Pataraia's Fixpoint Theorem

July 25, 2012

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A map $f : P \to P$ is called order preserving (or monotone) if for every pair of elements $x, y \in P$ if $x \leq y$ then $f(x) \leq f(y)$.

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Hence the least upper bound of $E(C)$ belongs to $E(C)$. Let m denote the least upper bound.

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- $m(\perp)$ is the least fixpoint of f.

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• f is Scott continuous implies that f is a monotone map.

THANK YOU DITO!