ON BITOPOLOGICAL CLOPEN SETS

By

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Introduction

In a bitopological space (briefly, BS) (X, τ_1, τ_2) we use the following notations: the interior and the closure of a subset A of X with respect to the topology τ_i are denoted by $\tau_i intA$ and $\tau_i clA$, respectively, where $i \in \{1, 2\}$.

If O is open in the τ_i, then we write $O \in \tau_i$, while, for the τ_i-closed set F, we use the notation $F \in co\tau_i$.

We denote by $\tau_i^A = \{A \cap U | U \in \tau_i\}$ the topology induced on the set A from the τ_i.

The family of all τ_i-open neighborhoods of a subset M of X is denoted by $\sum_i^X (M)$.

Clopen sets in bitopological spaces appeared in the following references

1. **I. Reilly**, *Zero-dimensional bitopological spaces.* Indag. Math., 35 (1973), 127–131.

2. **G. Bezhanishvili, N.Bezhanishvili, D. Gabelaia, A. Kurz**, *Bitopological duality for distributive lattices and Heyting algebras.* Math. Struct. Comput. Sci., 20(3) (2010), 359–393.

Definition 1. A subset A of a BS (X, τ_1, τ_2) is called an (i, j)-clopen set if $A \in \tau_i \cap co\tau_j$, where $i, j \in \{1, 2\}, i \neq j$.

Denote by $(i, j) - Clp(X)$ the class of all (i, j)-clopen sets of a BS (X, τ_1, τ_2).

If $i = j$, we get the well known notion of general topology –the clopen set. Therefore, the class of $i - Clp(X)$ will denote the collection of all τ_i-clopen subsets of (X, τ_1, τ_2).

In bitopological spaces, considerations of so called $(1, 2)$-clopen and (2,1)-clopen sets seem to be not applied widely. Motivated by this gap in the bitopological case we try to develop some asymmetric constructions

Few Naive Observations

(1) $A \in (i, j) - Clp(X)$ if and only if $X \setminus A \in (j, i) - Clp(X)$.

(2) The following equation holds: $(1, 2) - Clp(X) \cap (2, 1) - Clp(X) = 1 - Clp(X) \cap 2 - Clp(X)$.

Let $A_\alpha \in (i, j) - Clp(X)$ for each $\alpha \in \Lambda$, $A = \bigcap_{\alpha \in \Lambda} A_\alpha$ and $B = \bigcup_{\alpha \in \Lambda} A_\alpha$. Then the following hold:

(3) $A \in co\tau_j$ and $B \in \tau_i$,

(4) $A, B \in (i,j) - Clp(X)$ if Λ is finite,

If in the standard definition of topological spaces (via open sets), we delete the word "finite" in the axiom:

"finite intersection of open sets is open",

then we come to the notion of Alexandroff's space.

Therefore we have:

(5) $A \in (i,j) - Clp(X)$ (resp. $B \in (i,j) - Clp(X)$) if (X, τ_i) (resp. (X, τ_j)) is an Alexandorff space.

Proposition 1. If A is a subset of a BS (X, τ_1, τ_2) and $B \in (i,j) - Clp(X)$, then $A \cap B$ is (i,j)-clopen in the subspace (A, τ_1^A, τ_2^A).

bf Question: What kind of Mappings Save Bitopological Clopens?

To answer this we need to remaind some definitions for BS.
A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be

(1) i-open (resp. i-continuous) if $f : (X, \tau_i) \to (Y, \gamma_i)$ is an open (resp. continuous) map.

(2) j-closed if $f : (X, \tau_j) \to (Y, \gamma_j)$ is a closed map.

(3) p-continuous if both $f : (X, \tau_1) \to (Y, \gamma_1)$ and $f : (X, \tau_2) \to (Y, \gamma_2)$ are continuous [J.C. Kelly, 1963].

(4) p-homeomorphism if f is bijective and both f and f^{-1} are p-continuous, where f^{-1} denotes the inverse to f.

Proposition 2. If a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is i-open and j-closed and $A \in (i,j) - Clp(X)$, then $f(A) \in (i,j) - Clp(Y)$.

A BS (X, τ_1, τ_2) is said to be (i,j)-stable [R.D. Kopperman, 1995] if any $A \in co\tau_i$ implies j-compactenss of A.

If (X, τ_1, τ_2) is (j,i)-stable and i-Hausdorff then $(i,j) - Clp(X) \subset i - Clp(X)$.

A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be $(i,j) - \Delta$ continuous if $f : (X, \tau_i) \to (Y, \gamma_j)$ is continuous.

Proposition 3. Let (X, τ_1, τ_2) be a (j,i)-stable BS and a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ be both i-open and $(i,j) - \Delta$-continuous. If a BS (Y, γ_1, γ_2) is j-T_2, then $f(A) \in (i,j) - Clp(Y)$ for each $A \in (i,j) - Clp(X)$.

Bitopological Connectedness and Continuous-like Maps

A BS (X, τ_1, τ_2) is said to be pairwise connected (briefly p-connected) if X could not be represented as the union of the disjoint sets $A \in \tau_1 \setminus \{\emptyset\}$ and $B \in \tau_2 \setminus \{\emptyset\}$ [W. Pervin, 1967].

(X, τ_1, τ_2) is p-connected $\Leftrightarrow X$ cannot be represented as the union of two nonempty disjoint $A \in (i,j) - Clp(X)$ and $B \in (j,i) - Clp(X) \Leftrightarrow$ There exists no nonempty proper (i,j)-clopen set.

Definition 2. A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be (i,j)-clopen-irresolute if $f^{-1}(V) \in (i,j) - Clp(X)$ for each $V \in (i,j) - Clp(Y)$, where $i \neq j$, $i, j \in \{1, 2\}$.

If a map is both $(1,2)$-clopen-irresolute and $(2,1)$-clopen-irresolute then it is called to be p-clopen-irresolute. Every p-continuous map is p-clopen-irresolute but the converse is not always true as shown by the following example.

Example 1. Let us consider a set $X = \{m, n, p, q, k\}$, together with topologies $\tau_1 = \{\emptyset, X\} \cup \{\{m\}, \{n, p\}, \{m, n, p\}\}$ and $\tau_2 = \{\emptyset, X\} \cup \{\{q, k\}\}$. Then we observe that $(1,2) - Clp(X) = \{\emptyset, X, \{m, n, p\}\}$ and $(2,1) - Clp(X) = \tau_2$. Moreover, let $Y = \{a, b, c, d\}$ be endowed with the following topologies $\gamma_1 = \{\emptyset, Y\} \cup \{\{a, b\}, \{c\}, \{a, b, c\}\}$ and $\gamma_2 = \{\emptyset, Y\} \cup \{\{c, d\}\}$, then $(1,2) - Clp(Y) = \{\emptyset, Y, \{a, b\}\}$ and $(2,1) - Clp(Y) = \gamma_2$. If we define a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ via the equations:$f(m) = f(n) = a, f(p) = b, f(q) = c, f(k) = d$, then it is p-clopen-irresolute. But $f : (X, \tau_1) \to (Y, \gamma_1)$ is not continuous and f is not p-continuous.

It is known that the p-connectedness is preserved under p-continuous surjections [Pe]. The following proposition is an improvement of this result.

Proposition 4. The p-connectedness is preserved by (i,j)-clopen-irresolute surjections.

Recall that a BS (X, τ_1, τ_2) is said to be (i,j)-zero dimensional if a basis $\mathbf{B}(\tau_i)$ for the topology τ_i is formed with $co\tau_j$, i.e. $\mathbf{B}(\tau_i) = co\tau_j$ [Re]. It is obvious that a BS (X, τ_1, τ_2) is (i,j)-zero dimensional if and only if $\mathbf{B}(\tau_i) = (i,j) - Clp(X)$.

Theorem 1. If (X, τ_1, τ_2) is an (i,j)-zero dimensional and i-T_1 BS, then $card(A) \leq 1$ for every p-connected subspace (A, τ_1^A, τ_2^A).

Definition 3. The subset $\bigcap\{U(x) | x \in U(x) \in (i,j) - Clp(X)\}$ of a BS (X, τ_1, τ_2) is called the (i,j)-quasi-component of a point $x \in X$ and is denoted by $(i,j) - Q_x$.

Proposition 5. Let x be a point in a BS (X, τ_1, τ_2). Then the following hold:

(1) $(i,j) - Q_x \in co\tau_j \setminus \{\emptyset\}$.

(2) If $y \in (i,j) - Q_x$, then $(i,j) - Q_y \subset (i,j) - Q_x$.

Proposition 6. If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-clopen-irresolute, then $f((i,j) - Q_x) \subset (i,j) - Q_{f(x)}$ for each $x \in X$.

Corollary Let $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ be a map.

(1) If f is p-continuous, then $f((i,j) - Q_x) \subset (i,j) - Q_{f(x)}$.

(2) If f is a p-homeomorphism, then $f((i,j) - Q_x) = (i,j) - Q_{f(x)}$.

The greatest p-connected subset containing a point $x \in X$ is called the p-component of x in a BS (X, τ_1, τ_2) and is denoted by $p - C_x$.

Theorem 2. In a BS (X, τ_1, τ_2), the following implication holds: $p - C_x \subset (1,2) - Q_x \cap (2,1) - Q_x$.

Definition 4. A BS (X, τ_1, τ_2) is said to be p-ultra-Hausdorff if for any pair of distinct points x_1,$x_2 \in X$ there exist $U_{x_1} \in (i,j) - Clp(X)$ and $V_{x_2} \in (j,i) - Clp(X)$ such that $x_1 \in U_{x_1}$, $x_2 \in V_{x_2}$ and $U_{x_1} \cap V_{x_2} = \emptyset$.

It should be especially noticed that if $i = j$ then the notion of p-ultra-Hausdorff coincides with the notion of ultra-Hausdorff for topological spaces, given in [R. Staum, 1974].

Example 2. Let X be a set with $card(X) \geq \aleph_0$, τ_d-discrete and $\tau_{cof.}$-cofinite (i.e. all finite subsets of X are closed, and vice versa) topologies on X, respectively. Then it is obvious that $(X, \tau_d, \tau_{cof.})$ is p-ultra-Hausdorff BS.

Proposition 7. A BS (X, τ_1, τ_2) is p-ultra-Hausdorff if and only if for any distinct points x_1, x_2, there exist $U \in (i,j) - Clp(X)$ such that $x_1 \in U, x_2 \notin U$ and $V \in (i,j) - Clp(X)$ such that $x_2 \in V, x_1 \notin V$.

Definition 5. A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be (i,j)-weakly clopen-continuous (resp. (i,j)-clopen-continuous) if for each $x \in X$ and each $V \in \sum_i^Y (f(x))$, there exists a set $U \in (i,j) - Clp(X)$ containing x such that $f(U) \subset \gamma_j cl(V)$ (resp. $f(U) \subset V$), where $i \neq j$, $i,j \in \{1,2\}$.

Example 3. Let us consider the set $X = \{m,n,p,q,k\}$, together with topologies $\tau_1 = \{\emptyset, X\} \cup \{\{m\},\{n,p\},\{m,n,p\}\}$ and $\tau_2 = \{\emptyset, X\} \cup \{\{q,k\}\}$. Then we observe that $(1,2) - Clp(X) = \{\emptyset, X, \{m,n,p\}\}$ and $(2,1) - Clp(X) = \{\emptyset, X, \{q,k\}\}$. Moreover, let $Y = \{a,b,c\}$ be endowed with the following topologies $\gamma_1 = \{\emptyset, Y\} \cup \{\{a\}\}$ and $\gamma_2 = \{\emptyset, Y\} \cup \{\{c\}\}$. If we define a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ via the equations:$f(m) = f(n) = a, f(p) = b, f(q) = f(k) = c$, then f is $(1,2)$-weakly clopen-continuous. But it is not $(1,2)$-clopen-continuous.

Proposition 8. If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-weakly clopen-continuous (resp. (i,j)-clopen-continuous) and A is a subset of X, then $f|_A : (A, \tau_1^A, \tau_2^A) \to$

(Y, γ_1, γ_2) is (i,j)-weakly clopen-continuous (resp. (i,j)-clopen-continuous).

Proposition 9. If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is

(1) (i,j)-weakly clopen-continuous and (X, τ_j) is Alexandorff, then f is (i,j)-clopen-irresolute.

(2) If a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-clopen irresolute and (Y, γ_1, γ_2) is (i,j)-zero dimensional, then f is (i,j)-clopen-continuous.

Corollary Let (X, τ_j) be an Alexandroff topological space and (Y, γ_1, γ_2) is an (i,j)-zero dimensional BS. Then a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-clopen-continuous if and only if it is (i,j)-clopen irresolute.

Results on (i,j)-Clopen-Compactness

In 1974 R. Staum introduced the property of the topological spaces named by mild compactness for investigation of the algebra of bounded continuous functions into a non-archimedian filds. Independently, A. Sostak (1976) rediscovered such topological property named as clopen-compactness. I want to develop asymmetric analog of this property and give to the end of the talk some nice results.

Definition 5. A subset K of a BS (X, τ_1, τ_2) is said to be (i,j)-clopen-compact relative to X if every cover of K by (i,j)-clopen sets of X has a finite subcover.

It should be noticed that every i-compact subset of (X, τ_1, τ_2) is (i,j)-clopen-compact relative to X.

Proposition 10.. If a BS (X, τ_1, τ_2) is (i,j)-zero dimensional and a subset K of X is (i,j)-clopen-compact relative to X, then K is a τ_i-compact subset of X.

Theorem 3. For a BS (X, τ_1, τ_2), the following are equivalent:

(1) X is p-ultra-Hausdorff;

(2) For each set K of X which is (i,j)-clopen compact relative to X, $K = \cap\{V | K \subset V \in (i,j) - Clp(X)\}$;

(3) For each $x \in X$, $x = \cap\{V | x \in V \in (i,j) - Clp(X)\}$.

Corollary If a BS (X, τ_1, τ_2) is p-ultra-Hausdorff and K is (i,j)-clopen-compact relative to X, then $K \in co\tau_j$.

Problem: Unfortunately, at this time I have not examples demonstrating (i,j)-clopen compact sets.

Thank You!