Deciding Admissibility

Jeroen P. Goudsmit
Utrecht University
Topological Methods in Logic 4
June 26^{th} 2014

A / $\triangle$ admissible

σA is derivable

$A \;/\; \Delta$ admissible

σC is derivable for some $C \in \Delta$

σA is derivable

$A \mathrel{|\!\sim} \Delta$ admissible

σC is derivable for some $C \in \Delta$

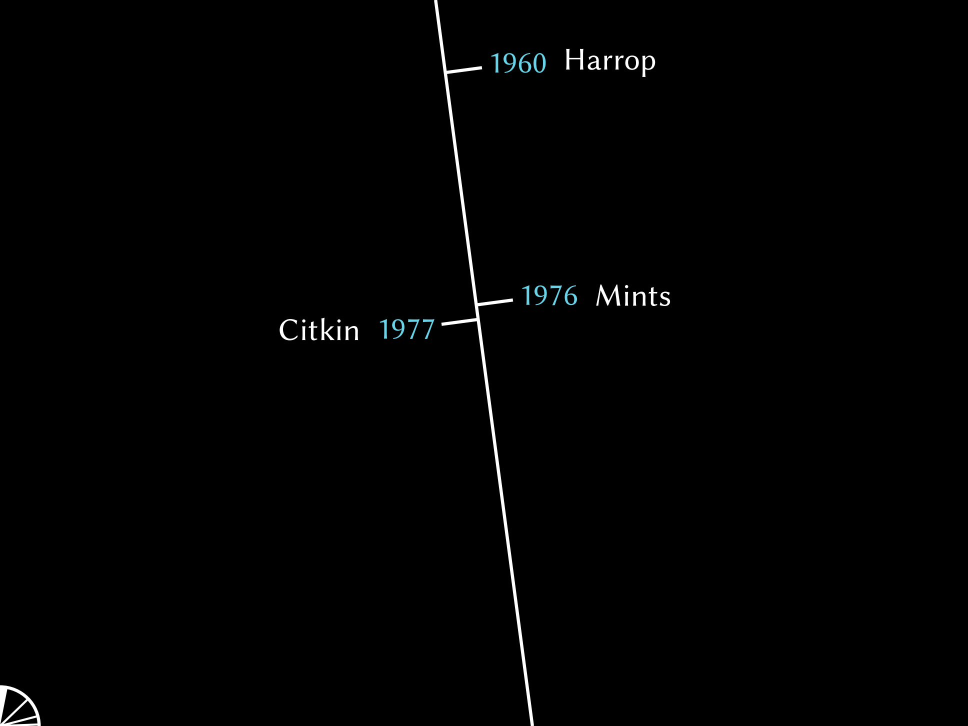
$$\frac{\neg C \rightarrow A \vee B}{(\neg C \rightarrow A) \vee (\neg C \rightarrow B)}$$

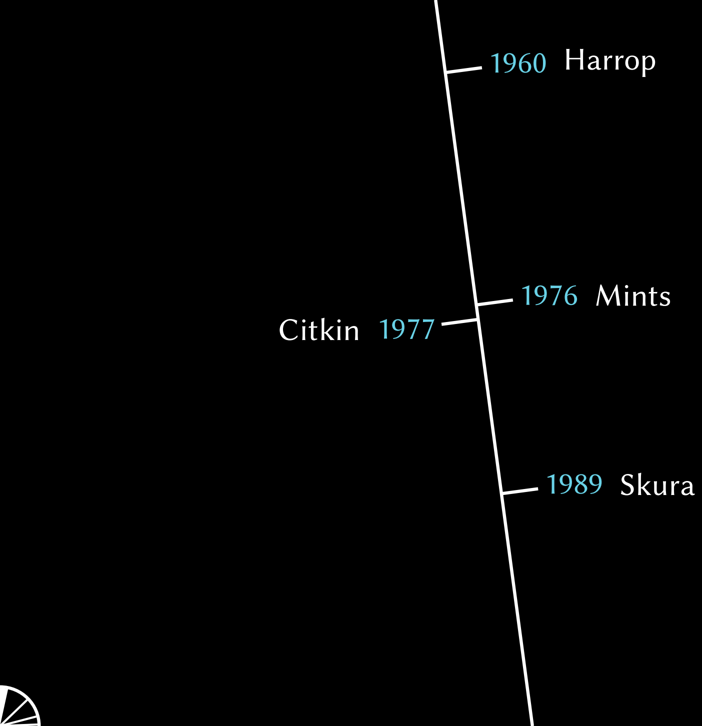
$$\frac{(A \to C) \to A \vee B}{\big((A \to C) \to A\big) \vee \big((A \to C) \to B\big)}$$

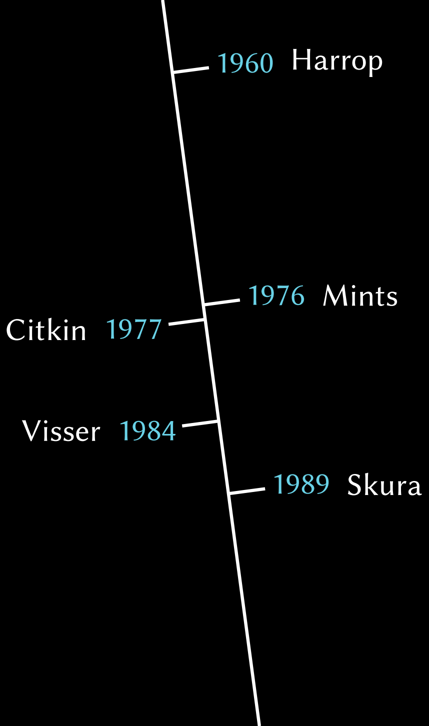
1960 Harrop

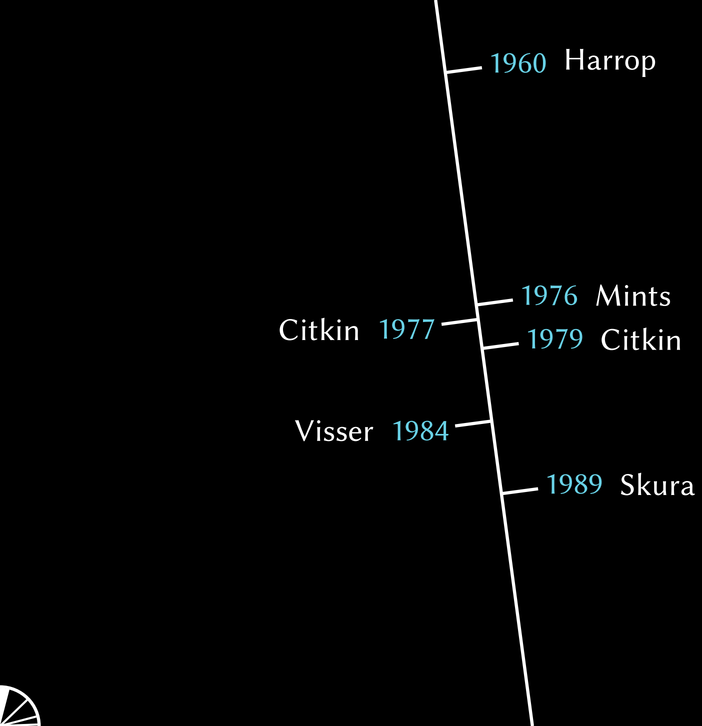
1960 Harrop

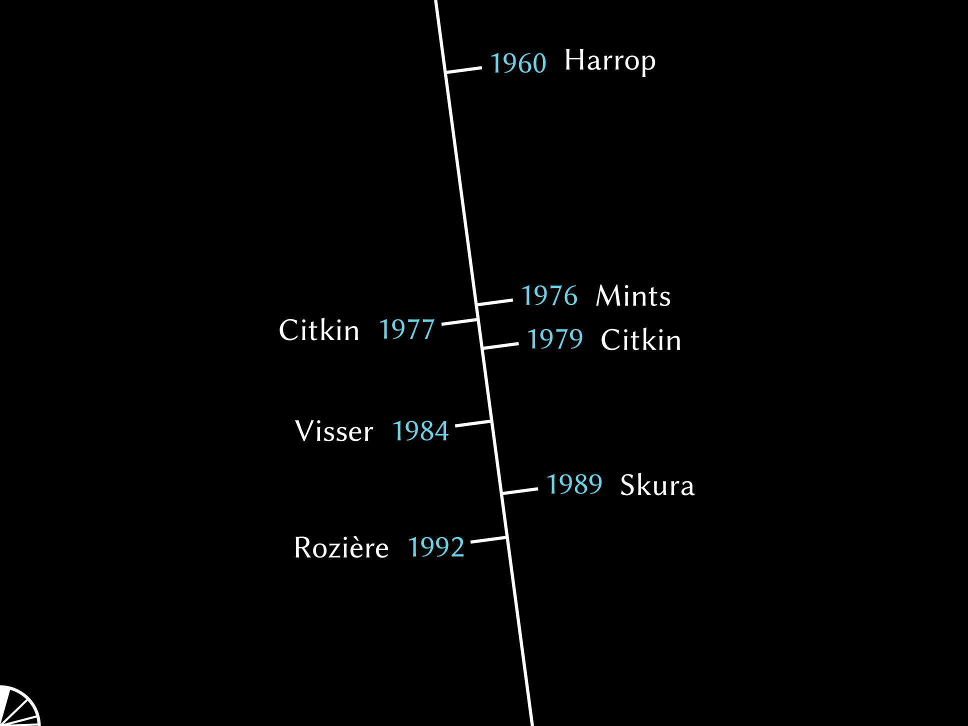
1976 Mints

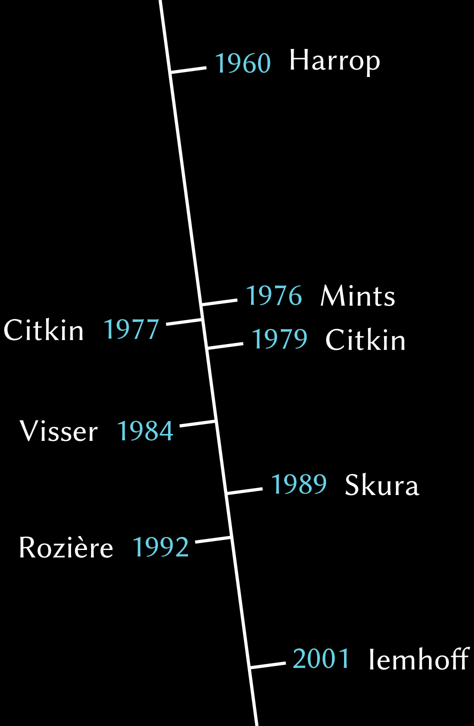

1960 Harrop
1976 Mints
Citkin 1977

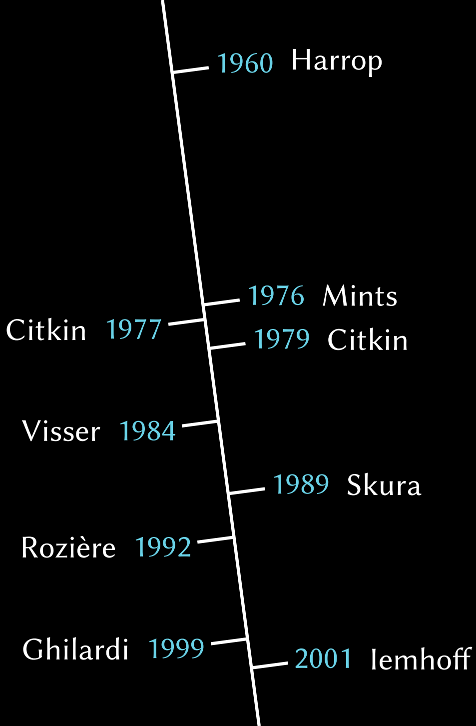

1960 Harrop
1976 Mints
Citkin 1977
1989 Skura

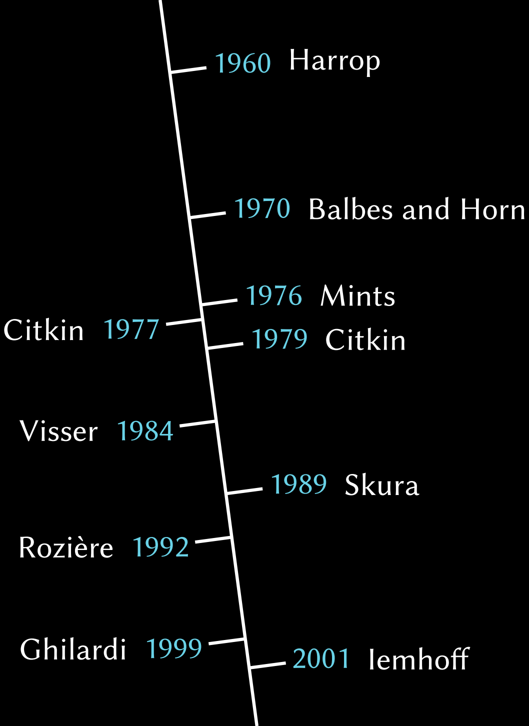

1960 Harrop
1976 Mints
Citkin 1977
Visser 1984
1989 Skura

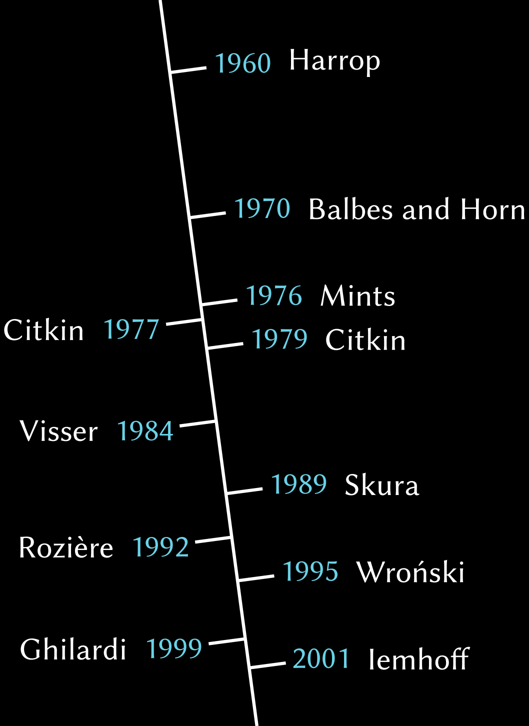

1960 Harrop
1976 Mints
Citkin 1977
1979 Citkin
Visser 1984
1989 Skura

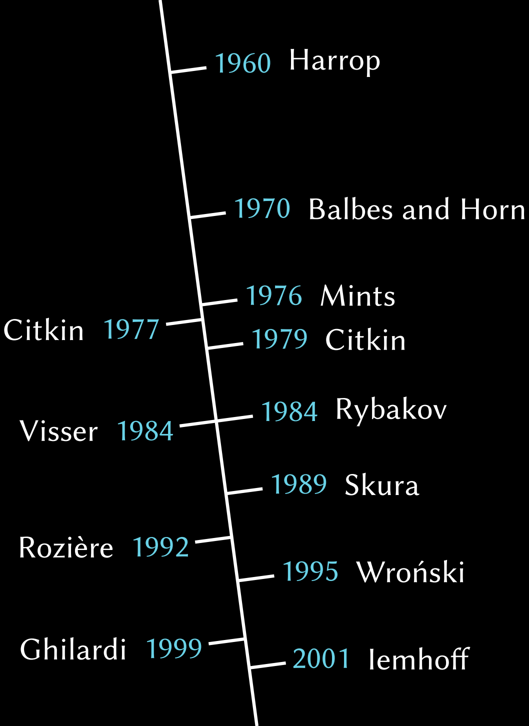

1960 Harrop
1976 Mints
Citkin 1977
1979 Citkin
Visser 1984
1989 Skura
Rozière 1992


1960 Harrop
1976 Mints
Citkin 1977
1979 Citkin
Visser 1984
1989 Skura
Rozière 1992
2001 Iemhoff


1960 Harrop
1976 Mints
Citkin 1977
1979 Citkin
Visser 1984
1989 Skura
Rozière 1992
Ghilardi 1999
2001 Iemhoff


1960 Harrop
1970 Balbes and Horn
1976 Mints
Citkin 1977
1979 Citkin
Visser 1984
1989 Skura
Rozière 1992
Ghilardi 1999
2001 Iemhoff

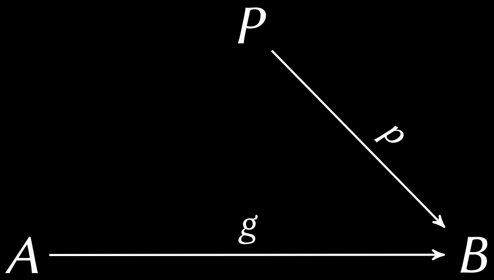

1960 Harrop
1970 Balbes and Horn
1976 Mints
Citkin 1977
1979 Citkin
Visser 1984
1989 Skura
Rozière 1992
1995 Wroński
Ghilardi 1999
2001 Iemhoff


1960 Harrop
1970 Balbes and Horn
1976 Mints
Citkin 1977
1979 Citkin
Visser 1984
1984 Rybakov
1989 Skura
Rozière 1992
1995 Wroński
Ghilardi 1999
2001 Iemhoff

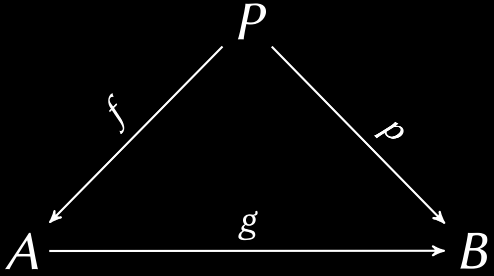
Projective

P

Projective

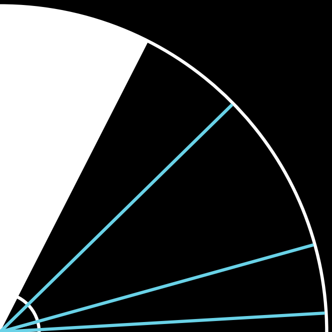


Projective

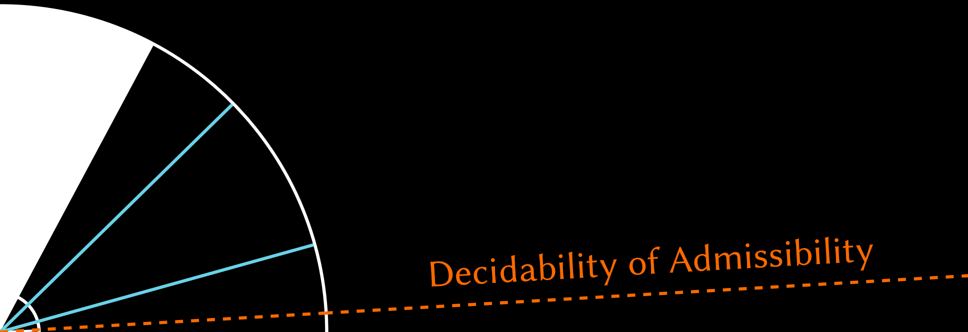


Overview

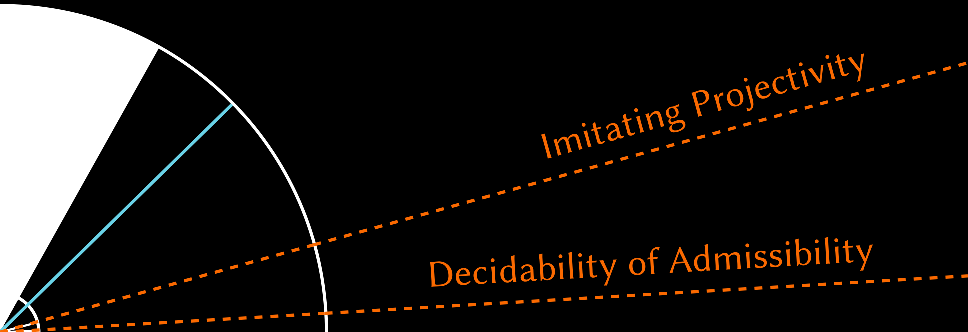
Overview



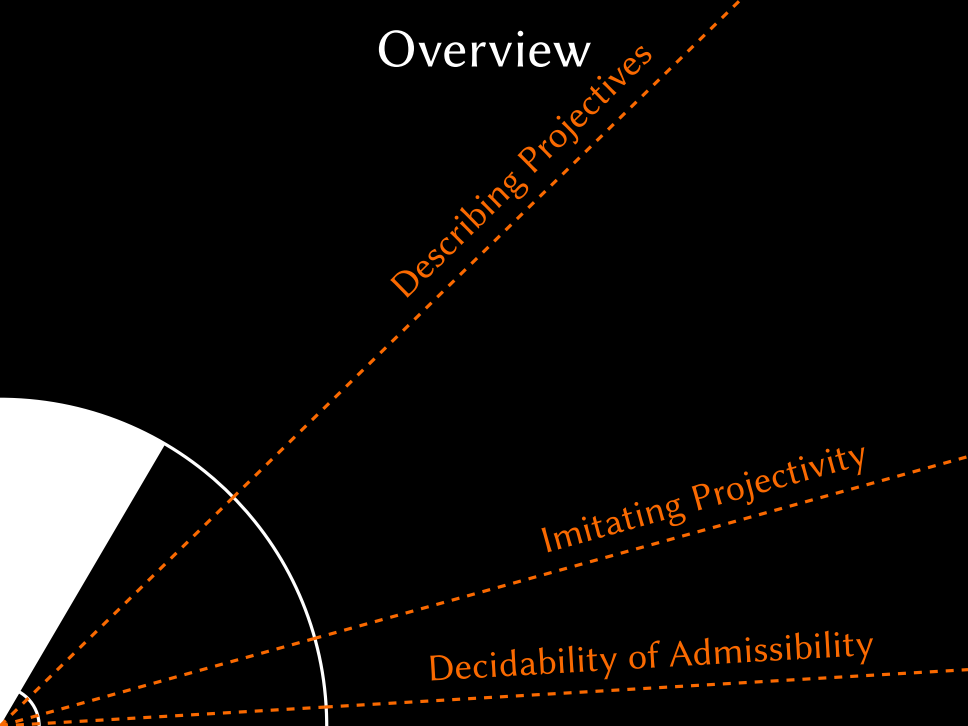
Overview



Overview



Overview



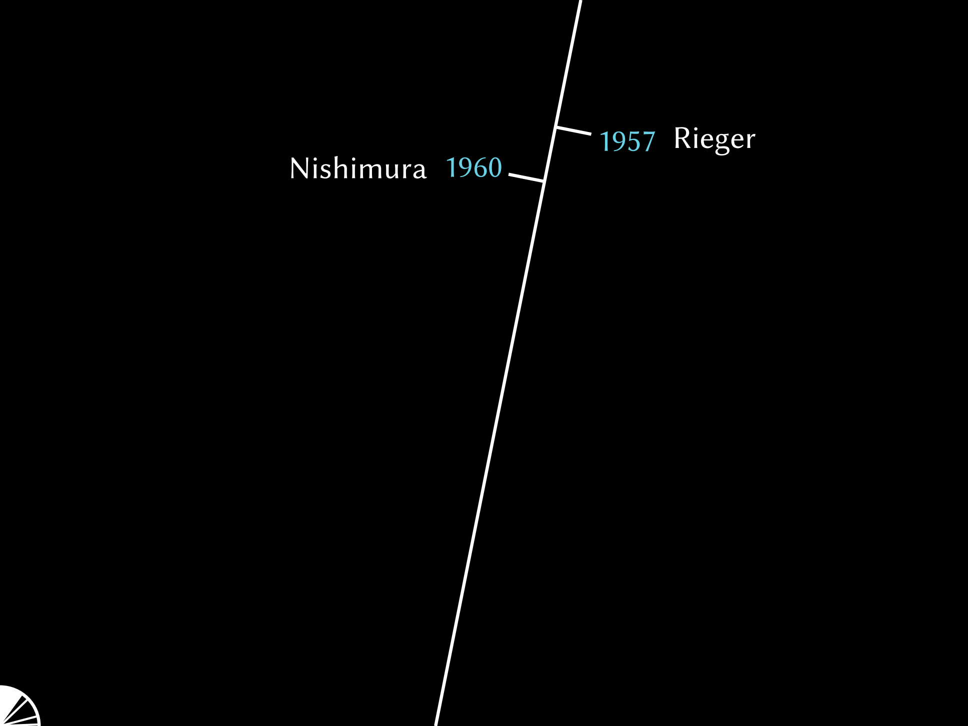
Describing Projectives
Imitating Projectivity
Decidability of Admissibility

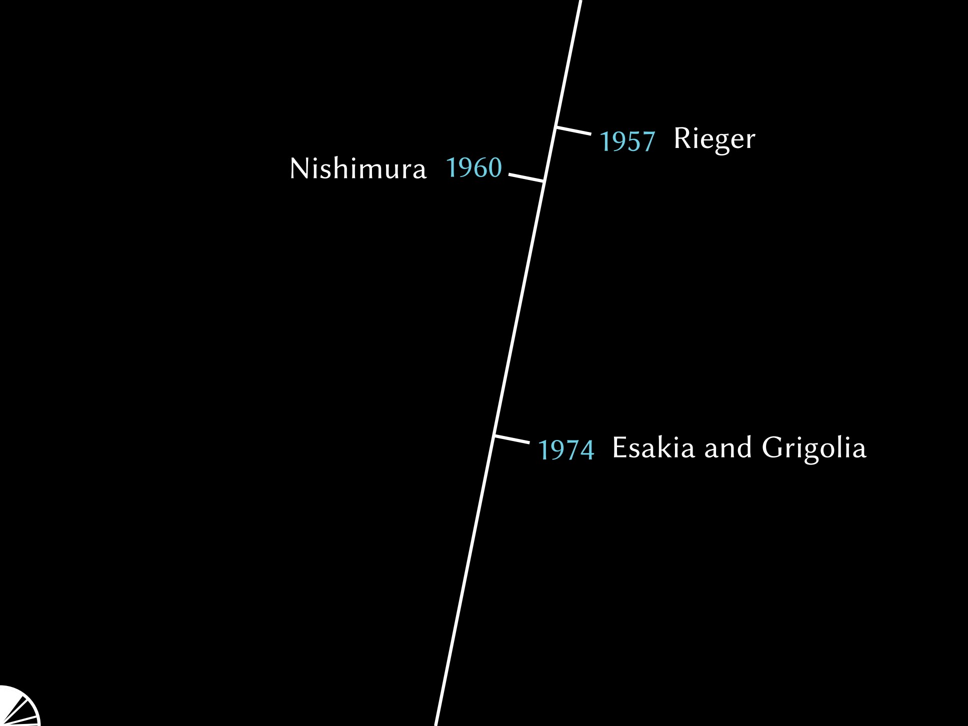
The universal model is the “smallest” model into which every image-finite model fits.

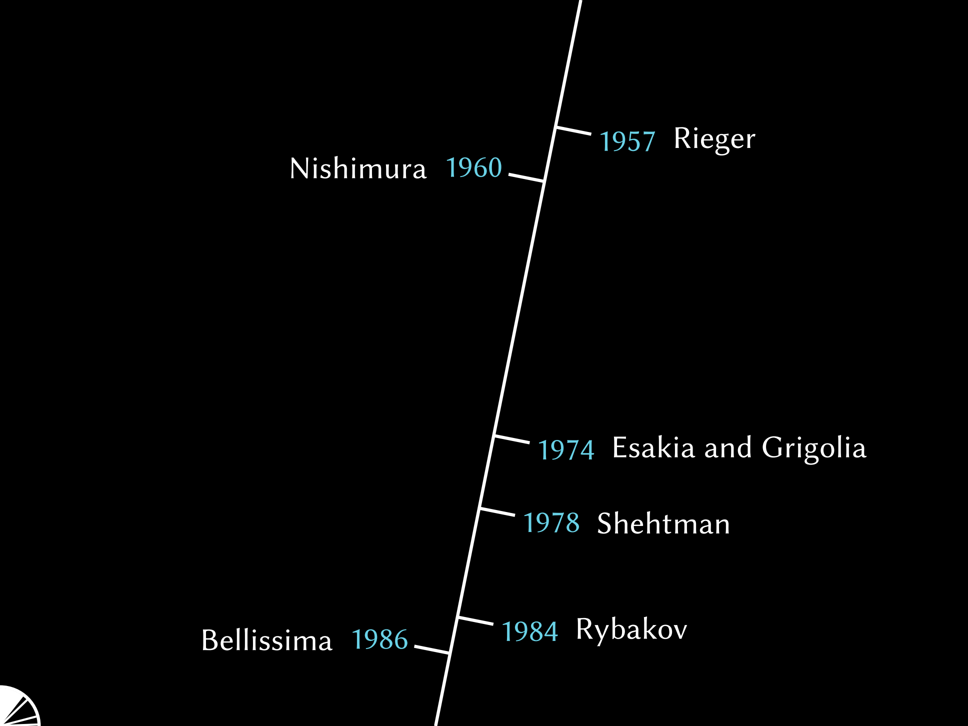
The universal model $\mathsf{U}(X)$ is the "smallest" model on X into which every image-finite model on X fits.

The universal model is complete.

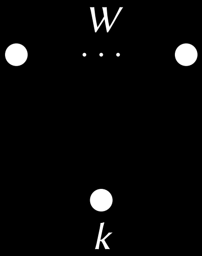
The universal model is complete:
$\mathsf{U}(X) \Vdash A$ iff $\vdash A$ for all $A \in \mathcal{L}(X)$.


1957 Rieger
Nishimura 1960

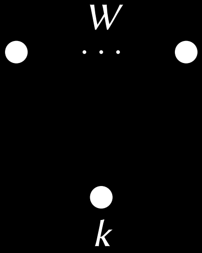

1957 Rieger
Nishimura 1960
1974 Esakia and Grigolia


1957 Rieger
Nishimura 1960
1974 Esakia and Grigolia
1978 Shehtman
1984 Rybakov
Bellissima 1986

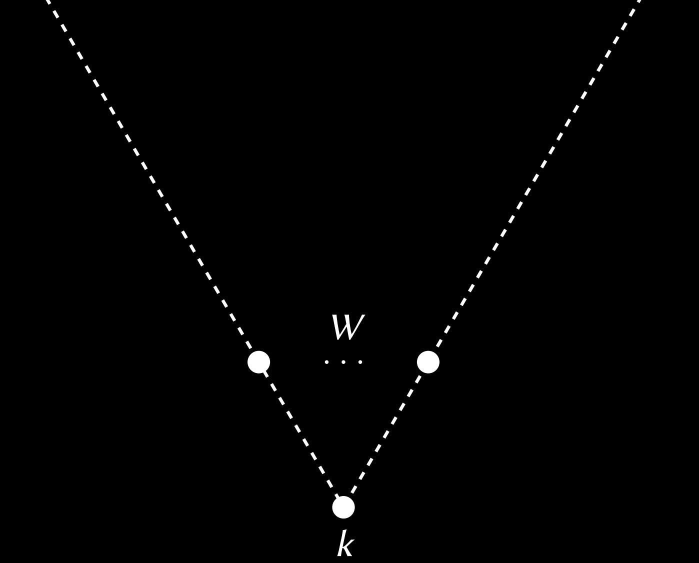

k


W
k

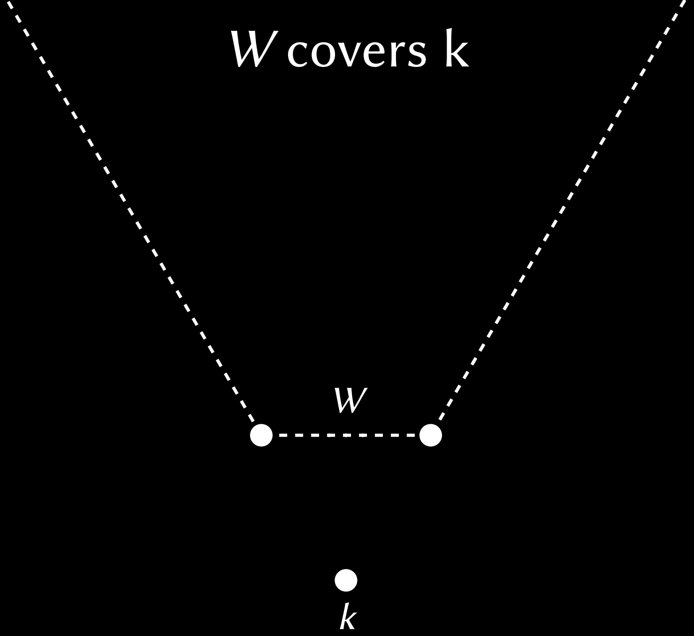
W covers k



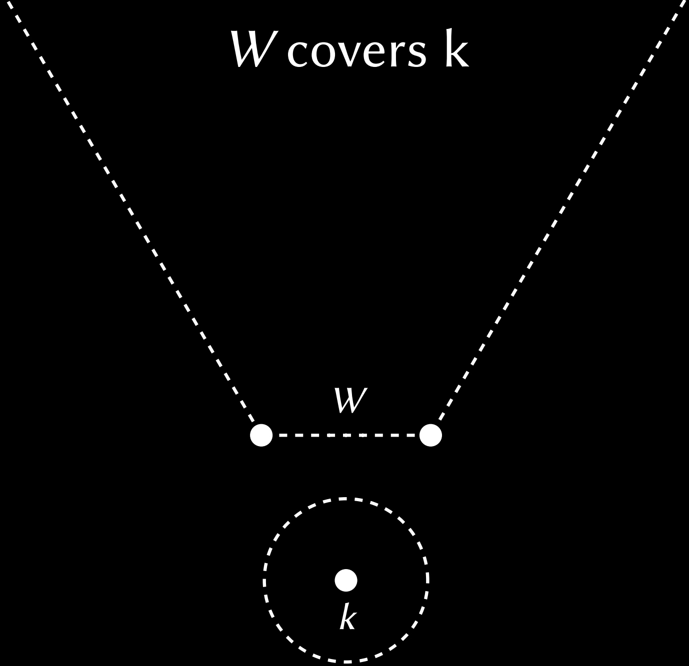
W covers k



W covers k



W covers k



If W covers k, then $k \Vdash A \rightarrow B$ iff
$W \Vdash A \rightarrow B$ and
$k \Vdash A$ implies $k \Vdash B$.

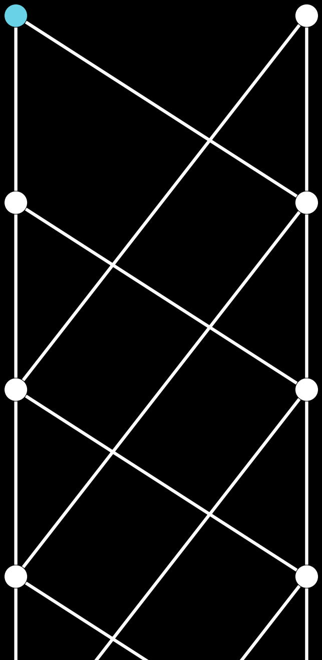
Ghilardi (1999)

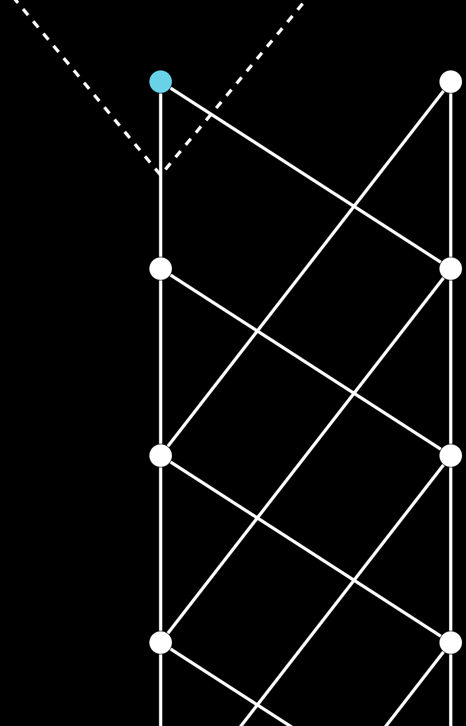
A formula is projective iff its corresponding upset has all covers.

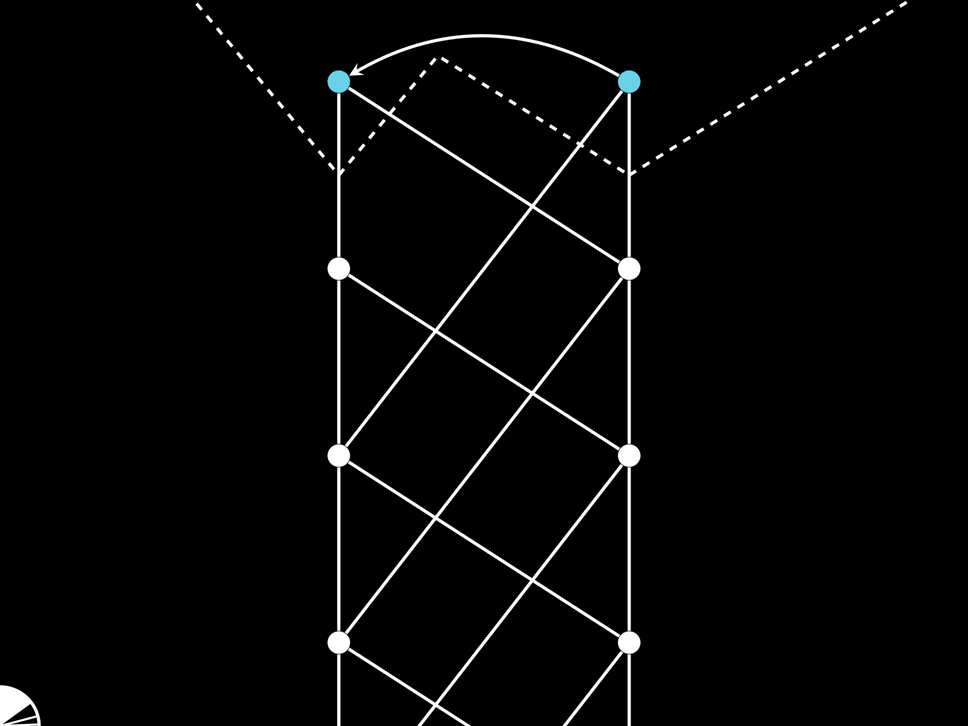
Bezhanishvili and de Jongh (2012)

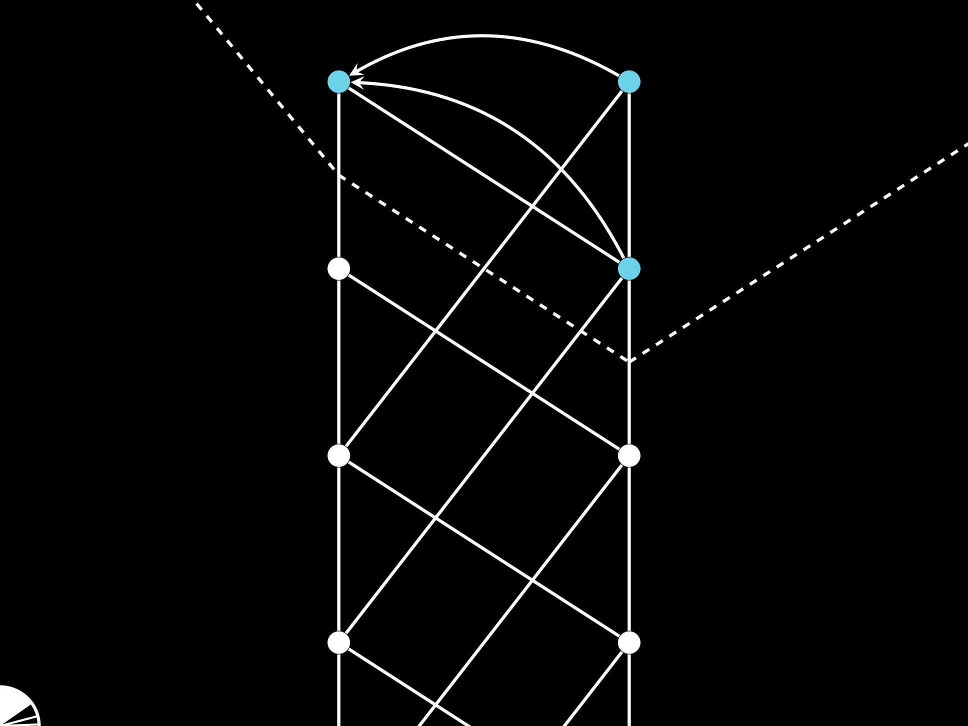
An upset has all covers iff it is the image of the universal model under a definable map.

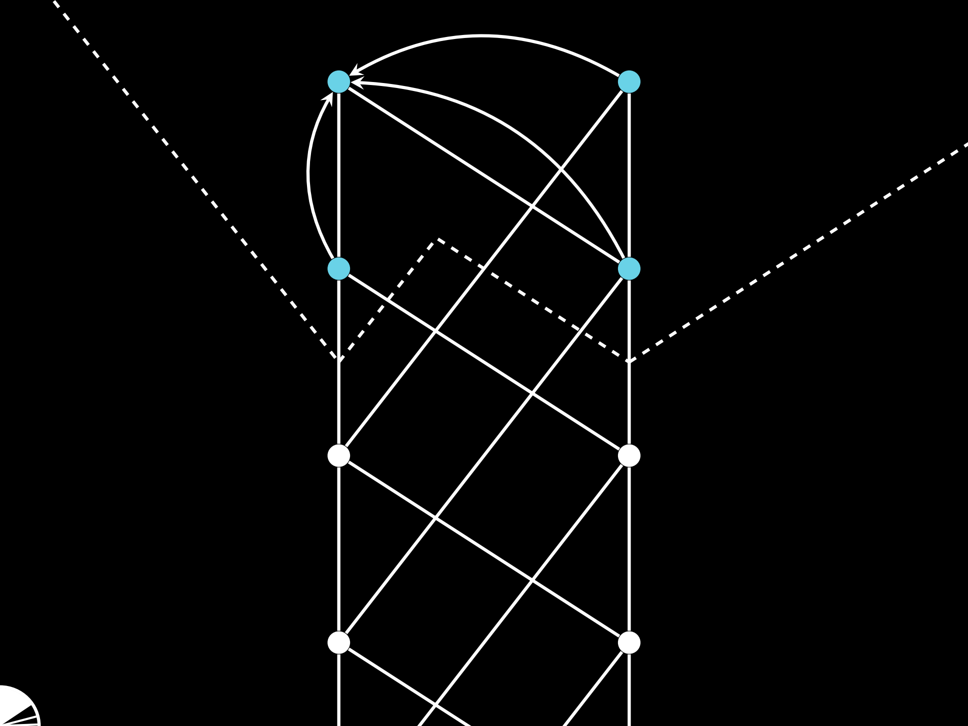
p

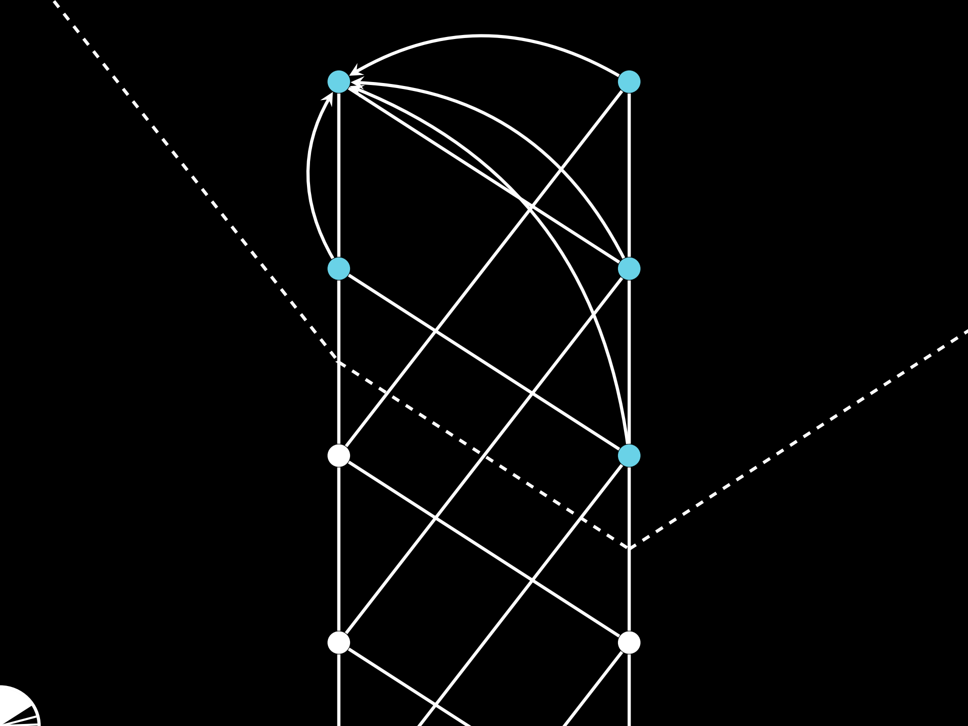


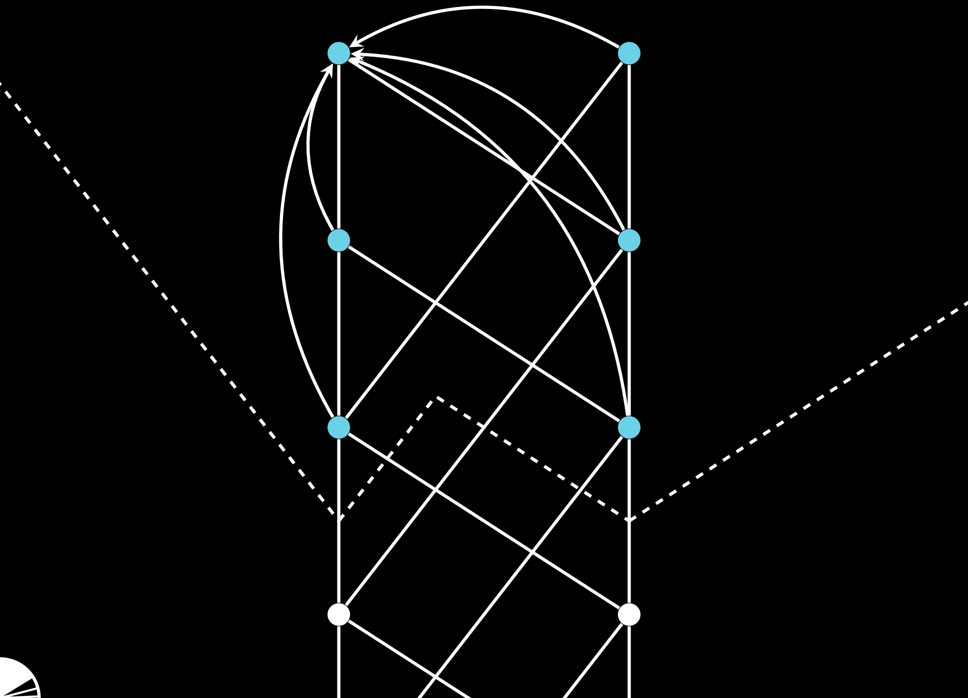


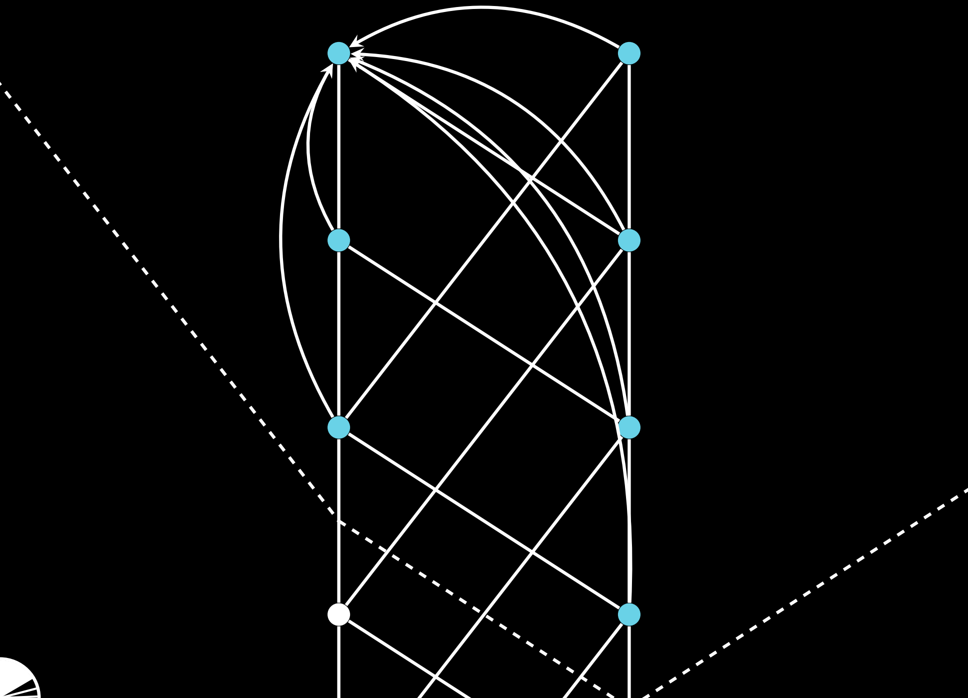


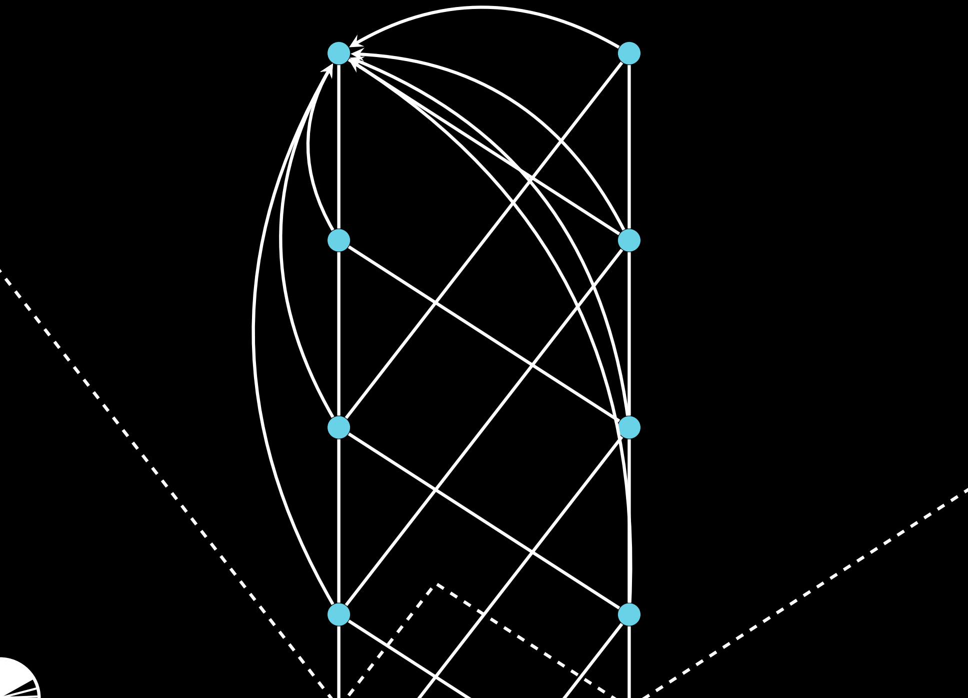


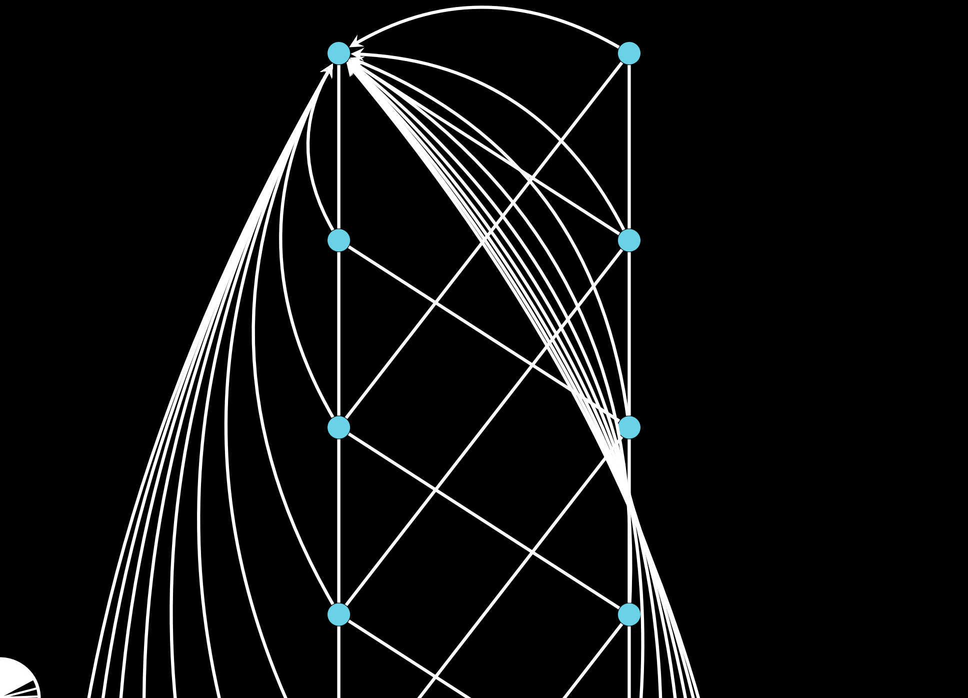




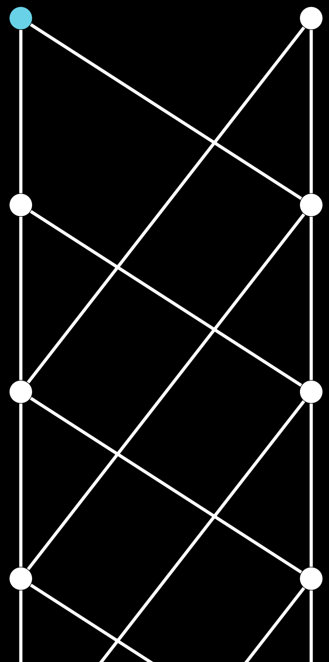


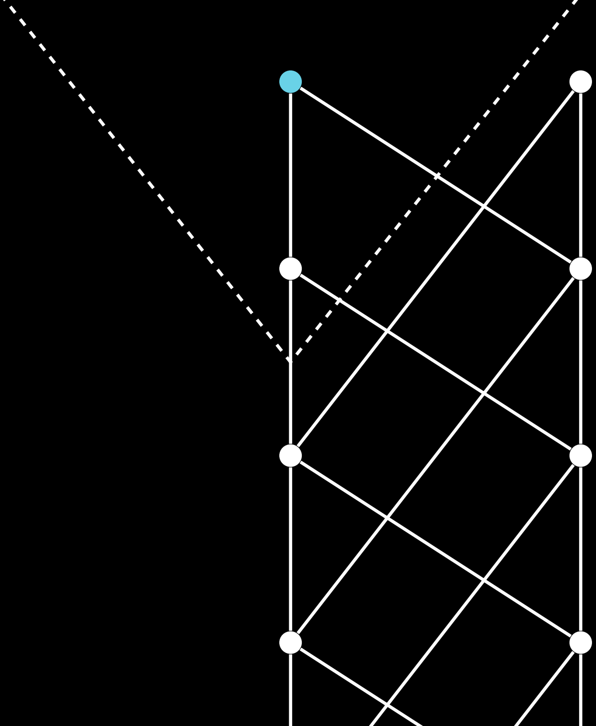


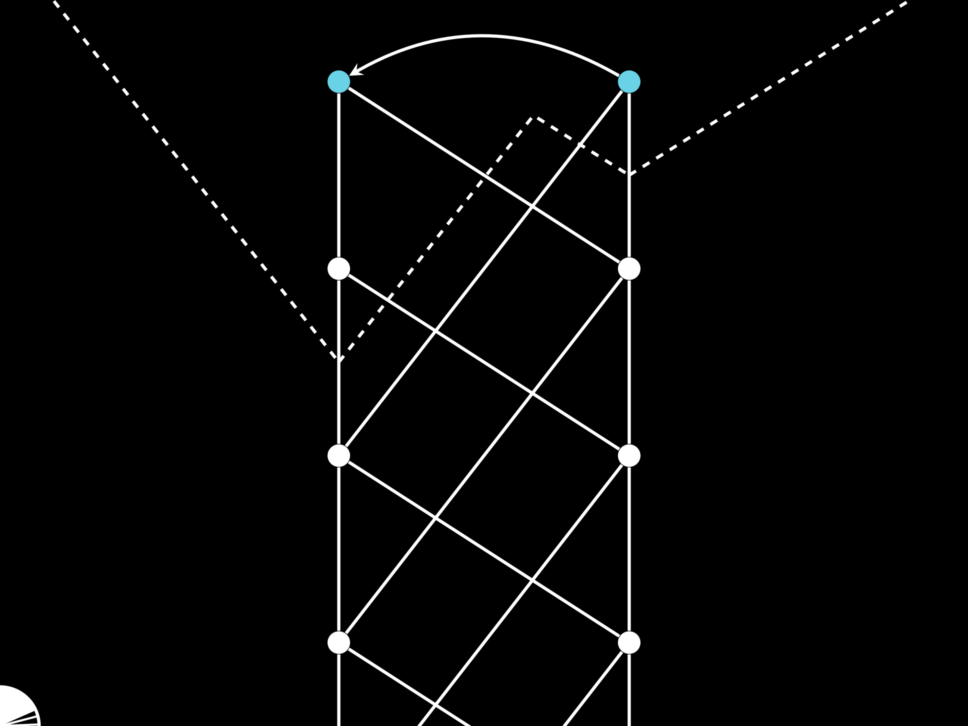


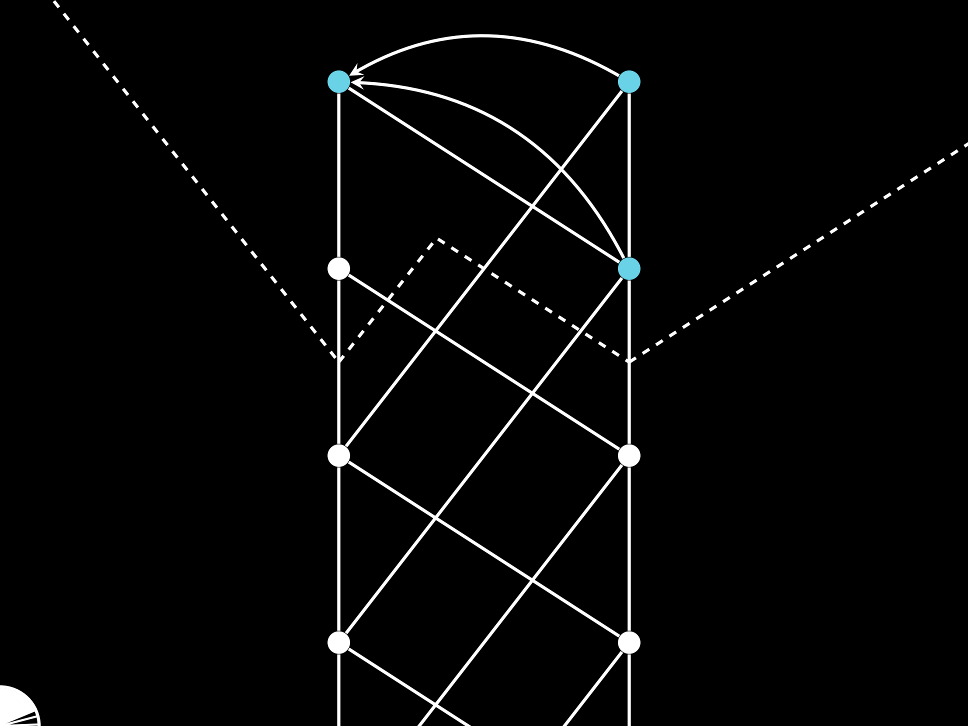


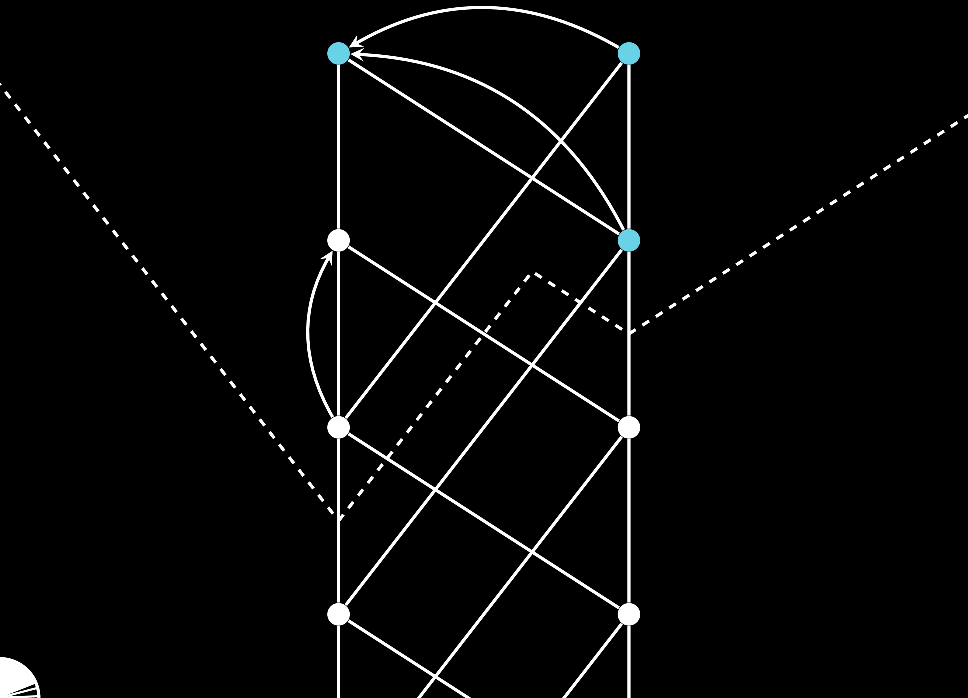
$\neg\neg p$

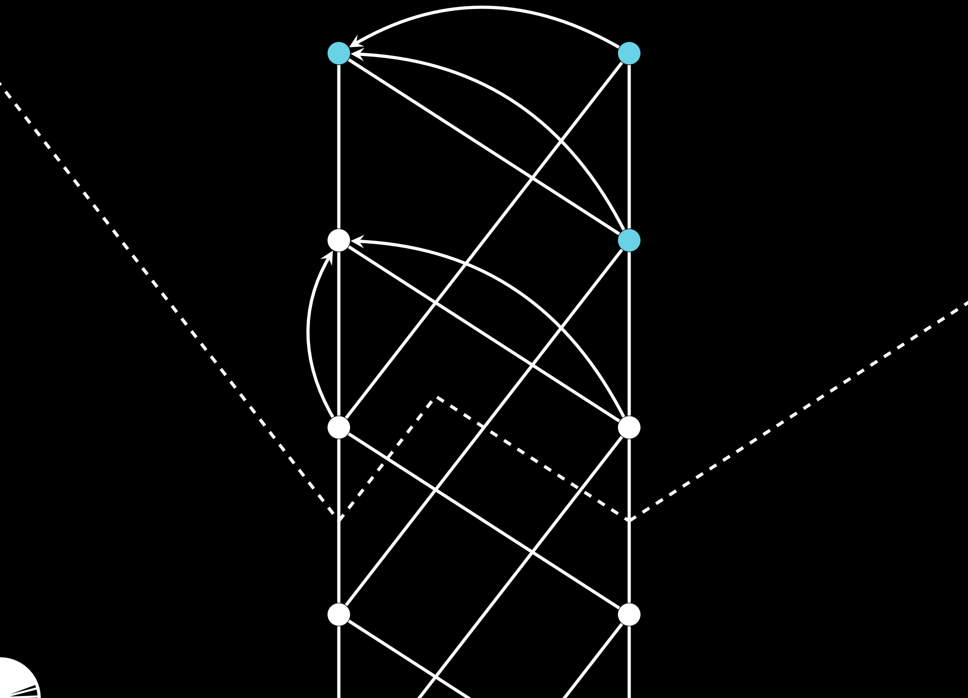


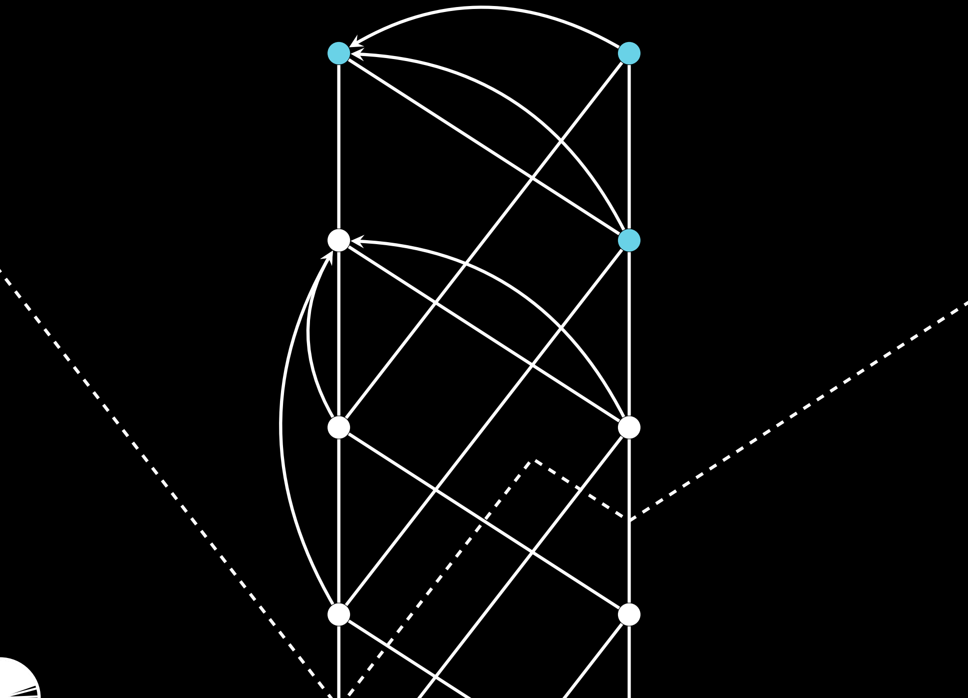


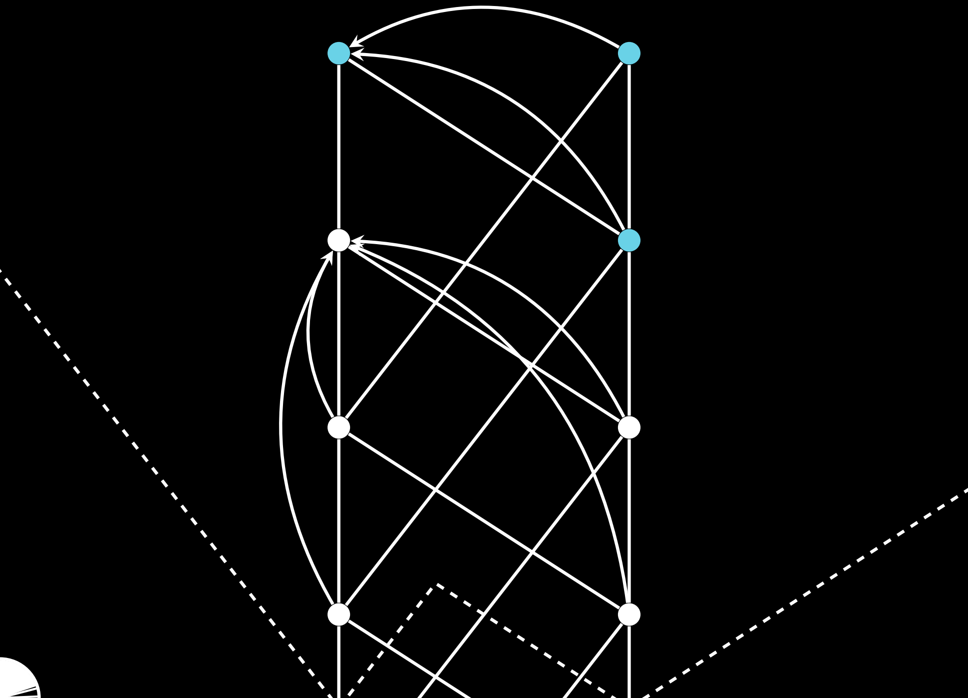


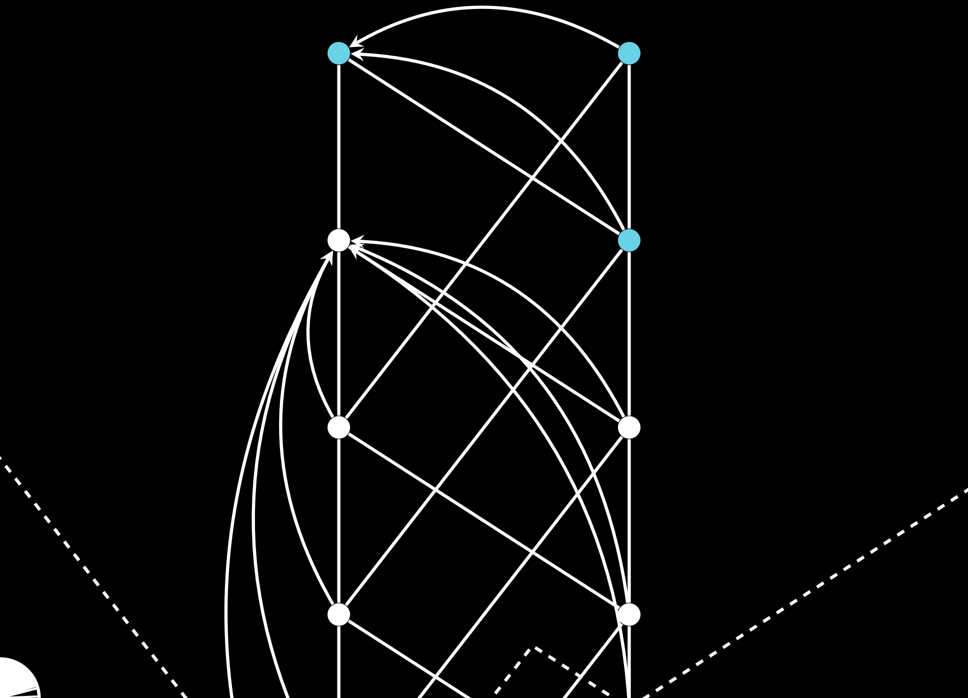


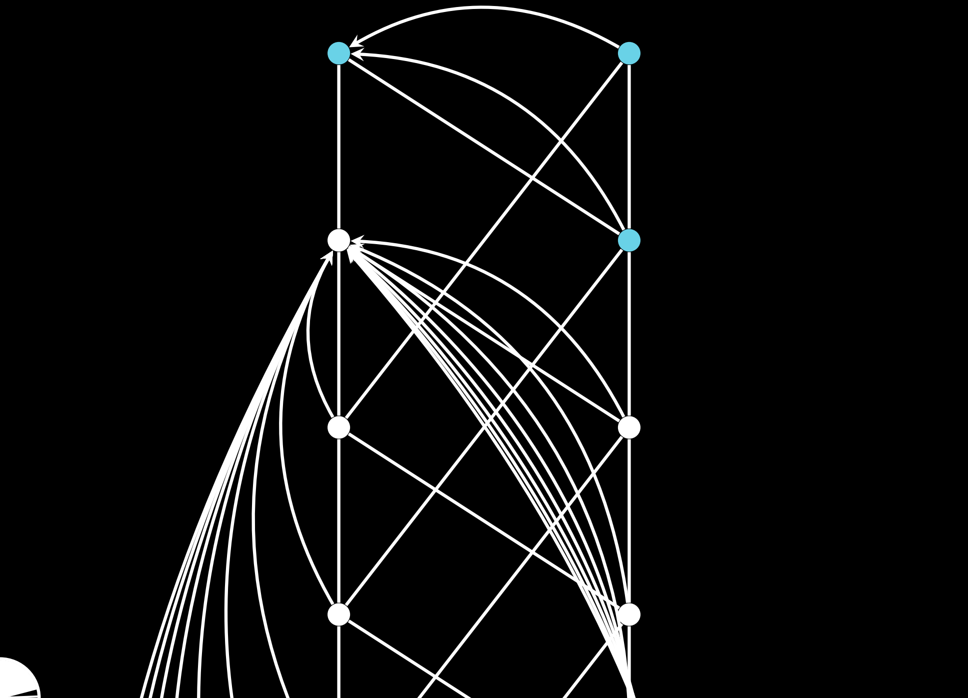












If W covers k, then $k \Vdash A \rightarrow B$ iff
$W \Vdash A \rightarrow B$ and
$k \Vdash A$ implies $k \Vdash B$.

Say that W *adequately covers* k w.r.t. Σ when:

- $k \leq w$ for all $w \in W$;
- $k \Vdash A \to B$ iff $W \Vdash A \to B$ and $k \Vdash A$ implies $k \Vdash B$ for all $A \to B \in \Sigma$.

An upset $V \subseteq \mathsf{U}(X)$ is *definable* when there is a formula $\mathrm{def}\, V \in \mathcal{L}(X)$ such that:

$$k \Vdash \mathrm{def}\, V \text{ iff } k \in V \text{ for all } k \in \mathsf{U}(X).$$

Theorem
Let $U \subseteq \mathsf{U}(X)$ be adequately covered w.r.t. Σ. There exists a definable, monotone map $f : \mathsf{U}(X) \to U$ such that $f \restriction U = \mathrm{id}_U$ and for all $A \in \Sigma$:

$$k \Vdash A \text{ iff } \quad f(k) \Vdash A \quad \text{for all } k \in \mathsf{U}(X).$$

Theorem

Let $U \subseteq \mathsf{U}(X)$ be adequately covered w.r.t. Σ. There exists a definable, monotone map $f : \mathsf{U}(X) \to U$ such that $f \restriction U = \mathrm{id}_U$ and for all $A \in \Sigma$:

$$(u \circ f), k \Vdash A \text{ iff } u, f(k) \Vdash A \quad \text{for all } k \in \mathsf{U}(X).$$

Corollary

In the above, there is a substitution σ such that:

$$\vdash \sigma(A) \text{ iff } U \Vdash A \text{ for all } A \in \Sigma.$$

Corollary

In the above, there is a substitution σ *such that:*

$$\vdash \sigma(A) \textit{ iff } U \Vdash A \textit{ for all } A \in \Sigma.$$

Proof.

Take $\sigma(x) := \mathrm{def}\, f^{-1}\{k \in U \mid u, k \Vdash x\}$.

Corollary

In the above, there is a substitution σ such that:

$$\vdash \sigma(A) \textit{ iff } U \Vdash A \textit{ for all } A \in \Sigma.$$

Proof.

Take $\sigma(x) := \text{def} f^{-1} \{k \in U \mid u, k \Vdash x\}$. See:

$$\vdash \sigma(A) \text{ iff } u, \mathsf{U}(X) \Vdash \sigma(A) \text{ iff } (u \circ f) \Vdash A.$$

Corollary

In the above, there is a substitution σ such that:

$$\vdash \sigma(A) \textit{ iff } U \Vdash A \textit{ for all } A \in \Sigma.$$

Proof.

Take $\sigma(x) := \text{def} f^{-1}\{k \in U \mid u, k \Vdash x\}$. See:

$$\vdash \sigma(A) \text{ iff } u, \mathsf{U}(X) \Vdash \sigma(A) \text{ iff } (u \circ f) \Vdash A.$$

The latter is equivalent to $U \Vdash A$ when $A \in \Sigma$.

Corollary

In the above, there is a substitution σ such that:

$$\vdash \sigma(A) \text{ iff } U \Vdash A \text{ for all } A \in \Sigma.$$

Proof.

Take $\sigma(x) := \mathrm{def}\, f^{-1}\{k \in U \mid u, k \Vdash x\}$. See:

$$\vdash \sigma(A) \text{ iff } u, \mathsf{U}(X) \Vdash \sigma(A) \text{ iff } (u \circ f) \Vdash A.$$

The latter is equivalent to $U \Vdash A$ when $A \in \Sigma$. ■

Theorem

Let $\Sigma \subseteq \mathcal{L}(X)$ be an finite adequate set and let $v : \mathsf{U}(X) \to \mathsf{P}(Y)$ be a valuation. There exists a $U \subseteq \mathsf{U}(X + \Sigma)$ such that:

1. *$U \Vdash A$ iff $v \Vdash A$ for all $A \in \Sigma$;*
2. *U is adequately covered w.r.t. Σ;*
3. *for all $k, l \in U$ have $k \leq l$ iff $k \Vdash A$ implies $l \Vdash A$ for all $A \in \Sigma$.*

Theorem

Admissibility is decidable.

Theorem

Admissibility is decidable.

Proof.

Remark that $A \mathrel{|\!\sim} B$ iff $u \Vdash A$ implies $u \Vdash B$ for all $U \subseteq 2^{\Sigma}$ with $u : U \to \mathrm{P}(X)$ defined by $u(k) = k \cap x$ that are adequately covered w.r.t. $\Sigma = \mathrm{Sub}\{A, B\}$.

Theorem

Admissibility is decidable.

Proof.

Remark that $A \mathrel{|\!\sim} B$ iff $u \Vdash A$ implies $u \Vdash B$ for all $U \subseteq 2^{\Sigma}$ with $u : U \to \mathrm{P}(X)$ defined by $u(k) = k \cap x$ that are adequately covered w.r.t. $\Sigma = \mathrm{Sub}\{A, B\}$. ■

References I

- Balbes, R. and A. Horn (1970). "Injective and Projective Heyting Algebras". In: *Transactions of the American Mathematical Society* 148.2, pp. 549–559. ISSN: 00029947. DOI: 10.2307/1995388. JSTOR: 1995388.
- Bellissima, F. (1986). "Finitely Generated Free Heyting Algebras". In: *The Journal of Symbolic Logic* 51.1, pp. 152–165. ISSN: 00224812. DOI: 10.2307/2273952. JSTOR: 2273952.
- Bezhanishvili, N. and D. H. J. de Jongh (2012). "Extendible Formulas in Two Variables in Intuitionistic Logic". In: *Studia Logica* 100 (1), pp. 61–89. ISSN: 0039-3215. DOI: 10.1007/s11225-012-9389-8.
- Citkin, A. (1977). "On Admissible Rules of Intuitionistic Propositional Logic". In: *Mathematics of the USSR-Sbornik* 31.2, pp. 279–288. DOI: 10.1070/SM1977v031n02ABEH002303.

References II

Citkin, A. (1979). "О Проверке Допустимости нéкоторых Правил Интуиционистской Логике". Russian. In: *V-th All-Union Conference in Mathematical Logic*. English translation of title: On verification of admissibility of some rules of intuitionistic logic. Novosibirsk, p. 162.

Esakia, L. and R. Grigolia (1974). "Christmas trees. On free cyclic algebras in some varieties of closure algebras". In: *Bulletin of the Section of Logic* 4.3. ISSN: 0138-0680.

Ghilardi, S. (1999). "Unification in Intuitionistic Logic". In: *The Journal of Symbolic Logic* 64.2, pp. 859–880. ISSN: 00224812. JSTOR: 2586506.

Harrop, R. (1960). "Concerning Formulas of the Types $A \to A \vee B$, $A \to (\exists x)B(x)$ in Intuitionistic Formal Systems". In: *The Journal of Symbolic Logic* 25.1, pp. 27–32. ISSN: 00224812. JSTOR: 2964334.

References III

- Iemhoff, R. (2001). “On the Admissible Rules of Intuitionistic Propositional Logic”. In: *The Journal of Symbolic Logic* 66.1, pp. 281–294. ISSN: 00224812. JSTOR: 2694922.
- Mints, G. E. (1976). “Derivability of admissible rules”. In: *Journal of Mathematical Sciences* 6 (4), pp. 417–421. ISSN: 1072-3374. DOI: 10.1007/BF01084082.
- Nishimura, I. (1960). “On Formulas of One Variable in Intuitionistic Propositional Calculus”. In: *The Journal of Symbolic Logic* 25.4, pp. 327–331. ISSN: 00224812. DOI: 10.2307/2963526.
- Rieger, L. (1957). “Заметка о т. наз. свободных алгебрах с замыканиями”. Russian. In: *Czechoslovak Mathematical Journal* 7.1. A remark on the s.c. free closure algebras, pp. 16–20. ISSN: 0011-4642; 1572-9141/e. URL: http://hdl.handle.net/10338.dmlcz/100226.
- Rozière, P. (1992). “Règles admissibles en calcul propositionnel intuitionniste”. PhD thesis. Université de Paris VII.

References IV

Rybakov, V. V. (1984). "A criterion for admissibility of rules in the model system S4 and the intuitionistic logic". In: *Algebra and Logic* 23 (5), pp. 369–384. ISSN: 0002-5232. DOI: 10.1007/BF01982031.

Shehtman, V. B. (1978). "Rieger-Nishimura lattices". English. In: *Soviet Mathematics Doklady* 19.4. Translation from Doklady Akademii Nauk SSSR 241, 1288-1291 (1978), pp. 1014–1018. ISSN: 0197-6788.

Skura, T. F. (1989). "A complete syntactical characterization of the intuitionistic logic". In: *Reports on Mathematical Logic* 23, pp. 75–80.

Visser, A. (1984). "Evaluation, provably deductive equivalence in Heyting's arithmetic of substitution instances of propositional formulas". In: *Logic Group Preprint Series* 4. URL: http://phil.uu.nl/preprints/lgps/number/4.

Wroński, A. (1995). "Transparant Unification Problem". In: *Reports on Mathematical Logic* 29, pp. 105–107.