# Concerning maximal $\ell$-ideals of rings of continuous integer-valued functions

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**Workshop on Topological Methods in Logic IV: ToLo4**

**Tbilisi, Georgia**

**(23 June 2014)**

Throughout, $L$ stands for a frame.

1. An element $a \in L$ is complemented if $a \vee a^* = 1$. The set
$$BL = \{x \in L \mid x \text{ is complemented}\}$$
is called the Boolean part of $L$.

2. $L$ is zero-dimensional if it is $\bigvee$-generated by $BL$. That is, if every element in $L$ is a join of complemented elements.

3. The frame $\zeta L = \mathfrak{J}(BL)$ is the coreflection of $L$ into compact zero-dimensional frame. The join map $j_L \colon \zeta L \to L$ is the coreflection map.

4. The right adjoint of $j_L$ is the map
$$r_L(a) = \{x \in BL \mid x \leq a\}.$$

Now $L$ stands for a zero-dimensional frame. All that follows can be found in the article

B. Banaschewski
*On the function ring functor in pointfree topology*
Appl. Categor. Struct. **13** (2005), 305–328.

The elements of the ring $\mathfrak{Z}L$ are maps $\alpha : \mathbb{Z} \to L$ such that

$$\alpha(k) \wedge \alpha(l) = 0 \text{ if } k \neq l \quad \text{and} \quad \bigvee\{\alpha(m) \mid m \in \mathbb{Z}\} = 1_L.$$

The ring and lattice operations are derived from those of $\mathbb{Z}$ as follows:

1. for any $\diamond = +, \cdot, \vee, \wedge$: $(\alpha \diamond \beta)(m) = \bigvee\{\alpha(k) \wedge \beta(l) \mid k \diamond l = m\}$.
2. $(-\alpha)(m) = \alpha(-m)$.
3. for any $k \in \mathbb{Z}$, $\mathbf{k}(m) = 1_L$ if $m = k$ and $\mathbf{k}(m) = 0_L$ if $m \neq k$.

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For any $\alpha \in \mathfrak{Z}L$ and $k \in \mathbb{Z}$, define

$$\alpha(k, -) = \bigvee\{\alpha(m) \mid m \geq k + 1\}.$$

Then, for any $\alpha, \beta \in \mathfrak{Z}L$,

$$\alpha \leq \beta \iff \alpha(k, -) \leq \beta(k, -) \text{ for every } k \in \mathbb{Z}.$$

Note:

$$\mathfrak{Z}2 \cong \mathbb{Z}.$$

This appears to have first been observed by Banaschewski.

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Note:

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Associated with the ring $\mathfrak{Z}L$ is the cozero map

$$\operatorname{coz}\colon \mathfrak{Z}L \to L \quad \text{given by} \quad \operatorname{coz}\alpha = \bigvee\{\alpha(m) \mid 0 \neq m \in \mathbb{Z}\}.$$

Some of the properties of this cozero map that we shall use include:

1. $\operatorname{coz}\mathbf{1} = 1.$
2. $\operatorname{coz}\alpha = 0 \iff \alpha = \mathbf{0}.$
3. $\operatorname{coz}(\alpha\beta) = \operatorname{coz}\alpha \wedge \operatorname{coz}\beta$ for all $\alpha, \beta \in \mathfrak{Z}L$.
4. $a \in BL \iff a = \operatorname{coz}\alpha,$ for some $\alpha \in \mathfrak{Z}L$.

By a point of a frame $L$ we mean a prime element, that is, an element $p < 1$ such that, for any $a, b \in L$,

$$a \wedge b \leq p \implies a \leq p \quad \text{or} \quad b \leq p.$$

We denote by $\mathrm{Pt}(L)$ the set of all points of $L$.

Let $L$ be a completely regular frame and $\beta L$ be is its Stone-Čech compactification. Write $\rho_L : L \to \beta L$ for the right adjoint of the coreflection map $\beta L \to L$.

An ideal $I$ of an $\ell$-ring $A$ is said to be an $\ell$-ideal (Gillman and Jerison say it is absolutely convex) if, for all $a, b \in A$,

$$|a| \leq |b| \ \ \& \ \ b \in I \implies a \in I.$$

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$\mathcal{R}L$ is a reduced $f$-ring with bounded inversion, so its maximal $\ell$-ideals are precisely the maximal ideals, and they are the ideals

$$\boldsymbol{M}^I = \{\alpha \in \mathcal{R}L \mid \varrho_L(\operatorname{coz}\alpha) \leq I\},$$

for $I \in \mathrm{Pt}(\beta L)$.

Reasoning by analogy, we would expect sets of the form

$$N^I = \{\alpha \in \mathfrak{Z}L \mid \eta_L(\operatorname{coz}\alpha) \leq I\},$$

for $I \in \mathrm{Pt}(\zeta L)$, to be the maximal $\ell$-ideals of $\mathfrak{Z}L$.

We dispense with the notation $N^I$ by observing that this set is just a special case of the sets

$$\operatorname{coz}^{-1}[J] = \{\alpha \in \mathfrak{Z}L \mid \operatorname{coz}\alpha \in J\},$$

for $J \in \zeta L$, which are easily checked to be (possibly improper) ideals of $\mathfrak{Z}L$.

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For any ideal $P$ of $\mathfrak{Z}L$ we set

$$I_P = \{\operatorname{coz}\alpha \mid \alpha \in P\}.$$

Simple calculations show that

$$I_P \in \zeta L \qquad \text{and} \qquad P \subseteq \operatorname{coz}^{-1}[I_P].$$

We denote by $\mathrm{Max}_\ell(\mathfrak{Z}L)$ the set of all maximal $\ell$-ideals of $\mathfrak{Z}L$.

If $c \in BL$, then the characteristic function of $c$, denoted $\gamma_c$, is the element of $\mathfrak{Z}L$ given by

$$\gamma_c(m) = \begin{cases} c & \text{if } m = 1 \\ c^* & \text{if } m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

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## Proposition

*For any zero-dimensional frame $L$,*

$$\mathrm{Max}_\ell(\mathfrak{Z}L) = \mathrm{Min}(\mathfrak{Z}L) = \{\mathrm{coz}^{-1}[I] \mid I \in \mathrm{Pt}(\zeta L)\}.$$

Proof Outline.

- Let $Q \in \mathrm{Max}_\ell(\mathfrak{Z}L)$; take $\alpha \geq \mathbf{0}$ in $Q$.
- Write $a = \mathrm{coz}\,\alpha$, and observe that

$$\mathrm{coz}(\alpha\gamma_{a^*}) = \mathrm{coz}\,\alpha \wedge \mathrm{coz}(\gamma_{a^*}) = a \wedge a^* = 0,$$

  whence $\alpha\gamma_{a^*} = \mathbf{0}$.
- Show that $\mathbf{1} \leq \alpha + \gamma_{a^*}$.
- Finally, for arbitrary $\tau \in Q$, $\tau^2 \geq \mathbf{0}$, so $\tau^2\varphi = 0$ for some $\varphi \notin Q$. Then $(\tau\varphi)^2 = \mathbf{0}$ and so $\tau\varphi = \mathbf{0}$.

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□

## Outline cont.

- Let $I \in \mathrm{Pt}(\zeta L)$, and take an $\ell$-ideal $Q$ such that $\mathrm{coz}^{-1}[I] \subseteq Q$.
- Show $I = I_Q$. This requires verifying that, for any $a \in Q$, $\mathrm{coz}\, \alpha \neq 1$. Enough to show that for $\alpha \geq \mathbf{0}$, $\mathrm{coz}\, \alpha = 1$ implies $\mathbf{1} \leq \alpha$.
- From all this one can deduce that $\mathrm{coz}^{-1}[I] = Q$.

$\square$

An ideal $I$ of a ring $A$ is called a $d$-ideal if, for every $a, b \in A$

$$a \in I \quad \text{and} \quad \operatorname{Ann}(a) = \operatorname{Ann}(b) \quad \Longrightarrow \quad b \in I.$$

Equivalently, $I$ is a $d$-ideal if and only if

$$\forall a \in I, \ \operatorname{Ann}^2(a) \subseteq I.$$

Recall that in $C(X)$ an ideal $I$ is called a $z$-ideal if, for any $f, g \in C(X)$,

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An ideal $I$ of $\mathcal{R}L$ is a $z$-ideal if, for any $\alpha, \beta \in \mathcal{R}L$,

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In the article

G. Mason
*$z$-Ideals and prime ideals*
J. Algebra. **26** (1973), 280 - 297.

the author defines an ideal $I$ of a commutative ring $A$ with identity to be a $z$-ideal if, for any $a, b \in A$,

$$a \in I \quad \text{and} \quad \mathfrak{M}(a) = \mathfrak{M}(b) \quad \Longrightarrow \quad b \in I.$$

**Example**

In $\mathfrak{Z}\mathbf{2}$ any constant function $\mathbf{k}$ with $|\mathbf{k}| \neq \mathbf{1}$ is not invertible, and is therefore contained in some maximal ideal $M$. Since $\operatorname{coz} \mathbf{k} = \operatorname{coz} \mathbf{1}$, $M$ cannot be a $z$-ideal topologically; but it is a $z$-ideal algebraically.

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## Lemma

*Let L be a zero-dimensional frame. For any* $\alpha \in \mathfrak{Z}L$, $\mathrm{Ann}^2(\alpha) = \mathrm{coz}^{-1}[r_L(\mathrm{coz}\,\alpha)]$.

## Corollary

*For any zero-dimensional frame L, the sets d-ideals and topological z-ideals of* $\mathfrak{Z}L$ *coincide.*

Denote by $\mathrm{Max}_d(\mathfrak{Z}L)$ the set of maximal $d$-ideals of $\mathfrak{Z}L$.

## Lemma

*Let L be a zero-dimensional frame. Then*

$$\mathrm{Max}_d(\mathfrak{Z}L) = \{\mathrm{coz}^{-1}[I] \mid I \in \mathrm{Pt}(\zeta L)\}.$$

An ideal $I$ of a ring is called pure if

$$\forall a \in I \quad \exists b \in I \quad \text{such that} \quad a = ab.$$

In the text *Stone Spaces*, Peter Johnstone calls these ideals “neat", and proves that they form a frame in any ring.

**Remark**

In the rings $\mathcal{R}L$, pure ideals are exactly the ideals $\text{coz}^{-1}[I]$ for $I \in \beta L$.

We write $\text{Max}_p(\mathfrak{Z}L)$ for the set of all maximal pure ideals of $\mathfrak{Z}L$.

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## Remark

In the rings $\mathcal{R}L$, pure ideals are exactly the ideals $\mathrm{coz}^{-1}[I]$ for $I \in \beta L$.

We write $\mathrm{Max}_p(\mathfrak{Z}L)$ for the set of all maximal pure ideals of $\mathfrak{Z}L$.

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*Let L be a zero-dimensional frame. Then*

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We now summarise.

**Proposition**

*For any zero-dimensional frame $L$,*

$$\mathrm{Max}_\ell(\mathfrak{Z}L) = \mathrm{Min}(\mathfrak{Z}L) = \mathrm{Max}_d(\mathfrak{Z}L) = \mathrm{Max}_p(\mathfrak{Z}L) = \{\mathrm{coz}^{-1}[I] \mid I \in \mathrm{Pt}(\zeta L)\}.$$

**Example**

Maximal $d$-ideals in $\mathfrak{Z}L$ need not be maximal ideals. Indeed, $\mathrm{Pt}(\zeta 2) = \{0\}$. Since $\mathrm{coz}^{-1}[\{0\}] = \{0\}$, this ideal is not maximal because $\mathfrak{Z}2 \cong \mathbb{Z}$.

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Here are some noteworthy corollaries to this proposition.

An $f$-ring is called $\ell$-semisimple if the intersection of its maximal $\ell$-ideals is the zero ideal. Suppose $\alpha \in \mathfrak{Z}L$ is in every maximal $\ell$-ideal. Then $\alpha \in \mathrm{coz}^{-1}[I]$ for every $I \in \mathrm{Pt}(\mathfrak{Z}L)$, which implies $r_L(\mathrm{coz}\,\alpha) \leq I$ for every $I \in \mathrm{Pt}(\zeta L)$. By spatiality of $\zeta L$, this implies $\mathrm{coz}\,\alpha = 0$, hence $\alpha = \mathbf{0}$. We therefore have the following.

## Corollary

*$\mathfrak{Z}L$ is $\ell$-semisimple for any zero-dimensional frame $L$.*

A ring $A$ is said to be normal if

$$ab = 0 \text{ in } A \quad \Longrightarrow \quad \mathrm{Ann}(a) + \mathrm{Ann}(b) = a.$$

For reduced $f$-rings normality is equivalent to the condition that every maximal $\ell$-ideal contains a unique minimal prime ideal. We therefore have the following result.

**Corollary**

*$\mathfrak{Z}L$ is normal for any zero-dimensional frame $L$.*

A ring $A$ is said to be projectable if, for every $a \in A$,

$$\mathrm{Ann}(a) + \mathrm{Ann}^2(a) = A.$$

Let $\mathrm{Spec}_d(\mathfrak{Z}L)$ denote the set of all prime $d$-ideals in $\mathfrak{Z}L$. Recall that in a projectable reduced ring prime $d$-ideals are minimal prime.

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Using the result that, for any $\alpha \in \mathfrak{Z}L$,

$$\mathrm{Ann}(\alpha) = \mathrm{coz}^{-1}[r_L((\mathrm{coz}\,\alpha)^*)] \quad \text{and} \quad \mathrm{Ann}^2(\alpha) = \mathrm{coz}^{-1}[r_L(\mathrm{coz}\,\alpha)],$$

we arrive at the following corollary.

## Corollary

*$\mathfrak{Z}L$ is projectable for any zero-dimensional frame $L$, and hence*

$$\mathrm{Max}_\ell(\mathfrak{Z}L) = \mathrm{Spec}_d(\mathfrak{Z}L).$$

Recall that if $\phi\colon A \to B$ is a ring homomorphism and $I$ is an ideal of $B$, then the ideal $\phi^{-1}[I]$ is called the contraction of $I$ by $\phi$.

Not every morphism $h\colon L \to M$ in **CRFrm** has the property that the induced ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ contracts maximal $\ell$-ideals to maximal $\ell$-ideals. Here is a counterexample.

## Example

Let $L$ be a non-$P$-frame and $h\colon L \to \mathfrak{B}L$ be the Booleanisation map $x \mapsto x^{**}$. Then $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}(\mathfrak{B}L)$ does not contract every maximal $\ell$-ideal to a maximal $\ell$-ideal.

In $\mathfrak{Z}L$ we have the following.

## Corollary

*For any morphism $h\colon L \to M$ in* **ODFrm**, *the ring homomorphism $\mathfrak{Z}h\colon \mathfrak{Z}L \to \mathfrak{Z}M$ contracts maximal $\ell$-ideals to maximal $\ell$-ideals.*

## Idea of proof.

Show that for any $I \in \mathrm{Pt}(\zeta M)$,

$$(\mathfrak{Z}h)^{-1}[\mathrm{coz}^{-1}[I]] = \mathrm{coz}^{-1}[(\zeta h)_*(I)],$$

where $\zeta h$ is the unique frame homomorphism making the square below commute.

$$\begin{array}{ccc} \zeta L & \xrightarrow{\ \zeta h\ } & \zeta M \\ {\scriptstyle j_L}\downarrow & & \downarrow{\scriptstyle j_M} \\ L & \xrightarrow{\ \ h\ \ } & M \end{array}$$

## Corollary

*There is no $L$ for which $\mathfrak{Z}L$ is von Neumann regular.*

## Outline of proof.

1. Let $p \geq 2$ in $\mathbb{Z}$. For any $\alpha \in \mathfrak{Z}L$ we have

$$
\begin{aligned}
(\boldsymbol{p}\alpha)(1) &= \bigvee\{\boldsymbol{p}(k) \wedge \alpha(m) \mid km = 1\} \\
&= \big(\boldsymbol{p}(1) \wedge \alpha(1)\big) \vee \big(\boldsymbol{p}(-1) \wedge \alpha(-1)\big) \\
&= 0,
\end{aligned}
$$

   which shows that $\boldsymbol{p}\alpha \neq \mathbf{1}$.

2. $\boldsymbol{p}$ is not invertible, and so belongs to some maximal ideal $M$.

3. This ideal cannot be minimal prime because $\boldsymbol{p}$ is not annihilated by any non-member of $M$.

□

However

**Corollary**

*For any zero-dimensional frame L, the classical ring of quotients of $\mathfrak{Z}L$ is von Neumann regular.*

## Lemma

*For any zero-dimensional frame L,*

$$\mathrm{PId}(\mathfrak{Z}L) = \{\mathrm{coz}^{-1}[I] \mid I \in \zeta L\}.$$

## Proposition

*For any zero-dimensional frame L we have the following.*

1. *The map*

$$\zeta L \to \mathrm{PId}(\mathfrak{Z}L) \quad \text{given by} \quad I \mapsto \mathrm{coz}^{-1}[I]$$

   *is a frame isomorphism.*

2. *The map*

$$\zeta L \to \mathfrak{O}(\mathrm{Max}_{\ell}(\mathfrak{Z}L)) \quad \text{given by} \quad I \mapsto \mathcal{U}(\mathrm{coz}^{-1}[I])$$

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THANK YOU