Some Modal Logics Arising from Subspaces of the Real Line

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Topological Semantics

MT44 Two topological semantics are introduced

- c-semantics $\dashrightarrow$ diamond is closure
- d-semantics $\dashrightarrow$ diamond is derivative (limit point operator)
- d-semantics is strictly more expressive than c-semantics; $\overline{A} = A \cup dA$

Theorem

The c-logic of any separable metrizable dense-in-itself (dii) space is **S4**.

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S4 is the c-logic of the real line $\mathbb{R}$, the space of rational numbers $\mathbb{Q}$ and the Cantor discontinuum **C**.

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Definition

The key component of defining the forcing relation for a given valuation:

$$x \models \Diamond\varphi \text{ iff } \forall U_x,\ \exists y \in U_x - \{x\},\ y \models \varphi \qquad \text{(Diamond version)}$$

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Theorem and Corollary (VS90)

T: **K4D** $=$ **K4** $+ \Diamond\top$ is the d-logic of any zero-dimensional separable dense-in-itself metrizable space.

C: **K4D** is the d-logic of both $\mathbb{Q}$ and **C**.

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Geometric approach to obtain $L_d(\mathbb{Q}) =$ **K4D**.

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Outline of Method

Goal

Obtain a copy of $\mathbb{Q}$ that allows for 'easy' utilization of results for Kripke frames.

Process

1. Define a dense strict linear order, $<$, without endpoints on the set of (finite) strings of nonzero integers, Σ.
2. By Cantor's theorem, $(\Sigma, <)$ and $\mathbb{Q}$ are (order-)isomorphic.
3. Equip Σ with the order topology, τ, induced by $<$.
   Recall a basis for τ is $\{(\sigma, \lambda) : \sigma, \lambda \in \Sigma\}$ where $(\sigma, \lambda) = \{\kappa \in \Sigma : \sigma < \kappa < \lambda\}$.

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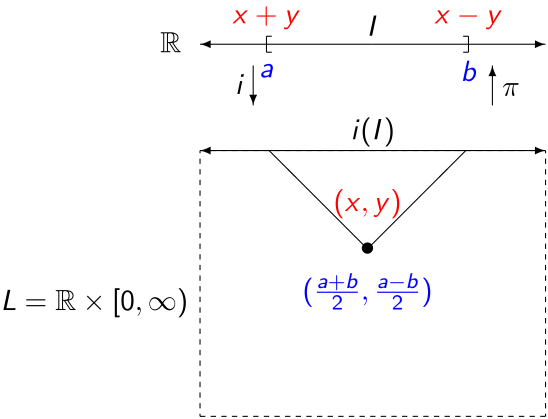

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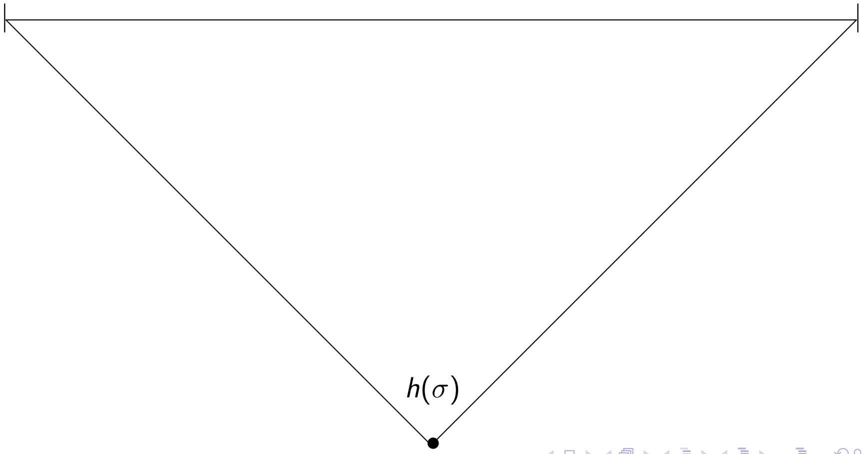
Define recursively $h : \Sigma \to L$ as depicted below:
Set $h(\Lambda) = (0, -1)$ and assume $h(\sigma)$ is defined.



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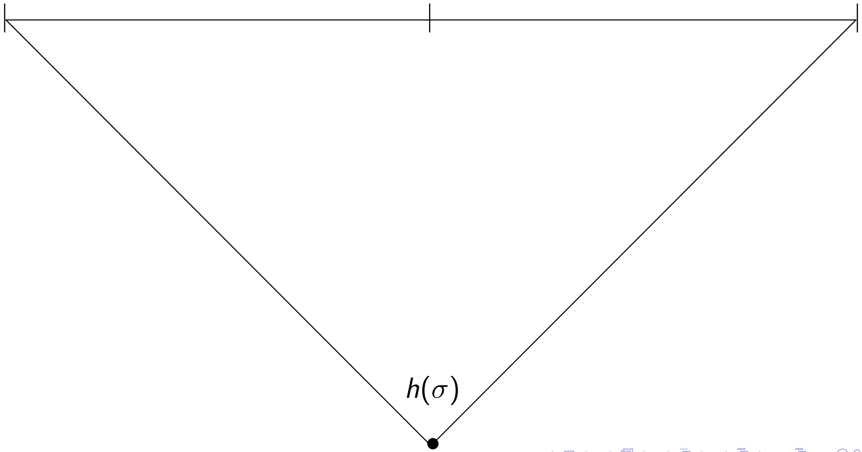

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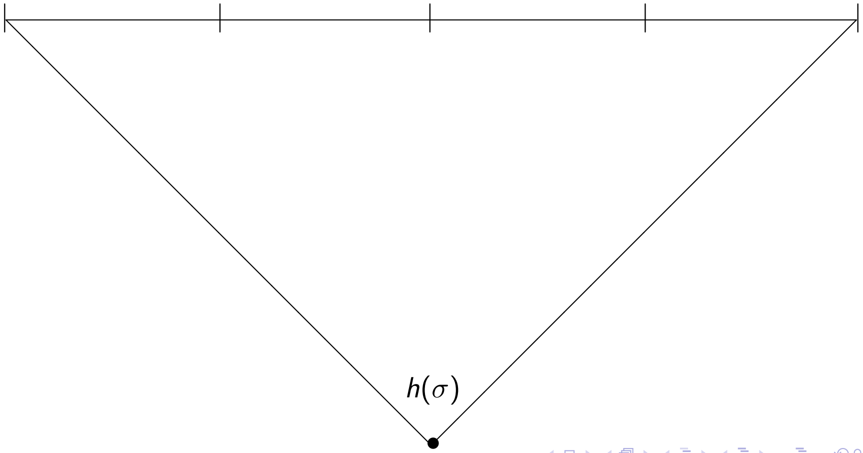

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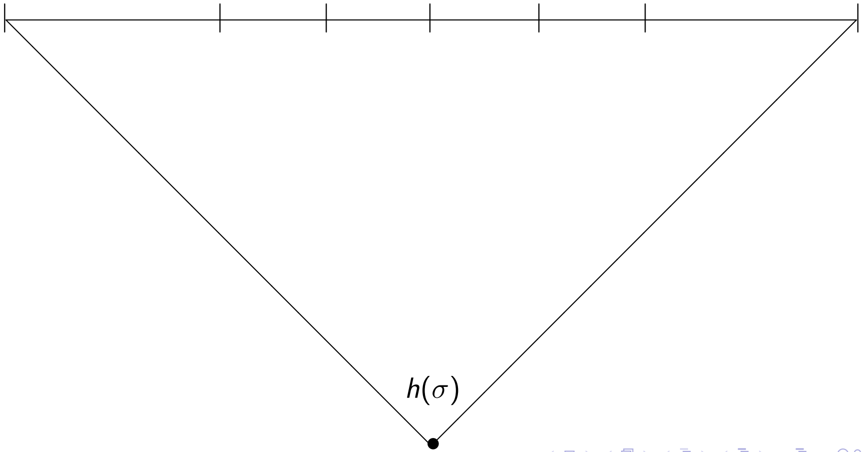

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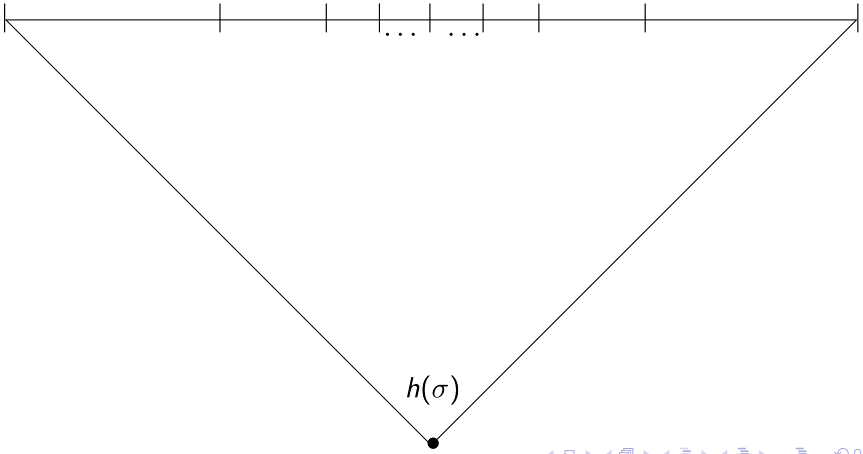

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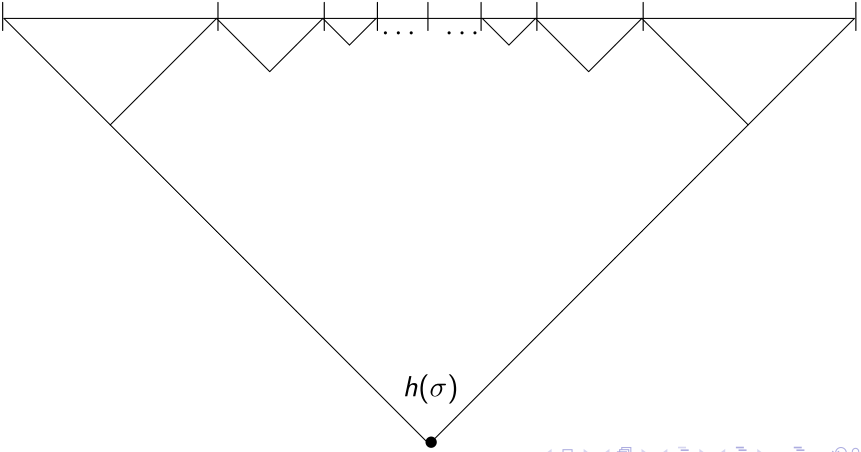

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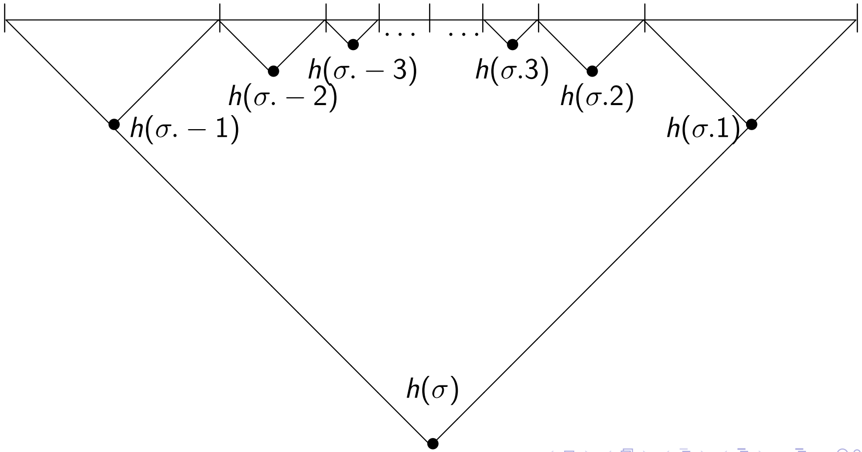
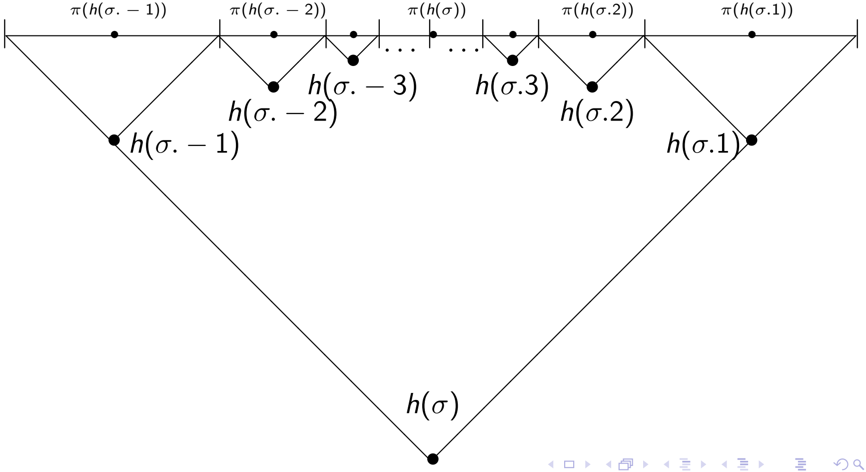


Ordering Σ

Define $<$ on Σ via the projection into $\mathbb{R}$ as depicted below:
$\sigma < \lambda$ iff $\pi(h(\sigma)) < \pi(h(\lambda))$ in $\mathbb{R}$.



Correct Maps

Definition (BEG05)

A d-morphism is a function $f : X \to W$ from a space (X, τ) to a frame (W, R), such that $\forall A \subseteq W$:

$$d(f^{-1}(A)) = f^{-1}(R^{-1}(A)).$$

Theorem (BEG05)

An onto d-morphism preserves validity; equivalently reflects refutation.

$$X \models \varphi \text{ implies } (W, R) \models \varphi. \qquad \text{(Preserve Validity)}$$

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Countable Rooted **K4**-Frames

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Let (W, R) be transitive, rooted and countable.
There are $X \subseteq \Sigma$ and onto d-morphism $f : X \to W$.
Hence, (W, R) is a d-morphic image of a subspace of $\mathbb{Q}$.

Corollary

Let $\mathcal{C}$ be a countable collection of countable rooted **K4**-frames, $\exists X \subseteq \mathbb{Q}$ so that $L_d(X) \subseteq L(\mathcal{C})$.

Remark

When using this method to realize subspaces of $\mathbb{Q}$:

- Completeness always holds.
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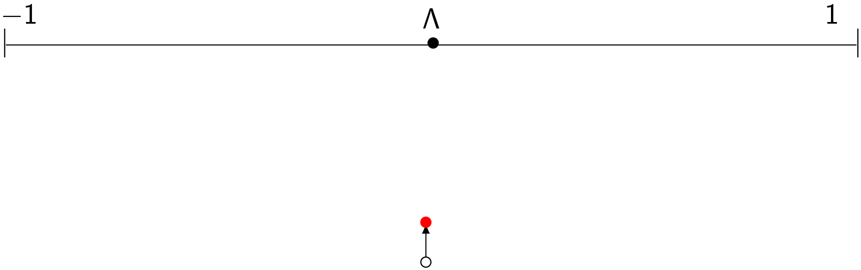

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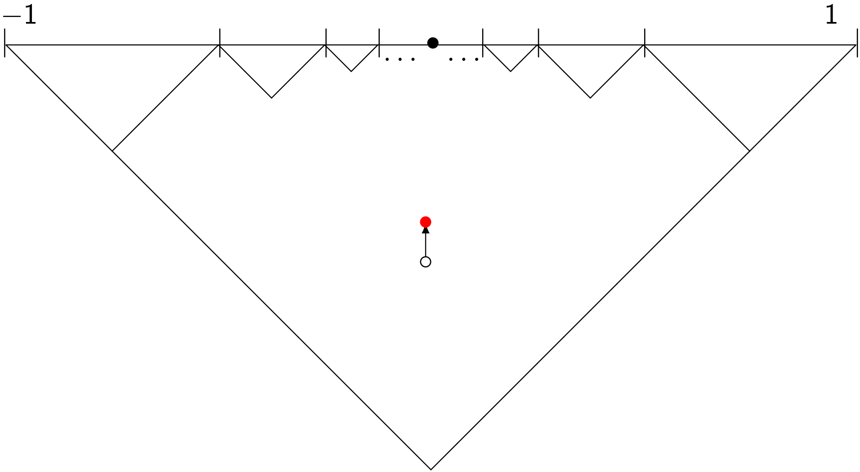

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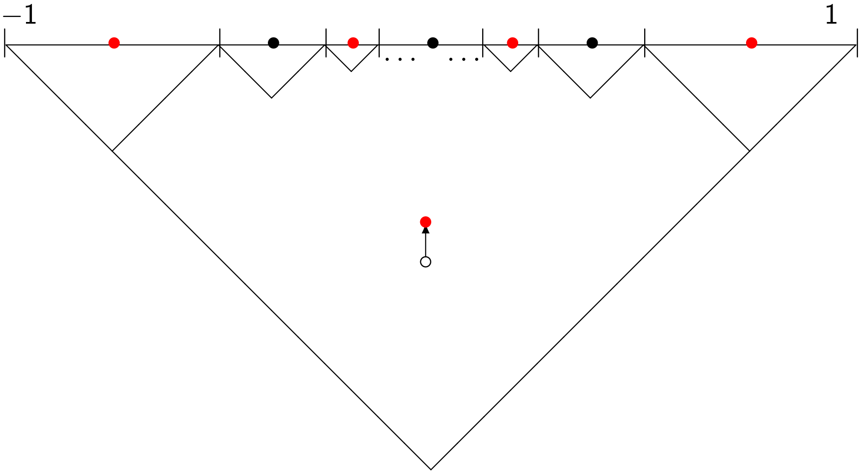

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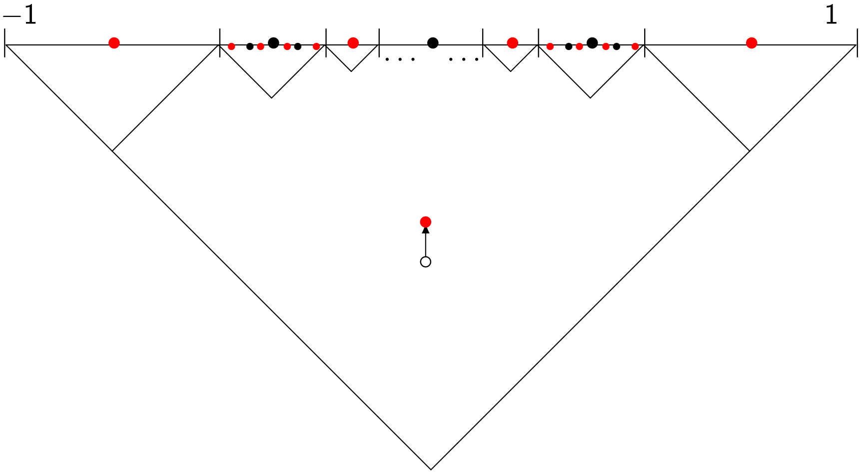

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Lemma

Let:

X be T_d,

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2. **L** containing **K4** and axiomatized by variable free formulas

$$\mathbf{K4D} = \mathbf{K4} + \Diamond\top$$
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$$\mathbf{K4\Xi}_n = \mathbf{K4} + \Diamond^{n}\Box\bot \to \Diamond\neg\Diamond^{+}\Box\bot$$

3. Arbitrary intersection of logics extending **K4** by variable free formulas; e.g. $\mathbf{GL} = \bigcap \mathbf{GL}_n, \bigcap \mathbf{K4\Delta}_n, \bigcap \mathbf{K4\Xi}_n$

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Some d-Logics Arising from Subspaces of $\mathbb{Q}$

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The following logics are the d-logic of some subspace of $\mathbb{Q}$.

1. **K4**
2. **L** containing **K4** and axiomatized by variable free formulas

$$\begin{aligned}
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More d-Logics Arising from Subspaces of $\mathbb{Q}$

Main Results

1. Subspaces of $\mathbb{Q}$ give rise to continuum many d-logics over K4.
2. There exist continuum many d-logics of subspaces of $\mathbb{Q}$ that are not finitely axiomatizable.
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Scattered Spaces, **GL** and $\mathbf{GL}_n$

Recall

1. X is scattered if every nonempty subspace has an isolated point.
   - X is scattered iff $\exists\alpha,\ d^{\alpha}(X) = \varnothing$.
   - If X is scattered then the isolated points, $\mathrm{Iso}(X)$, are dense.
2. $\mathbf{GL}_n = \mathbf{K4} + \Box^n\bot$

$$X \models \Box^n\bot \text{ iff } d^n(X) = \varnothing. \qquad \text{(n-scattered)}$$

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Weakly Scattered Spaces and wGL

Definition

- A T_d space X is weakly scattered if $\mathrm{Iso}(X)$ is dense; i.e. $\overline{\mathrm{Iso}(X)} = X$. E.g. $\beta(\mathbb{N})$.
- **wGL** $=$ **K4** $+ \Diamond^+\Box\bot$.

Results

wGL $\subsetneq$ **GL**.

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- A T_d space X is semi-scattered when $\mathrm{int}(\overline{\mathrm{Iso}X})$ is scattered.
- **sgl** $= \Box(\Box(p \vee \chi) \to (p \vee \chi)) \to \Box(p \vee \chi) \vee \chi$ where $\chi = \Diamond^+\Box^+\Diamond\top$ and **sGL** $=$ **K4** $+$ **sgl**

Theorem

1. $X \models$ **sgl** iff X is semi-scattered.
2. $X \models \Diamond^n\Box\bot \to \Diamond\neg\Diamond^+\Box\bot$ iff $d^n(\mathrm{Iso}X) \subseteq d(X - \overline{\mathrm{Iso}X})$.
3. **sGL** $= \bigcap$ $\mathbf{K4\Xi}_n$ (Recall $\mathbf{K4\Xi}_n =$ **K4** $+ \Diamond^n\Box\bot \to \Diamond\neg\Diamond^+\Box\bot$); so ... **sGL** is the d-logic of a subspace of $\mathbb{Q}$.
4. A finite frame $(W, R) \models$ **sgl** iff $(R^+)^{-1}(\mathrm{imax}W) - R^{-1}(\mathrm{rmax}W)$ is irreflexive.

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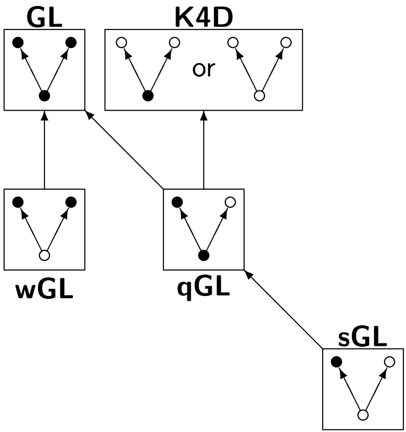
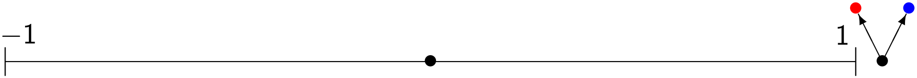
Two Forks Separate **GL**, **wGL**, **qGL**, **sGL** and **K4D**

A picture says it all ...



Pictures of Two Forks and Associated Subspace of $\mathbb{Q}$



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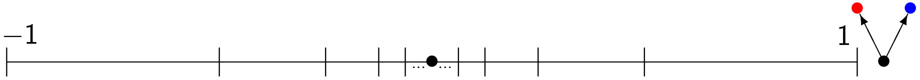

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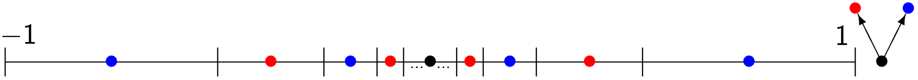

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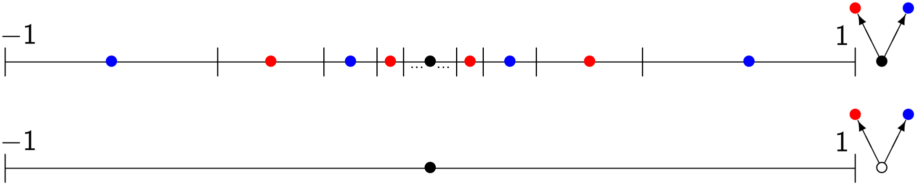

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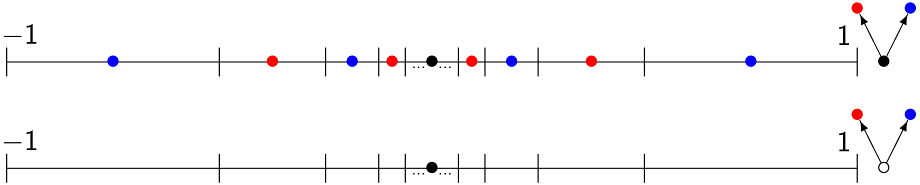

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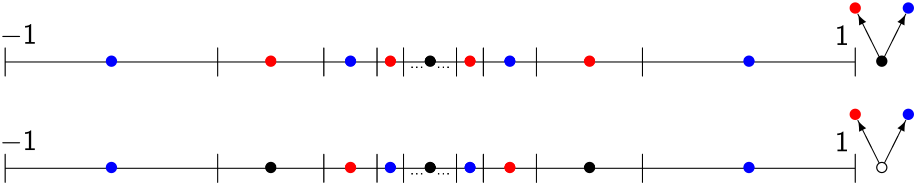

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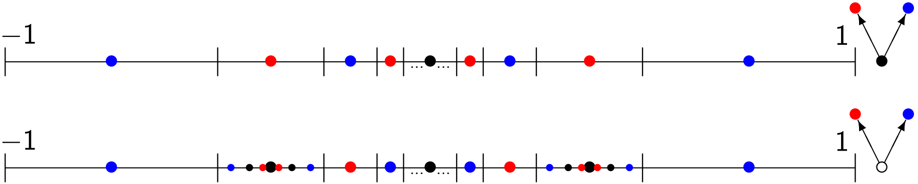

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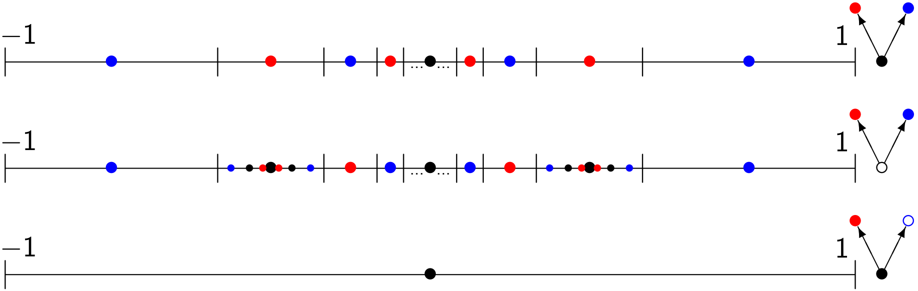

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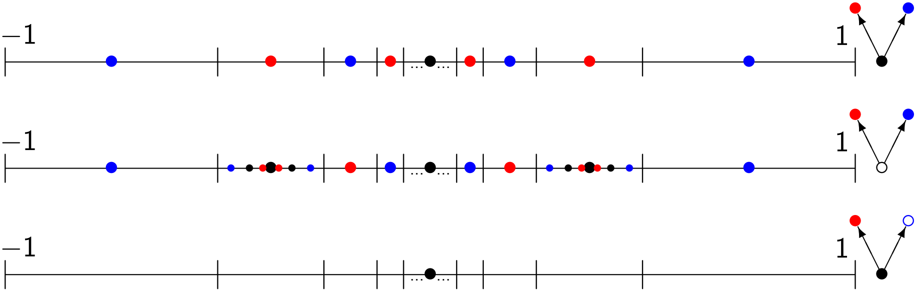

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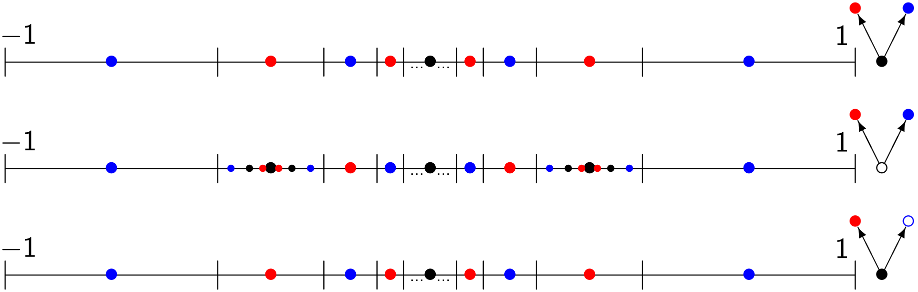

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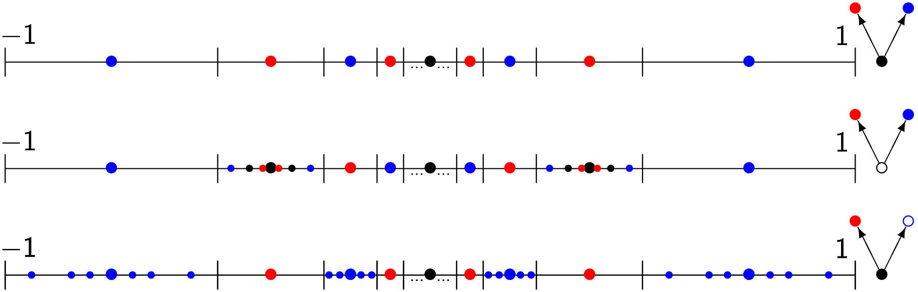

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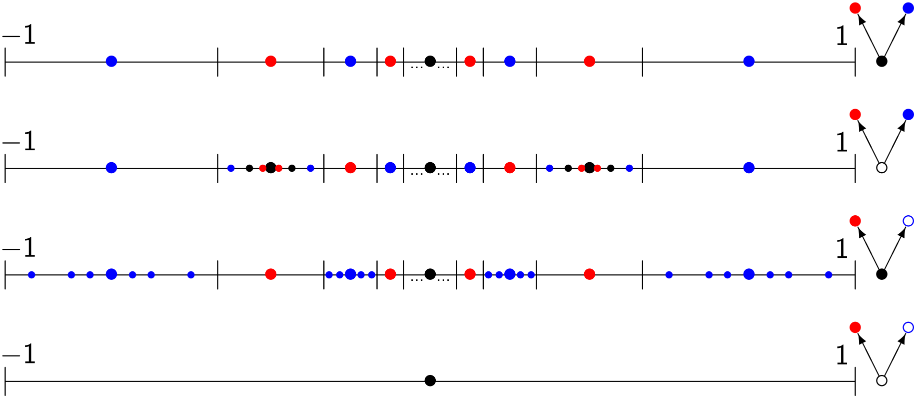

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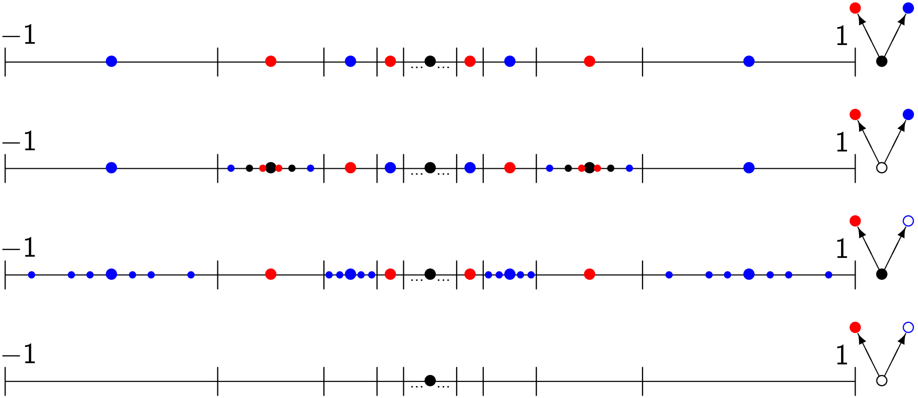

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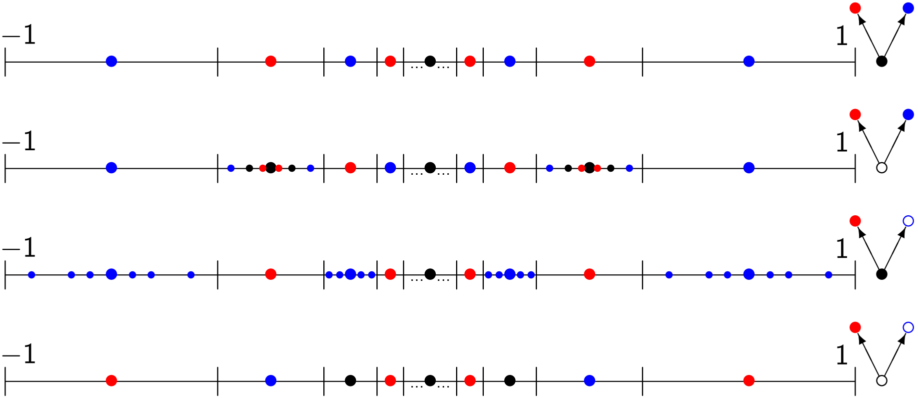

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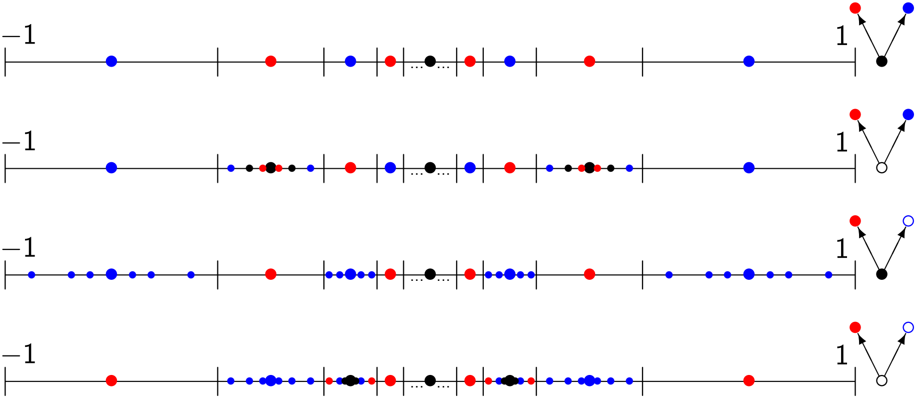

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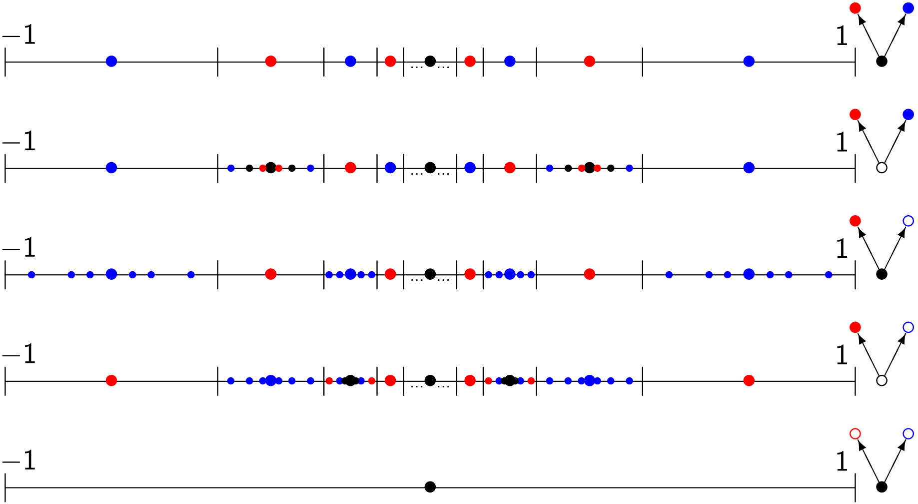

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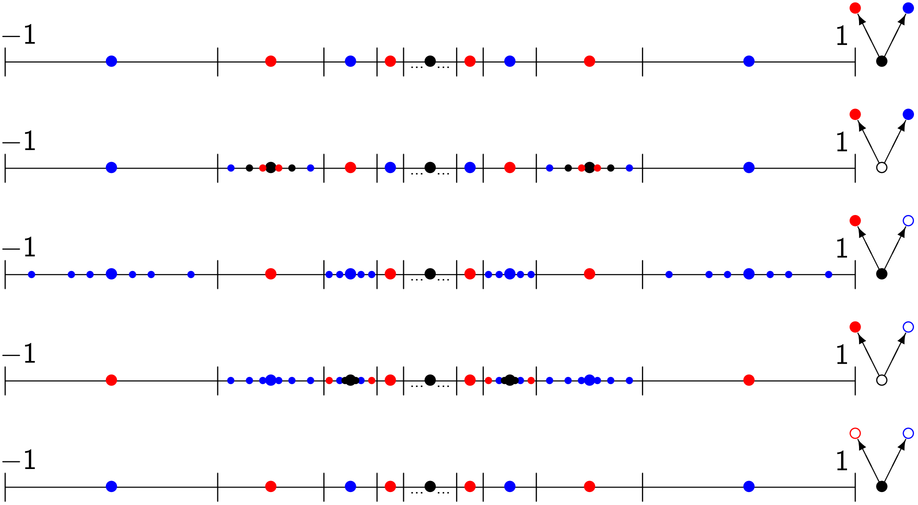

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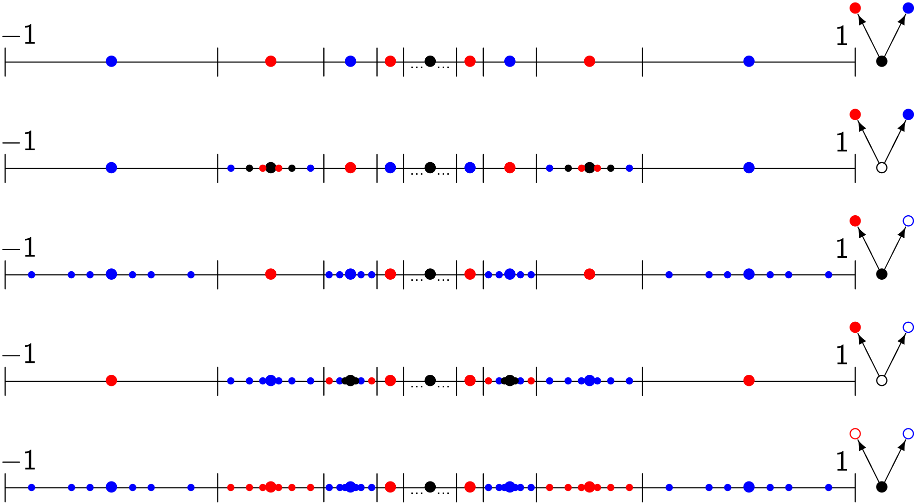

Finite Model Property

Definition (recall)

A logic $\mathbf{L}$ has the finite model property (FMP) provided any nontheorem φ of $\mathbf{L}$ is refuted on some finite $\mathbf{L}$-frame $\mathfrak{F}_\varphi$.

Theorem (CZ97)

$L(\mathfrak{G})$ does not have the FMP.

Put $\alpha_i = \Box^{i+1}\bot \wedge \Diamond^i \top$

$n \models \alpha_i$ iff $n = i$

$\mathfrak{G} \models \neg\mathbf{gl} \wedge \Diamond\alpha_i \to \neg\mathbf{gl} \wedge \Diamond\alpha_{i+1}$

Only $\omega \not\models \mathbf{gl}$

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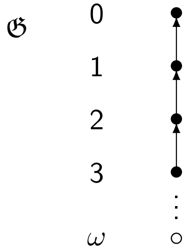

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An Interval of Logics Without FMP

Theorem (CZ97)

$$L_0 = \mathbf{K4} + \{\neg \mathbf{gl} \wedge \Diamond \alpha_i \to \neg \mathbf{gl} \wedge \Diamond \alpha_{i+1} : i \in \omega\}$$
$$I = [L_0, L(\mathfrak{G})]$$

1. No $L \in I$ has the FMP.
2. I is uncountable.
3. Infinitely many $L \in I$ are finitely axiomatizable.

Apply Construction

Apply construction to $\mathfrak{G}$ to build $X \subseteq \mathbb{Q}$.

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$L_d(X) \in I$ and so ... $L_d(X)$ does not have the FMP.

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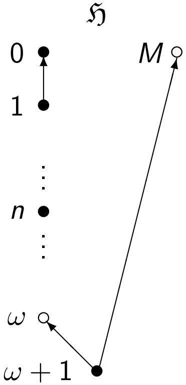
A Motivating Frame

Add two points to $\mathfrak{G}$:



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USING **sgl**:
ANOTHER INTERVAL OF LOGICS WITHOUT FMP

THEOREM

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L_1 &= \mathbf{K4} + \{\neg \mathbf{sgl} \wedge \Diamond \alpha_i \rightarrow \neg \mathbf{sgl} \wedge \Diamond \alpha_{i+1} : i \in \omega\} \\
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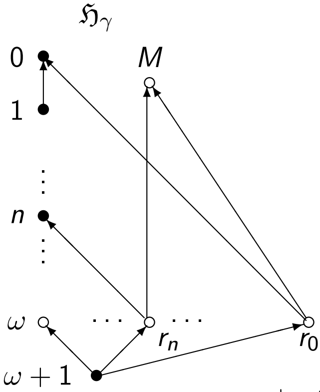

$\forall n \in \omega$,
r_n occurs exactly
when $n \in \gamma \subseteq \omega$

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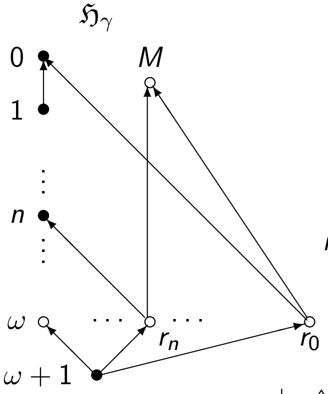

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Apply Construction

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$L_d(X_\gamma) \in J$ and so ... $L_d(X_\gamma)$ does not have the FMP.

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Theorem (Recall)

There exist continuum many d-logics of subspaces of $\mathbb{Q}$ that do not have the FMP.

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