ord(x) = \infty$.

An *MV* -algebra A is called *perfect* if for every nonzero element $x \in A$ $ord(x) = \infty$ if and only if $ord(\neg x) < \infty$.

Theorem 1. *An* 1*-generated free MV (C)-algebra* $F_{MV(C)}(1)$ *is isomorphic to* C^2 *with free generator* $(c, \neg c)$.


1
¬c
2c
c
0
C
(¬c,0)
(0,¬c)
(c,0)
(0,c)
C2



$R^*(C^2)$

Taking into account an order of generators of C_n it is generated by $c_{\varphi i(1)}, c_{\varphi i(2)}, \ldots , c_{\varphi i(n)}$

for any $i \in \{1, \ldots , n!\}$, where

$$\varphi_i : \{1, \ldots , n\} \to \{1, \ldots , n\}$$

is any bijection. We have $n!$ diferent ordered generators that generate C_n.

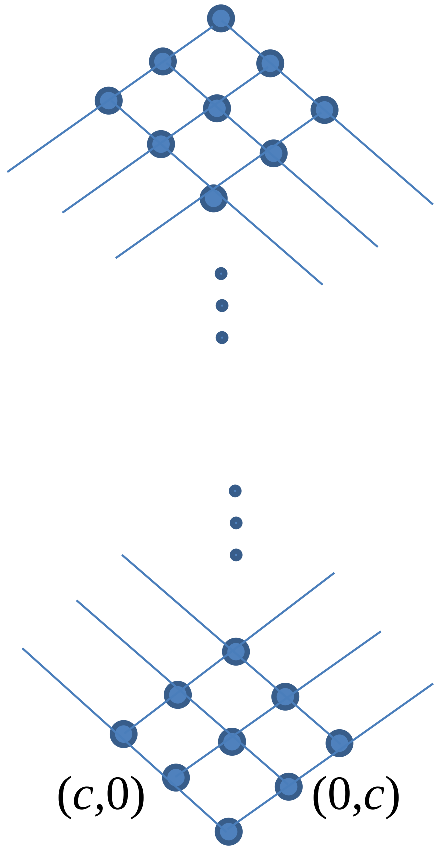
Let $\mathbf{a}_i = (c_{\varphi i(1)}, c_{\varphi i(2)}, \ldots , c_{\varphi i(n)})$.

Let B_n be the subalgebra of the algebra $C_n^{n!}$ generated by n generators

$$\mathbf{b}_i = (\pi_i(a_1), \pi_i(a_2), \dots, \pi_i(a_{n!})),$$

$i = 1, \dots, n$.

The generators $\mathbf{b}_1, \dots, \mathbf{b}_n$ belong to Rad($C_n^{n!}$). Therefore the algebra B_n is perfect. If $n = 2$, then $B_2 = R^*(C^2)$



$R^*(C^2)$

Let us consider subalgebra A_k of the algebra $\Pi_{i=1}{}^{\infty} D_i^{(k)}$, where $D_i^{(k)} = C_k$ $(1 \leq k < n)$, generated by

$$\mathbf{d}_{j(k)} = (u_{1k}^{(j)}, u_{2k}^{(j)}, u_{3k}^{(j)}, \dots , u_{ik}^{(j)} , \dots),$$

$j = 1, \dots , n,$ where $u_{ik}^{(1)}, \dots , u_{ik}^{(n)}$ generate $D_i^{(k)}$, $(u_{ik}^{(1)}, \dots , u_{ik}^{(n)}) \neq (u_{jk}^{(1)}, \dots , u_{jk}^{(n)})$ for $i \neq j$.

Let B be a subalgebra of

$$B_n \times A_1 \times \ldots \times A_{n-1}$$

generated by

$$\mathbf{g}_1 = (\mathbf{b}_1, \mathbf{d}_1^{(1)}, \ldots, \mathbf{d}_1^{(n-1)}),$$

$$\ldots$$

$$\mathbf{g}_n = (\mathbf{b}_n, \mathbf{d}_n^{(1)}, \ldots, \mathbf{d}_n^{(n-1)}).$$

Theorem 2. *n-generated free MV (C)-algebra $F_{\mathbf{MV(C)}}(n)$ is isomorphic to B^{2^n} with free generators*

$$G_1 = (\mathbf{g}_1^{\varepsilon_{11}}, \mathbf{g}_1^{\varepsilon_{21}}, \dots, \mathbf{g}_1^{\varepsilon_{2^n 1}}),$$
$$G_2 = (\mathbf{g}_2^{\varepsilon_{12}}, \mathbf{g}_2^{\varepsilon_{22}}, \dots, \mathbf{g}_2^{\varepsilon_{2^n 2}}),$$
$$\dots$$
$$G_n = (\mathbf{g}_n^{\varepsilon_{1n}}, \mathbf{g}_n^{\varepsilon_{2n}}, \dots, \mathbf{g}_n^{\varepsilon_{2^n n}}),$$

where $\mathbf{g}_i^{\varepsilon} = \begin{cases} \mathbf{g}_i, & \varepsilon = 1 \\ \neg\, \mathbf{g}_i, & \varepsilon = 0 \end{cases}$.

Spectral Duality

A topological sace X is called *spectral* if X a compact T_0-space, every non-empty irreducible closed subset of X is the closuer of a unique point, and the set $\mathcal{D}(X)$ of compact open subsets of X constitutes a basis for the topology of X and is closed under finite union and intersection.

Spectral Duality

The category **D** of bounded distributive lattices and the category *Spec* of spectral spaces are dually equivalent.

Since **D** is dually equivalent to both the category spectral spaces *Spec* and the category of Priestley spaces *PS,* it follows that *the categories Spec* and *PS* are equivalent.

MV-spaces

An *MV* -space is Priestley space *X* such that *R(x)* is a chain for any $x \in X$ and a morphism between *MV* -spaces is a strongly isotone map (or an *MV*- morphism), i. e. a continuous map $\varphi : X \to Y$ such that $\varphi(R(x)) = R(\varphi(x))$ for *all* $x \in X$.

Hence *MV* -space forms a root system.

We denote the category of *MV* -spaces plus *MV* -morphisms by *MVS*.

Let *MVSC* be a subcategory of the category *MVS*, the objects of which are such kind of *MV* -spaces X for which there exist *MV (C)*-algebras *A* such that $\mathrm{M}(A) = X$, where $\mathrm{M}(A)$ is the set of all prime *MV* –filters of *A.*

Belluce's functor

We define binary relation $\equiv^*$ on *MV*-algebra *A* by the following stipulation:

$$x \equiv^* y \text{ iff } supp(x) = supp(y),$$

where *supp*(*x*) is defined as the set of all prime filters of A containing the element *x*. Then, it is a congruence with respect to $\otimes$ and $\vee$. The resulting set $\beta^*(A)(= A/\equiv^*)$ of equivalence classes is a bounded distributive lattice (which we call also the Belluce lattice of A).

We stress that β^* defines a covariant functor from the category of MV-algebras to the category of bounded distributive lattices. $\mathrm{M}(A)$ and $\mathrm{P}(\beta^*(A))$ are homeomorphic.

So, in the sequel we will use notation $\mathrm{P}(A)$ instead of $\mathrm{M}(A)$.

Duality

Theorem 3. $\beta^*(F_{\mathbf{MV(C)}}(n))$ *is a Gödel algebra*.

Duality

Recall that **MV(C)$^{\mathbf{G}}$ = LSP**$\{C_n : n \in \omega\}$ is the class of algebras generated from $\{C_n : n \in \omega\}$ by the operators of direct product, subalgebras and direct limit.

So $F_{MV}(C)(n) \in$ ***MV(C)^{G}.*** This class is a full subcategory of the category of *MV(C)-algebras* ***MV(C).*** We can consider **MV(C)$^{\mathbf{G}}$** as the category the objects of which are the algebras from **MV(C)$^{\mathbf{G}}$**.

Duality

Taking into account that **GA** is locally finite and any algebra can be represented as direct limit of finitely generated subalgebras, we have that **GA = LSP**$\{\beta^*(C_n) : n \in \omega\}$**.**

Duality

Theorem 4. *Let we have two categories:* $\mathbf{MV(C)^G}$ *and* GS. *Then there exist contravariant functor*

$$\mathcal{P} : \mathbf{MV(C)^G} \to GS$$

and contravariant functor

$$\mathcal{H} : GS \to \mathbf{MV(C)^G}$$

such that $\mathcal{H}(\mathcal{P}(A))$ *is isomorphic to* A *for any object* $A \in \mathbf{MV(C)^G}$ *and* $\mathcal{P}(\mathcal{H}(X))$ *is homeomorphic to* X *for any object* $X \in GS$, *i. e. the functors* $\mathcal{P}$ *and* $\mathcal{H}$ *are dense.*

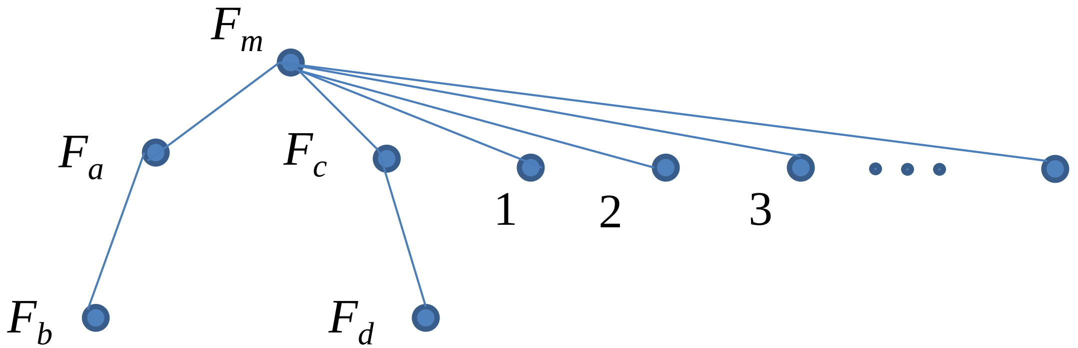
Moreover, the functor $\mathcal{P} : \mathbf{MV(C)^G} \to GS$ *is full, but not faithfull and the functor* $\mathcal{H} : GS \to \mathbf{MV(C)^G}$ *is faithfull, but not full.*

Duality

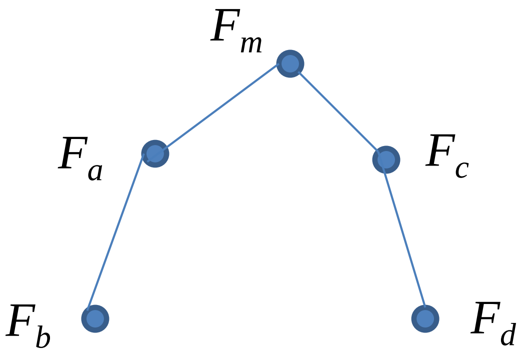
The category *GS* of Göedel spaces is dual equivalent to the category **GA** *of Gödel algebras, i. e. there exist two functors* G ***:*** **GA** $\rightarrow$ *GS and* HS *: GS* $\rightarrow$ **GA.** *So, we have composition of two contravariant functors*
$HS \circ P$ *:* **MV(C)^G^** $\rightarrow$**GA** *and* $H \circ G$ ***:*** **GA** $\rightarrow$ **MV(C)^G^.**

From the above we have the following

Theorem 5. *Covariant functors*
$HS \circ P$ *:* $\mathbf{MV(C)^G}$ $\rightarrow$**GA** *and* $H \circ G$ ***:*** **GA** $\rightarrow$ $\mathbf{MV(C)^G}$ *are dense. Moreover,* $HS \circ P$ *coincides with Belluce functor* β* *defined on the* $\mathbf{MV(C)^G}$**.**



H(B_2)



$$\mathrm{H}(\mathrm{R}^*(C_2^{\;2}))$$

Coproduct in $MV(C)^G$

Suppose **V** is a class of algebras and $A,B \in \mathbf{V}$. *The* ***V****-coproduct of* A and B is an algebra $A \sqcup B \in \mathbf{V}$ with algebra homomorphisms

$i_A : A \to A \sqcup B$, $i_B : B \to A \sqcup B$,
such that $i_A(A) \cup i_B(B) \subset A \sqcup B$ generates $A \sqcup B$, satisfying the following universal property: for every algebra $D \in \mathbf{V}$ with algebra homomorphisms $f : A \to D$ *and* $g : B \to D$, there exists an algebra homomorphism
$h : A \sqcup B \to D$ such that $h \circ i_A = f$ and $h \circ i_B = g$.

Coproduct in $\mathbf{MV(C)}^G$

If we change in the definition of coproduct the requirement that the algebra homomorphisms to be injective, then we have the definition of *free product.*

Theorem 7. *In the class* $\mathbf{MV(C)}^G$ *a coproduct coincides with free product.*

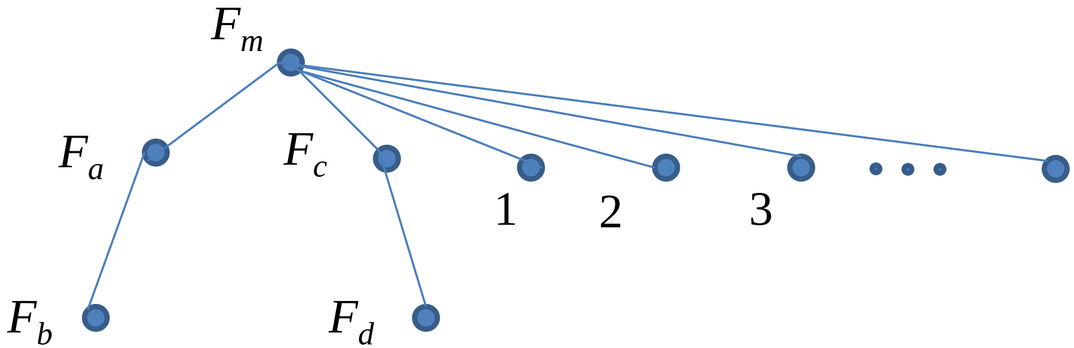
Coproduct in $MV(C)^G$

Notice that the coproduct coincides with the free product in the variety of abelian ℓ*-groups with strong unit* [D. Mundici, 1988].

Coproduct in $MV(C)^G$

Theorem 8. *The algebra B(2) is isomorphic to the coproduct* $C_1 \sqcup C_1$**.**

Theorem 9. *The algebra B(m) is isomorphic to the coproduct* $C_1 \sqcup \ldots \sqcup C_1$ *(m times).*



H$(B(2))$

Projective algebras in $MV(C)^G$

Theorem 10.

(i) *The MV (C)-algebra* C_n *is projective.*

(ii) *The MV (C)-algebra* $C_1 \sqcup \ldots \sqcup C_1$ *(m times) is projective.*

(iii) *The MV (C)-algebra*

$B_n = Rad(C_n^{n!}) \cup \neg Rad(C_n^{n!})$

is projective for any $n \in Z^+$.

Projective algebras in $MV(C)^G$

Theorem 11. *Any homomorphic image of the MV (C)-algebra* $B_n = Rad(C^{n!}{}_n) \cup \neg Rad(C^{n!}{}_n)$ *is projective for any* $n \in Z^+$.

In other words the algebra $A = C_{n(1)} \times \ldots \times C_{n(k)}$, *where* $n(1), \ldots, n(k) \leq n$ *are positive integers, is projective.*

Projective algebras in $\mathbf{MV(C)^G}$

Theorem 12. *Let $A \in$* ***$MV(C)^G$****. If $\mathcal{P}(A)$ is finite, then A is projective in* **$\mathbf{MV(C)^G}$**.

Projective algebras in $MV(C)^G$

MV -algebra A is finitely presented iff

A is isomorphic to $F_{\mathbf{MV}}(m)/[u)$ *for some principal filter generated by* $u \in F_{\mathbf{MV}}(m)$

[R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici (2000); A. Di Nola and R. Grigolia (2003)].

Theorem 13. *Any finitely presented algebra* $A \in \mathbf{MV(C)}^G$ *is projective.*

THANK YOU!