On Algebraic Analysis of Temporal Heyting Calculus

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Introduction

IPC = intuitionistic propositional calculus

Grz = Grzegorczyk modal system

GL = Gödel-Löb modal system (provability logic)

$$\mathbf{IPC} \xrightarrow{\text{Gödel}} \mathbf{Grz} \xrightarrow{\text{Splitting}} \mathbf{GL}$$

Introduction

$\Lambda(\mathbf{IPC})$ = lattice of superintuitionistic logics

$\Lambda(\mathbf{Grz})$ = lattice of logics above **Grz**

$\Lambda(\mathbf{GL})$ = lattice of logics above **GL**

Theorem (Blok-Esakia)

$\Lambda(\mathbf{IPC}) \cong \Lambda(\mathbf{Grz})$

Introduction

$$\Lambda(\mathbf{IPC}) \not\cong \Lambda(\mathbf{GL})$$

Introduction

Kuznetsov-Muravitsky logic **KM** is an intuitionistic modal logic (in the language with "$\Box$") containing **IPC** and the following axioms:

- $\Box(p \to q) \to (\Box p \to \Box q)$
- $p \to \Box p$
- $\Box p \to (q \vee (q \to p))$
- $(\Box p \to p) \to p$ (intuitionistic analogue of the Gödel-Löb axiom)

Theorem (Kuznetsov-Muravitsky)

$\Lambda(\mathbf{KM}) \cong \mathbf{\Lambda}(\mathbf{GL})$

Modalized Heyting Calculus

mHC-modalized Heyting calculus introduced by Esakia

mHC $=$ **KM** without axiom $(\Box p \to p) \to p$

Algebraic models of **mHC** are frontal Heyting algebras

Frontal Heyting Algebras

Definition

A frontal Heyting algebra is an algebra $(H, \vee, \wedge, \rightarrow, \Box, 0, 1)$ such that:

1. H is a Heyting algebra
2. $\Box(x \wedge y) = \Box x \wedge \Box y$
3. $x \leq \Box x$
4. $\Box x \leq y \vee (y \rightarrow x)$

The class of frontal Heyting algebras is denoted by **fHA**.

Frontal Esakia Spaces

Definition

A frontal Esakia space is a triple $(X, \leq, R)$, where $(X, \leq)$ is an Esakia space and R is a binary relation on X that satisfies the following conditions:

1. $xRy \Rightarrow x \leq y$
2. $x < y \Rightarrow xRy$
3. U is clopen upset $\Rightarrow \Box_R U$ is clopen upset.

where $\Box_R U = X - R^{-1}(X - U)$

Instead of 1. and 2. we can write: $< \subseteq R \subseteq \leq$

Following Esakia, we call the triple $(X, \leq, R)$ a transit.

Frontal Esakia Spaces

Definition

fES is the category whose objects are transits and whose morphisms are functions $f : (X_1, \leq, R_1) \to (X_2, \leq, R_2)$ such that:

1. $f : (X_1, \leq) \to (X_2, \leq)$ is a continious p-morphism (Esakia morphism)
2. for every $x \in X_1$: $f(R_1(x)) = R_2 f(x)$

Representation Theory

Construction:

For every frontal Heyting algebra $(H, \Box)$, let H_* be the transit consisting of prime filters of Heyting algebra H;
$x \leq y \Leftrightarrow x \subseteq y$;
$xRy \Leftrightarrow \Box^{-1}x \subseteq y$.

For every transit $(X, \leq, R)$, let $(X^*, \Box_R)$ be the frontal Heyting algebra, where X^* is the Heyting algebra of clopen upsets and $\Box_R U = X - R^{-1}(X - U)$.

Representation Theory

Theorem (Castiglioni, Sagastume and San Martin)

fHA *is dually equivalent to* ***fES***.

Temporal Heyting Calculus

Let $\Diamond_R U = R(U)$
In general $\Diamond_R U$ may not be clopen, so $\Diamond_R$ is not defined on X^*
If $\Diamond_R U$ is clopen, then $\Diamond_R$ becomes a diamond-like operator on X^*.

In such a case $\Box, \Diamond$ become temporal operators on X^*
$\Box \longrightarrow$ always
$\Diamond \longrightarrow$ before

This yields the notion of temporal Heyting calculus **tHC** introduced by Esakia

Algebraic models of **tHC** are temporal Heyting algebras.

Temporal Heyting Algebras

Definition

A temporal Heyting algebra is an algebra $(H, \vee, \wedge, \to, \Box, \Diamond, 0, 1)$ such that:

1. $(H, \vee, \wedge, \to, \Box, 0, 1)$ is a frontal Heyting algebra
2. $x \leq \Box\Diamond x$
3. $\Diamond\Box x \leq x$
4. $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$

The class of temporal Heyting algebras is denoted by **tHA**.

Temporal Esakia Spaces

Definition

A temporal Esakia space is a transit $(X, \leq, R)$ such that $U \in X^* \Rightarrow R(U) \in X^*$.

Let **tES** be the category whose objects are transits and whose morphisms are functions $f : (X, \leq, R) \to (X', \leq', R')$ such that:

(1) f is an **fES**-morphism
(2) $yRf(x) \Rightarrow \exists z : zRx$ and $y \leq f(z)$

Representation Theory

For a temporal Heyting algebra $(H, \Box, \Diamond)$ we actually have two relations on the Esakia space of H, one is $R_\Box$ and the other is $R_\Diamond$, where

$$xR_\Box y \Leftrightarrow \Box^{-1}x \subseteq y.$$
$$xR_\Diamond y \Leftrightarrow x \subseteq \Diamond^{-1}y.$$

Lemma

$R_\Box = R_\Diamond$.

Consequently, we write R instead of $R_\Box$ and $R_\Diamond$.

Representation Theory

As we know $\varphi : H \to (H_*)^*$ is an isomorphism, where

$\varphi(a) = \{x \in X : a \in x\}$

Lemma

$\varphi(\Box a) = \Box_R \varphi(a)$
$\varphi(\Diamond a) = \Diamond_R \varphi(a)$

Thus, if $H \in$ **tHA**, then $H_* \in$ **tES**.

Representation Theory

Lemma

If $(X, \leq, R)$ *is a temporal Esakia space, then* $(X^*, \Box_R, \Diamond_R)$ *is a temporal Heyting algebra.*

Representation Theory

Theorem

tHA *is dually equvalent to* ***tES****.*

$\Diamond -$ *Filters*

Definition

Let $(H, \Box, \Diamond)$ be a temporal Heyting algebra. We call a filter F of H a $\Diamond$-filter if it satisfies:

$a \to b \in F \Rightarrow \Diamond a \to \Diamond b \in F$

Theorem

$\Diamond - \textit{filters} = \textit{congruences of tHA}$

$\Theta \longmapsto F_\Theta = \{a : a\Theta 1\}$

$F \longmapsto \Theta_F$ where $a\Theta_F b$ iff $a \leftrightarrow b \in F$

R-Upsets

Definition

Let $(X, \leq, R) \in \mathbf{tES}$. We call an upset U of X an R-upset if the following holds:

$\forall x \notin U \ \forall y \in U \ xRy \Rightarrow \exists z \in U : xRzRy$

Theorem

$\Diamond - \textit{filters} \ = \textit{closed } R\textit{-upsets.}$

Corollary

$\textit{Congruences} = \Diamond - \textit{filters} \ = \textit{closed } R\textit{-upsets.}$

R-Upsets

Theorem

$\forall S \subseteq X$ there exists the least closed R-upset $\hat{S}$ containing S.

If S={x}, then $\hat{x} := \hat{S}$.

Simple and Subdirectly Irreducible tHA-Algebras

Definition

Let $(X, \leq, R)$ be a temporal Esakia space.

We call $x \in X$ an R-root provided $\hat{x} = X$.

We call X R-rooted provided there is at least one R-root in X.

Simple and Subdirectly Irreducible *tHA*-algebras

Theorem

Let $(H, \Box, \Diamond)$ be a temporal Heyting algebra and let $(X, \leq, R)$ be its dual temporal Esakia space.

1. *H is subdirectly irreducible iff X is R-rooted.*
2. *H is simple iff each $x \in X$ is an R-root.*

Thank You