Categories of Equational Theories

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(joint work with Stanisław Szawiel)

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The functor part of the free monoid monad

$$M : Set \longrightarrow Set$$

can be described by a series

$$M(X) = \sum_{n \in \omega} X^n$$

More generally a *(finitary) polynomial functor* on *Set* is a functor

$$P : Set \to Set$$

(isomorphic to one of) form

$$P(X) = \sum_{n \in \omega} A_n \times X^n$$

Inventors and/or early users of polynomial functors:
Y. Diers, G. C. Wraith, P.T. Johnstone, J-Y. Girard, A. Joyal, E. G. Manes, M. A. Arbib, F. Lamarche, P. Taylor, A. Carboni, B. Jay, J. R. B. Cockett, M. Abbott, T. Altenkirch, N. Ghani, J. Kock, N. Gambino, M. Hyland

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Theorem

For a finitary functor $P : Set \to Set$ the following are equivalent

- P is a polynomial functor (i.e. $P(X) \cong \sum_{n\in\omega} A_n \times X^n$ for a family of sets $\{A_n\}_n$);
- P preserves wide pullbacks;
- the category $Set \downarrow P$ is a presheaf topos.

- The right notion of a morphism of polynomial functors is a *cartesian natural transformation*
- **Poly** is the (monoidal) category of polynomial functors and cartesian natural transformations;
- We have a strict monoidal embedding

$$\mathbf{Poly} \to \mathbf{End}$$

End is the monoidal category on finitary endofunctors on *Set* and natural transformations.

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- **Mnd** - the category of finitary monads on *Set* is the category of monoids in **End**
- **PolyMnd** - the category of polynomial monads on *Set*.

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Remark The category **Poly** and hence **PolyMnd** does not have good closure properties (limits, colimits).

- *Sig* - the category of (algebraic) signatures Set^{ω};
- $A = \{A_n\}_{n\in\omega}$ a signature; A_n - set of n-ary operations;
- *Sig* is a monoidal category with substitution tensor

$$(A \otimes B)_n = \sum_{k,n_1,\ldots n_k,\sum_i n_i=n} A_{n_1} \times \ldots \times A_{n_k} \times B_k$$

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- We have a lax monoidal *symmetrization monad* (S_n - n-th symmetric group)

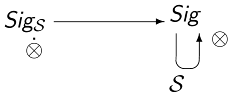
$$\mathcal{S} : Sig \to Sig$$

$$\mathcal{S}(A)_n = S_n \times A_n$$

  'all versions' of n-ary operations in A
- coherence morphism for $\mathcal{S}$ is the *'little combing'*

$$\phi : \mathcal{S}(A) \otimes \mathcal{S}(B) \to \mathcal{S}(A \otimes B)$$

$$\begin{tikzcd} Sig \arrow[loop below, "\otimes"] \\ \mathcal{S} \end{tikzcd}$$



$Sig_{\mathcal{S}}$
Sig
$\mathcal{S}$

$$
\begin{tikzcd}
\mathbf{Mon}(Sig_{\mathcal{S}}) \arrow[d] & \\
Sig_{\mathcal{S}} \arrow[r] & Sig \arrow[loop below, "\mathcal{S}"] \\
\otimes & \otimes
\end{tikzcd}
$$

- **Mon** - monoids

$$
\begin{tikzcd}
 & \mathbf{Mon}(Sig_{\mathcal{S}}) \arrow[d] & \\
\mathbf{Poly} \simeq & \underset{\otimes}{Sig_{\mathcal{S}}} \arrow[r] & Sig \otimes \arrow[loop below, "\mathcal{S}"]
\end{tikzcd}
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Polynomial and Analytic Monads
polynomial vs analytic

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\begin{tikzcd}
\mathbf{PolyMnd} \arrow[d] \arrow[r, phantom, "\simeq"] & \mathbf{Mon}(Sig_{\mathcal{S}}) \arrow[d] & \\
\mathbf{Poly} \arrow[r, phantom, "\simeq"] & Sig_{\mathcal{S}} \arrow[r] \arrow[d, phantom, "\otimes"] & Sig \arrow[loop below, "\mathcal{S}"] \arrow[r, phantom, "\otimes"] & {} \\
& {} &
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Rigid Operads **Symmetric Operads**

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- $\cdot : S_n \times B_n \longrightarrow B_n$ - left action of S_n on the set B_n, $n \in \omega$.
- we have for any set X a right action

$$X^n \times S_n \longrightarrow X^n$$

$$\langle \vec{x} : \underline{n} \rightarrow X, \sigma \rangle \mapsto \vec{x} \circ \sigma$$

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- Dividing $X^n \times B_n$ by the relation

$$\langle \vec{x} \circ \sigma, b \rangle \sim \langle \vec{x}, \sigma \cdot b \rangle$$

we get the tensor over S_n

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- and an *analytic functor*

$$X \mapsto \sum_{n \in \omega} X^n \otimes_n B_n$$

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Theorem

For a functor $F : \mathit{Set} \to \mathit{Set}$ the following are equivalent

- F is an analytic functor (i.e. $F(X) \cong \sum_{n\in\omega} X^n \otimes_n B_n$ for a family of actions of symmetric groups on sets $\{B_n\}_n$);
- F is finitary and weakly preserves wide pullbacks;
- F is a left Kan extension of a functor $B : \mathbb{B} \to \mathit{Set}$ along $\iota_{\mathbb{B}}$.

- The right notion of a morphism of analytic functors is a *weakly cartesian natural transformation*
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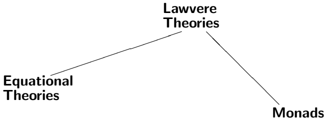
- **AnMnd** - the category of analytic monads on *Set*.

Categories of Equational Theories



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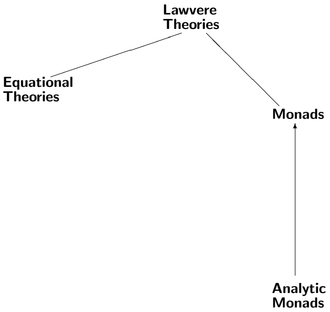

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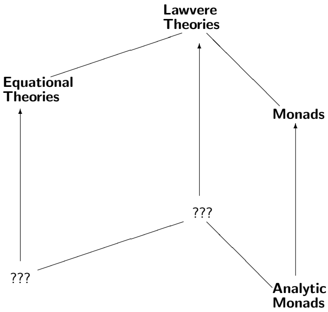

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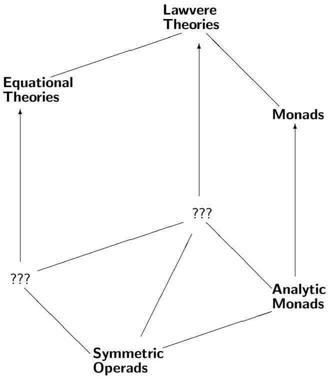

Categories of Equational Theories

$$\begin{tikzcd}
 & & \textbf{Lawvere Theories} & \\
\textbf{Equational Theories} \arrow[urr, no head] & & & \textbf{Monads} \arrow[ul, no head] \\
 & & \textbf{Analytic Lawvere Theories} \arrow[uu] & \\
\textbf{Linear regular Equational Theories} \arrow[uu] \arrow[urr, no head] & & & \textbf{Analytic Monads} \arrow[uu] \arrow[ul, no head] \\
 & \textbf{Symmetric Operads} \arrow[ul, no head] \arrow[uur, no head] \arrow[urr, no head] & &
\end{tikzcd}$$

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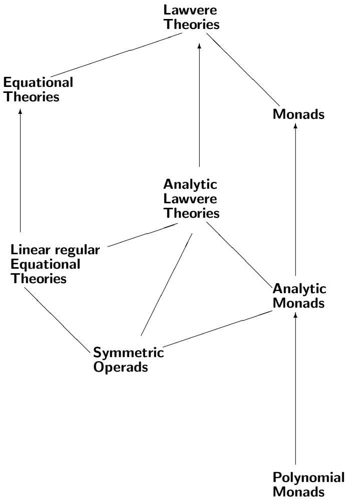

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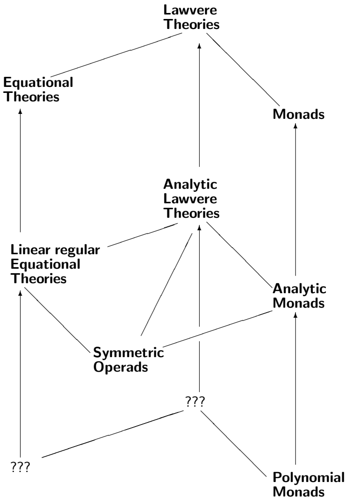

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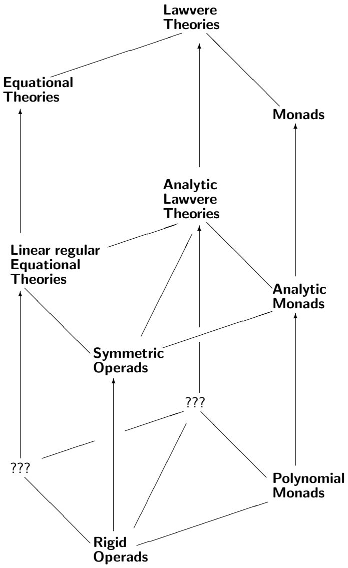

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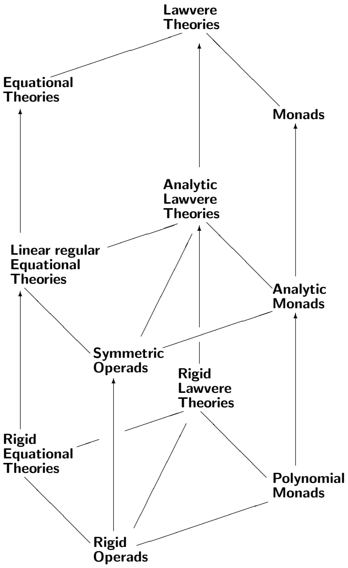

- $\mathbb{F}$ - skeleton of the category of finite sets; $\underline{n} = \{1, \ldots, n\}$
- $\mathbb{F}^{op}$ - the initial Lawvere theory
- the unique morphism into another theory Lawvere theory

$$\pi : \mathbb{F}^{op} \longrightarrow T$$

$$f : \underline{n} \longrightarrow \underline{m} \mapsto \langle \pi^m_{f(1)}, \ldots, \pi^m_{f(n)} \rangle : m \longrightarrow n$$

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- $Aut(n)$ is the set of automorphisms of n in T
- We have functions

$$\rho_n : S_n \times Aut(1)^n \longrightarrow Aut(n)$$

such that

$$(\sigma, a_1, \ldots, a_n) \mapsto a_1 \times \ldots \times a_n \circ \pi_\sigma$$

Simple automorphisms

We say that Lawvere theory T has *simple automorphisms* iff ρ_n is a bijection, for $n \in \omega$.

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Structural morphisms

The class of *structural morphisms* in T is the closure under isomorphism of the image under π of all morphisms in $\mathbb{F}$.

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The class of *structural morphisms* in T is the closure under isomorphism of the image under π of all morphisms in $\mathbb{F}$.

Analytic morphisms

A morphism in T is *analytic* iff it is right orthogonal to all structural morphisms.

Analytic Lawvere theory

Lawvere theory T is *analytic* iff

- T has simple automorphisms;
- structural and analytic morphisms form a factorization system in T.

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Rigid Lawvere theory

Lawvere theory T is *rigid* iff

- T is analytic;
- the actions of symmetric groups

$$S_n \times T(n,1) \longrightarrow T(n,1)$$

$$\langle \sigma, f \rangle \mapsto f \circ \pi_\sigma$$

are free on analytic morphisms.

Interpretations of Analytic Lawvere theories

An *analytic interpretation* of Lawvere theories $I : T \to T'$ is an interpretation of Lawvere theories that preserves analytic morphisms.

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Theorem

- The category of analytic Lawvere theories and analytic morphisms is equivalent the category of analytic monads.
- The category of rigid Lawvere theories and analytic morphisms is equivalent the category of polynomial monads.

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Theorem

The embedding of the category of analytic Lawvere theories into all Lawvere theories has a right adjoint which is monadic.

- $\vec{x}^n = x_1, \ldots, x_n$
- A term in context
$$t : \vec{x}^n$$
is *linear-regular* if every variable in $\vec{x}^n$ occurs in t exactly once.
- An equation
$$s = t : \vec{x}^n$$
is *linear-regular* iff both $s : \vec{x}^n$ and $t : \vec{x}^n$ are linear-regular terms in contexts.

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Linear-regular theory

A an equational theory T is *linear-regular* iff it has a set of linear-regular axioms.

- A linear-regular term in context

$$t(x_1, \ldots, x_n) : \vec{x}^n$$

is *flabby* in T iff

$$T \vdash t(x_1, \ldots, x_n) = t(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \vec{x}^n$$

for some $\sigma \in S_n$, $\sigma \neq id_n$.

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An example of a flabby term

In the theory T_{cm} of commutative monoids the term $x_1 \cdot x_2$ is flabby as

$$T \vdash x_1 \cdot x_2 = x_2 \cdot x_1$$

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Rigid theory

A an equational theory T is *rigid* iff it is linear-regular and has no flabby terms.

Linear-regular interpretation

An interpretation of equational theories $I : T \longrightarrow T'$ is *linear-regular* iff it interprets n-ary symbols in T as linear-regular terms in contexts $t : \vec{x}^n$ in T'.

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- The category of rigid theories and linear-regular interpretations is equivalent to the category of polynomial monads.

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Theorem[M.Bojanczyk, S.Szawiel, M.Z.]

The problem whether a finite set of linear-regular axioms defines a rigid equational theory is undecidable.

Monoids

The theory of monoids has two operations $\cdot$ and e, of arity 2 and 0, respectively, and equations

$$x_1 \cdot (x_2 \cdot x_3)) = (x_1 \cdot x_2) \cdot x_3, \quad x_1 \cdot e = x_1 = e \cdot x_1$$

By the form of these equations, this theory is strongly regular and hence rigid. In the Lawvere theory for monoids T_m a morphism

$$n \longrightarrow 1$$

is analytic iff it is of form

$$\langle x_1, \ldots x_n \rangle \mapsto x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)}$$

for some $\sigma \in S_n$.

Monoids with anti-involution

The theory of monoids with anti-involution in a theory of monoids that has an additional unary operation s and additional two axiom

$$s(x_1) \cdot s(x_2) = s(x_2 \cdot x_1), \quad s(s(x_1)) = x_1$$

This theory is not strongly regular but it is not difficult to see that it is rigid. In the Lawvere theory for monoids with anti-involution T_{mai} a morphism

$$n \longrightarrow 1$$

is analytic iff it is of form

$$\langle x_1, \ldots x_n \rangle \mapsto s^{\varepsilon_n}(x_{\sigma(1)}) \cdot \ldots \cdot s^{\varepsilon_n}(x_{\sigma(n)})$$

for some $\sigma \in S_n$ and $\varepsilon_i \in \{0, 1\}$.

Commutative monoids

The theory of commutative monoids is the theory of monoids with an additional axiom

$$m(x_1, x_2) = m(x_2, x_1)$$

Thus is it linear-regular but it is obviously not rigid. In the Lawvere theory for commutative monoids T_{cm} there is exactly one analytic morphism

$$n \longrightarrow 1$$

It is of form

$$\langle x_1, \ldots x_n \rangle \mapsto x_1 \cdot \ldots \cdot x_n$$

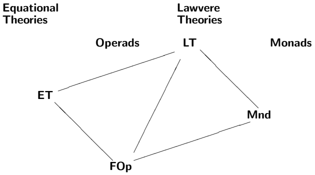
T_{cm} is the terminal analytic Lawvere theory.

Categories of Equational Theories (again)

Equational Theories **Lawvere Theories**

Operads **Monads**

Categories of Equational Theories (again)



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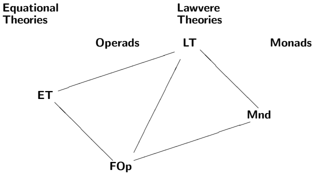

$\mathbb{F}$ - skeleton of the category of finite sets

$\iota_{\mathbb{F}} : \mathbb{F} \to Set$ - inclusion

$Lan_{\iota_{\mathbb{F}}} : Set^{\mathbb{F}} \to \mathbf{End}$ - equivalence of monoidal categories

$$\mathbf{FOp} \to \mathbf{Mnd}$$

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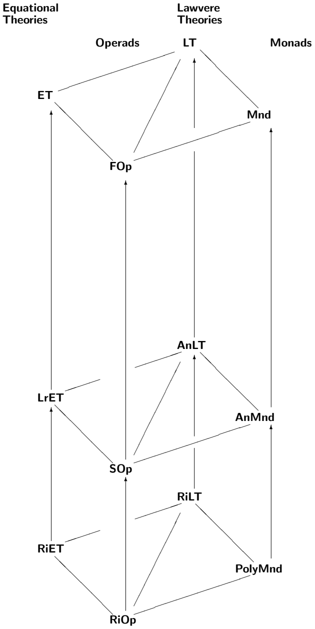

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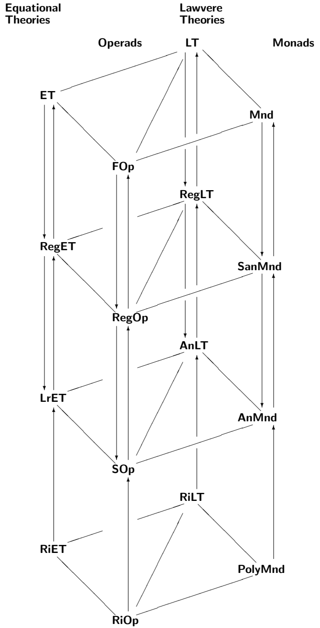

$\mathbb{F}$ - skeleton of the category of finite sets

$\iota_{\mathbb{F}} : \mathbb{F} \to Set$ - inclusion

$Lan_{\iota_{\mathbb{F}}} : Set^{\mathbb{F}} \to \mathbf{End}$ - equivalence of monoidal categories

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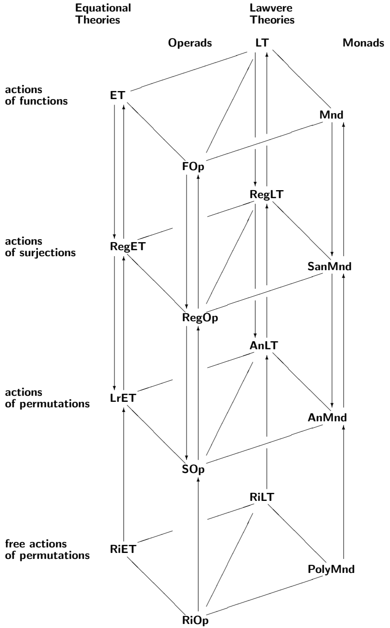

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- A term in context

$$t : \vec{x}^n$$

is *regular* if every variable in $\vec{x}^n$ occurs in t at least once.

- An equation

$$s = t : \vec{x}^n$$

is *regular* iff both $s : \vec{x}^n$ and $t : \vec{x}^n$ are regular terms in contexts.

A an equational theory T is *regular* iff it has a set of regular axioms.
An interpretation of equational theories $I : T \to T'$ is *regular* iff it interprets n-ary symbols in T as regular terms in contexts $t : \vec{x}^n$ in T'.

Examples of regular theories

- The theory of sup-semilattices: two operations $\vee$ and $\perp$, of arity 2 and 0, respectively, and equations

$$x_1 \vee (x_2 \vee x_3) = (x_1 \vee x_2) \vee x_3, \quad x_1 \vee \perp = x_1 = \perp \vee x_1,$$

$$x_1 \vee x_2 = x_2 \vee x_1, \quad x_1 \vee x_1 = x_1$$

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  It is the terminal regular theory.
- Monoids, monoids with involutions, abelian monoids, rigs without 0, commutative rigs without 0.
- Groups, rings, modules ARE NOT!

- $i : \mathbb{S} \longrightarrow \mathbb{F}$ is an inclusion of a subcategory with the same objects whose morphisms are surjections

$$\begin{tikzcd} Set^{\mathbb{F}} \arrow[rr, "Lan_{\iota_{\mathbb{F}}} \simeq"] & & \mathbf{End} \\ Set^{\mathbb{S}} \arrow[u, "Lan_i"] \arrow[rr, "Lan_{\iota_{\mathbb{S}}} \simeq"'] & & \mathbf{San} \arrow[u] \end{tikzcd}$$

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- **San** - the essential image of $Set^{\mathbb{S}} \to$ **End**; it is the category of semi-analytic functors and semi-cartesian natural transformations.
- the monoids in $Set^{\mathbb{S}}$ is the category of regular operads **RegOp**
- the monoids in **San** is the category of semi-analytic monads **SanMnd**.

- $\left[\begin{array}{c} Y \\ n \end{array} \right]$ - the set of injections from $\underline{n} = \{1, \ldots, n\}$ to the set Y

Regular operads and Semi-analytic monads

semi-analytic series, notation

- $\begin{bmatrix} Y \\ n \end{bmatrix}$ - the set of injections from $\underline{n} = \{1, \ldots, n\}$ to the set Y
- We have a right action of permutation group S_n

$$\begin{bmatrix} Y \\ n \end{bmatrix} \times S_n \longrightarrow \begin{bmatrix} Y \\ n \end{bmatrix}$$

$$\langle \vec{y}, \tau \rangle \mapsto \vec{y} \circ \tau$$

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- $A : \mathbb{S} \to Set$ functor then on A_n we have a left action of S_n

$$S_n \times A_n \longrightarrow A_n$$

$$\langle \tau, a \rangle \mapsto A(\tau)(a)$$

- Dividing $\left[\begin{array}{c} Y \\ n \end{array}\right] \times A_n$ by the relation

  $$\langle \vec{y} \circ \tau, a \rangle \sim \langle \vec{y}, A(\tau)(a) \rangle$$

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- ... and whole semi-analytic series

$$\hat{A}(Y) = \sum_{n \in \omega} \begin{bmatrix} Y \\ n \end{bmatrix} \otimes_n A_n$$

which IS functorial in Y!

- $f : X \longrightarrow Y$ - function, $[\vec{x}, a]$ an element of $\left[\begin{array}{c} X \\ n \end{array} \right] \otimes_n A_n$

- $f : X \to Y$ - function, $[\vec{x}, a]$ an element of $\left[\begin{array}{c} X \\ n \end{array} \right] \otimes_n A_n$
- We take the epi-mono factorization α, $\vec{y}$ of $f \circ \vec{x}$

$$\begin{tikzcd}
\underline{n} \arrow[r, tail, "\vec{x}"] \arrow[d, two heads, "\alpha"'] & X \arrow[d, "f"] \\
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- Thus we have defined $\hat{A}(f) : \hat{A}(X) \longrightarrow \hat{A}(Y)$

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- Thus we have a functor

$$(\hat{-}) : Set^{\mathbb{S}} \longrightarrow \mathbf{End}$$

Examples of semi-analytic functors

- The functor

$$\mathcal{P}_{\leq n} : Set \longrightarrow Set$$

associating to a set X the set of subsets of X with at most n-elements is not analytic, if $n > 2$, as it can be easily seen that it does not preserve weak pullbacks. However, it preserves pullbacks along monos and hence it is semi-analytic.

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- If U is a set, $n \in \omega$ then the functor $(-)^U_{\leq n} : Set \to Set$, associating to a set X the set of functions from U to X with an at most n-element image, is not analytic, if $|U| > n > 2$. Again it can be easily seen that it does not preserve weak pullbacks. However, it is semi-analytic.

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- The functor part of any monad on Set that comes from a regular equational theory (e.g. $\mathcal{P}_{<\omega}$) is semi-analytic.

Theorem

The following are three descriptions of the same (monoidal) category

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- the category **San** of semi-analytic functors, the essential image of the left Kan extension $Set^{\mathbb{S}} \longrightarrow End$;
- the essential image of the functor $(\hat{-}) : Set^{\mathbb{S}} \longrightarrow \mathbf{End}$;
- the category of finitary endofunctors on *Set* that preserve pullbacks along monos, with semi-cartesian natural transformations i.e. such that the naturality squares for monos are pullbacks. (= the category of finitary taut functors on *Set* E. G. Manes)

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The above monoidal category is equivalent (as a monoidal category) to

- the category $Set^{\$}$;

Projection morphisms

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Regular Lawvere theory

Lawvere theory T is *regular* iff

- T has simple automorphisms;
- projections and regular morphisms form a factorization system in T.

Interpretations of Regular Lawvere theories

A *regular interpretation* of Lawvere theories $I : T \longrightarrow T'$ is an interpretation of Lawvere theories that preserves regular morphisms.

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- the category **RegET** of regular equational theories and regular interpretations;
- the category **RegOp** of regular operads, i.e. monoids in $Set^{\mathbb{S}}$;
- the category **SanMnd** of semi-analytic monads, i.e. monoids in **San**;
- the category **RegLT** of regular Lawvere theories monads.

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Remark. A version of equivalence **RegET** $\simeq$ **SanMnd** is due to E. G. Manes (1998).

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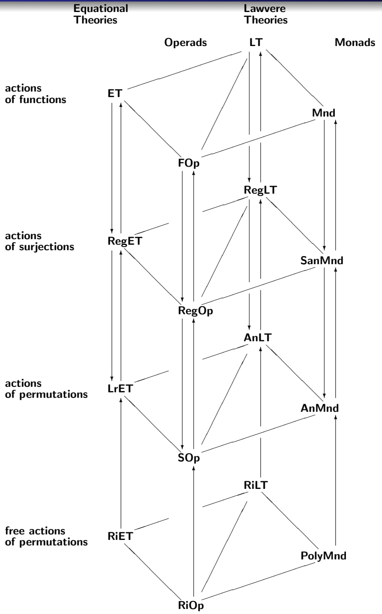

The end

Thank You for Your Attention!