Two papers: a scattered tale

T. Litak, U. of Leicester
(still, but not for much more)
Tbilisi, July 2012
...or how I learned to stop worrying and (reluctantly began to) love topos theory
Scattered toposes ♠

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Received 1 April 1999; received in revised form 1 July 1999
Communicated by I. Moerdijk

Abstract

A class of toposes is introduced and studied, suitable for semantical analysis of an extension of the Heyting predicate calculus admitting Gödel’s provability interpretation. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 03G30; 18B25

Keywords: Topos; Proof-intuitionistic logic; Heyting algebra; Fixed point
First steps in synthetic guarded domain theory: step-indexing in the topos of trees

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Abstract

We present the topos $S$ of trees as a model of guarded recursion. We study the internal dependently-typed higher-order logic of $S$ and show that $S$ models two modal operators, on predicates and types, which serve as guards in recursive definitions of terms, predicates, and types. In particular, we show how to solve recursive type equations involving dependent types. We propose that the internal logic of $S$ provides the right setting for the synthetic construction of abstract versions of step-indexed models of programming languages and program logics. As an example, we show how to construct a model of a programming language with higher-order store and recursive types entirely inside the internal logic of $S$.

The internal logic of $S$ is a standard many-sorted higher-order logic extended with modal operators on both types and terms. (Recall that terms in higher-order logic include both functions and relations, as the latter are simply Prop-valued functions.) This internal logic can then be used as a language to describe semantic models of programming languages with the features mentioned above. As an example which uses both recursively defined types and recursively defined relations in the $S$-logic, we present a model of $F_{\mu,\text{ref}}$, a call-by-value programming language with impredicative polymorphism, recursive types, and general ML-like references.

To situate our work in relation to earlier work, we now give a quick overview of the technical development of the present paper followed by a comparison to related work. We end the introduction with a summary of our contributions.
Two papers from different places...
...with different origins
and motivation
Esakia, Jibladze, Pataraia

- Intuitionistic Kuroda principle and related laws
- Kuznetsov-Muravitsky proof-intuitionistic logic KM
- Fixed point theorem for the Löb logic GL
- Simmons' point-free version of the Cantor-Bendixson derivative
- Johnstone's pioneering analysis of the topos-theoretic meaning of certain superintuitionistic principles (e.g., de Morgan)
- Papers of Blass and Freyd on well-orderings in toposes...
Birkedal, Mogelberg et al.
An abstract setting for...

- guarded recursive definitions (of predicates, dependent types...)
- step-indexed models of programming languages (with higher-order store, impredicative polymorphisms...)
- synthetic domain theory
- Banach-style, ultrametric approach to fixpoints (unique rather than least)
- ...

...
A lot of recent work in Theoretical CS...

- Escardo’s metric model of PCF
- Nakano (LiCS’00, TACS’01)
- Di Gianantonio, Miculan (FoSSaCS’04)
- Appel et al. (POPL’07)
- Krishnaswami and Benton (LiCS’11, ICFP’11, POPL’12) and Tabareau, Jabber et al. (TLDI’09, LiCS’12)
- Birkedal, Mogelberg et al. (LiCS’11, POPL’11, FiCS’10)...
  (all these papers use Löb-like modalities!)
The connection

- Copenhagen's “topos of trees” is a paradigm example of a scattered topos (even though, sadly, the APAL paper is not even quoted - seems almost completely unknown)

- Their “internal Banach fixed point theorem” strengthens that of Dito et al. - the main result in the last section of the APAL paper - for the special case of topos of trees...

- ... and this is just one of many other results they prove....
of course, they can build on an additional decade of developments

furthermore, they can also use additional facts true in the topos of trees but not in an arbitrary scattered topos

ey also fruitfully employ not just internal perspective on the modality (an operator on omega), but also external (an endofunctor on the topos)

missing in the Esakia et al. APAL paper
A sad aspect of the story: “road not taken” in the Tbilisi school

Still, the APAL paper unjustly overlooked despite its pioneering character

And still a lot of research which can be done!
Now for the details
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Int. modal frames

- Bozic & Dosen (SL'84)
  \([R]A \in \text{Up}(F)\) for any \(A \in \text{Up}F\)
  iff
  \(\leq ; R \subseteq R; \leq\)

- most common - a stronger condition
  \(\leq ; R ; \leq = R\)

- arises naturality via the Jonsson-Tarski construction

- harmless for validity
the latter also makes correspondence theory more natural

e.g., what is the counterpart of
\( \text{GL} := []([[]A \rightarrow A) \rightarrow [[]A?} \)

just like in the classical case!
transitivity + Noetheranity for \( R \)
(converse well-foundedness)
A very special case

- $R$ just the irreflexive (strict) part of $\leq$, i.e., $R = \leq - \Delta$

- This makes the derivative axiom true:

  $\Box A \rightarrow (B \rightarrow A) \lor B$

  or - equivalently -

  $\Box A \rightarrow (((B \rightarrow A) \rightarrow B) \rightarrow B)$

- In classical modal logic, always holds: corresponds to containment $\leq - \Delta \subseteq R$
But this way we also get the converse inclusion: \( R \subseteq \leq \)

This makes \( A \rightarrow []A \) valid.

This axiom is very exotic classically, much more harmless intuitionistically.

side note:
\[
A \rightarrow []A \land []([]A \rightarrow A) \rightarrow []A
\]
equivalent normally to
\[
([]A \rightarrow A) \rightarrow A
\]
equivalent normally to
\[
([]A \rightarrow A) \rightarrow []A
\]
In this way, we reconstructed the axioms for logic mHC:

- $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- $\Box A \rightarrow (((B \rightarrow A) \rightarrow B) \rightarrow B)$
- $A \rightarrow \Box A$

(the logic of all frames $(W, \leq)$ with $\Box$ interpreted wrt $<$, i.e., the strict intuitionistic ordes)
... but also for the logic \textbf{KM}:

\(\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box \Box B)\)

\(\Box A \rightarrow (((B \rightarrow A) \rightarrow B) \rightarrow B)\)

\(A \rightarrow \Box \Box A\)

\(\Box(\Box \Box A \rightarrow A) \rightarrow \Box A\)

(furthermore, we know we can merge these two)

(the logic of all Noetherian frames \((W, \leq)\) with \(\Box\) interpreted wrt <, i.e., the strict intuitionistic orders)
ultimate reaxiomatization of KM:

- \( [](A \rightarrow B) \rightarrow ([]A \rightarrow [])B \)
- \( []A \rightarrow (((B \rightarrow A) \rightarrow B) \rightarrow B) \)
- \( ([]A \rightarrow A) \rightarrow A \)

(the logic of all Noetherian frames \((W, \leq)\) with \(\[\]\) interpreted wrt <, i.e., the strict intuitionistic ordes)
Now for topological spaces...

- $[<]$, is easily seen to be induced by the Cantor-Bendixson (co-)derivative

$$\delta x := \{ A \in P(W) \mid \exists B \in T_x. B - \{x\} \subseteq A \}$$

- In the same way then mHC can be defined as the logic of all topological spaces, with $[]$ being interpreted as the co-derivative...

- ...and KM is the logic of all scattered ones (every non-empty subset contains an isolated point)
Can we be any more general than this?
YES WE CAN!
...not in the context of arbitrary Heyting algebras...

...but in arbitrary lattice-complete ones...
(also frames or locales depending what you want your morphisms to preserve and in what direction they should go)

point-free co-derivative: an idea of H. Simmons from early 1980's
Let $h, i$ be elements of a Heyting algebra $H$ s.t. $i \leq h$

$h$ is $i$-dense (or dense in $[i,T]$)
iff for all $j \in H$, $h \land j = i$ implies $j = i$ (Simmons)
iff exists $j \in H$ s.t. $h = (j \to i) \lor j$ (EJP)

point-free co-derivative

$\triangledown i := \bigwedge \{h \in H \mid i \leq h$ and $h$ is $i$-dense\} (Simmons)

$\triangledown i := \bigwedge \{(j \to i) \lor j \mid j \in H\} (EJP)$

remember the mHC axiom?

$\Box A \to (B \to A) \lor B$
Simmons: for T₀ spaces, coincides with the standard definition

If S is not T₀ then the constructed operations differ slightly from (\cdot)^{\uparrow}, \pi. The reason for this discrepancy is that for non-T₀-spaces the usual definition of isolated point does not quite capture the intended notion.
Can we push abstraction one more step further?
Propositional quantification!
ideas developed first by Leo in “Quantification in intuitionistic logic with provability smack”

Define an operator in QHC
(Heyting calculus with propositional quantifiers-C-H counterpart of Girard’s system F):
\[ \Diamond A := \forall P. (P \rightarrow A) \vee P \quad (P \text{ fresh for } A) \]

Always a normal modality satisfying the mHC laws

Which laws needed to ensure it is a KM modality? (i.e., that the Löb law holds as well for \( \Diamond \))

(Casari scheme) \[ \forall P. ((A \rightarrow \forall P A) \rightarrow \forall P A) \rightarrow \forall P A \]
also called “the relativized Kuroda principle”

\[(\forall P. \neg\neg A \to \neg\neg\forall P. A)\]
\[\leftrightarrow ((\neg \forall P. A) \to [\forall P. ((A \to \forall P. A) \to \forall P. A) \to \forall P. A])\]

is a theorem of QHC

\[(\forall P. \neg\neg A \to \neg\neg\forall P. A)\]
is the Kuroda principle
Recall the remark of Heyting [4, p.104] in connection with the formula $\neg\forall x p(x) \rightarrow \forall x \neg\neg p(x)$: "It is one of the most striking features of intuitionistic logic that the inverse implication does not hold, especially because the formula of the propositional calculus which results if we restrict $x$ to a finite set, is true". And further: "It has been conjectured [5, p.46] that the formula $\forall x \neg\neg p(x) \rightarrow \neg\neg\forall xp(x)$ is always true if $x$ ranges over a denumerable infinite species, but no way of proving the conjecture presents itself at present".
Using figurative style, we note that there is a curious analogy with the state of affairs in Modal logics above S4. Namely, the formula (Kur) plays the role of the McKisey formula (McK), the well-known formula of Dummett (Dum) behaves like the Minari formula

$$(Min) \quad \neg \forall x p \lor \{ \forall x[(p \rightarrow \forall x p) \rightarrow \forall x p] \rightarrow \forall x p \},$$

whereas the known connection $Grz = Dum + McK$ “transforms” into $Cas = Min + Kur.$
one place where QHC certainly lives in: Kripke-Joyal(-Beth) semantics of the Mitchell-Bénabou language in a topos

(a thought in passing: not the only place though any sound semantics for system F and extensions would do see, e.g., Girard’s “Proofs and Types” for more)
Theorem 2. The following conditions on an elementary topos $X$ are equivalent:

(i) the relativized Kuroda principle

$$\forall_x[(p(x) \Rightarrow \forall_y p(y)) \Rightarrow \forall_y p(y)] \Rightarrow \forall_x p(x)$$

holds in $X$;

(i') for any $\varphi$ and $p(x)$ with $\varphi \Rightarrow \forall_x p(x)$ one has

$$[\forall_x((p(x) \Rightarrow \varphi) \Rightarrow \varphi)] \Rightarrow [((\forall_x p(x)) \Rightarrow \varphi) \Rightarrow \varphi];$$

(ii) every closed subtopos of $X$ is $\bot$-scattered;

(iii) the Löb principle

$$(\Box \varphi \Rightarrow \varphi) \Rightarrow \varphi$$

holds in $X$;

(iv) the principle

$$[(\psi \Rightarrow \varphi) \Rightarrow \varphi] \Leftrightarrow [\Box \varphi \Rightarrow (\varphi \lor \psi)]$$

holds in $X$.

right-to-left is always the strong Löb, regardless of the actual definition of $\Box$...

...these are mostly syntactic derivations...
The Kuroda axiom easier to characterize

2. Scatteredness

**Theorem 1.** The following conditions on an elementary topos $\mathbf{X}$ are equivalent:

(i) the Kuroda principle

\[(\forall x \neg \neg p(x)) \Rightarrow \neg \neg \forall x p(x)\]  \hspace{1cm} (KP)

holds in $\mathbf{X}$;

(ii) the smallest dense subtopos $\text{sh}_{\neg \neg}(\mathbf{X})$ is open in $\mathbf{X}$;

(iii) $\mathbf{X}$ has a Boolean open dense subtopos;

(iv) interior of a dense subtopos of $\mathbf{X}$ is dense;

(v) $\neg \neg (\text{Boo})$ holds in $\mathbf{X}$. 
Two proposed characterizations for scatteredness

**Proposition 1.** A topos is scattered if and only if the internal lattice $\mathcal{N}$ of its nuclei is a Boolean algebra.

**Proposition 2.** A topos of sheaves on a topological space $X$ is scattered if and only if each of its nondegenerate subtoposes has a point.
Important examples

- Presheaves over well-founded orders
- Sheaves over scattered spaces
The main result in the last section of EJP APAL...

- ... another Dito's fixpoint theorem
- in fact, it does not even mention the modality
- ...or scatteredness, for that matter
The notion of an unchanging map

Let us call a map \( f : X \to X' \) unchanging if the equivalence relation

\[ R = \{(x, y) \in X \times X \mid f(x) = f(y)\} \]

is a dense element of the lattice \([\text{diagonal}(X), X \times X]\). (One might call equivalence relations \( R \) with this property undistinguishing.) In other words, \( f \) is unchanging if one has

\[ \forall_{x, y \in X} (fx = fy \Rightarrow x = y) \Rightarrow x = y. \]

Clearly any constant map, i.e. one which factors through a subterminal object, is unchanging. In a Boolean topos the converse also holds, however this is far from
It is certainly unique (subterminal)...  
...and pretty close to existing (max subterminal)...

if, e.g., X is injective, you’re lucky: Fix(f) is actually a global element...  
...but not always exactly exists
Connection with scatteredness
and the Löb law

Definition. Let $f : X \rightarrow X$ be an endomorphism in a scattered topos satisfying

$$\forall_{x,y \in X} (x = y) \Rightarrow f(x) = f(y).$$

(call it “internally contractive”?)

Internal contractiveness in a scattered topos implies being unchanging.
Theorem 2. The following conditions on an elementary topos $X$ are equivalent:

(i) the relativized Kuroda principle

$$\forall_x [(p(x) \Rightarrow \forall_y p(y)) \Rightarrow \forall_y p(y)] \Rightarrow \forall_x p(x) \quad \text{(rKP)}$$

holds in $X$;

(i') for any $\varphi$ and $p(x)$ with $\varphi \Rightarrow \forall_x p(x)$ one has

$$[\forall_x ((p(x) \Rightarrow \varphi) \Rightarrow \varphi)] \Rightarrow [((\forall_x p(x)) \Rightarrow \varphi) \Rightarrow \varphi];$$

(ii) every closed subtopos of $X$ is $\bot$-scattered;

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$$\Box \varphi \Rightarrow \varphi \Rightarrow \varphi$$

holds in $X$;

(iv) the principle

$$[(\psi \Rightarrow \varphi) \Rightarrow \varphi] \Leftrightarrow [\Box \varphi \Rightarrow (\varphi \lor \psi)]$$

holds in $X$.

right-to-left is always the strong Löb, regardless of the actual definition of $[]$

...now let $\varphi$ be $x=y$ and let $\psi$ be $fx = fy$...
Corollary: any $[] : \Omega \to \Omega$ satisfying the strong Löb law would do

but even being unchanging via being $\triangleright$-internally contractive does not guarantee that the fixpoint is a global element

a beautiful counterexample in the topos of presheaves on $\omega + 1$
...fast forward to Copenhagen...
First steps in synthetic guarded domain theory: step-indexing in the topos of trees

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The internal logic of $S$ is a standard many-sorted higher-order logic extended with modal operators on both types and terms. (Recall that terms in higher-order logic include both functions and relations, as the latter are simply Prop-valued functions.) This internal logic can then be used as a language to describe semantic models of programming languages with the features mentioned above. As an example which uses both recursively defined types and recursively defined relations in the $S$-logic, we present a model of $F_{\mu, \text{ref}}$, a call-by-value programming language with impredicative polymorphism, recursive types, and general ML-like references.

To situate our work in relation to earlier work, we now give a quick overview of the technical development of the present paper followed by a comparison to related work. We end the introduction with a summary of our contributions.
Birkedal, Mogelberg et al.
An abstract setting for...

- guarded recursive predicates
- guarded recursive dependent types
- step-indexed models of programming languages
- synthetic domain theory
- Banach-style, ultrametric approach to fixpoints (unique rather than least)
- ...

...
A lot of recent work in Theoretical CS

- Escardo’s metric model of PCF
- Nakano (LiCS’00, TACS’01)
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- Birkedal, Mogelberg et al. (LiCS’11, POPL’11, FiCS’10)
Topos of trees:
topos of presheaves on omega
(ω starts at 1 for them - they want numbers to encode downsets too)

Obviously scattered!

It’s possible to describe things pretty concretely
A subobject $A$ of $X$ is a family of subsets $A(n) \subseteq X(n)$ such that $r_n(A(n + 1)) \subseteq A(n)$. The subobject classifier has $\Omega(n) = \{0, \ldots, n\}$ and restriction maps $r_n(x) = \min(n, x)$. The characteristic morphism $\chi_A : X \to \Omega$ maps $x \in X(n)$ to the maximal $m$ such that $x|_m \in A(m)$ if such an $m$ exists and 0 otherwise.

The natural numbers object $N$ in $\mathcal{S}$ is the constant set of natural numbers.

An operator on predicates. There is a morphism $\triangleright : \Omega \to \Omega$ mapping $n \in \Omega(m)$ to $\min(m, n + 1)$. By setting $\chi_{\triangleright A} = \triangleright \circ \chi_A$ there is an induced operation on subobjects, again denoted $\triangleright$. This operation, which we also call
As they work in a very specific setting, they can afford a stronger “internal Banach theorem”

**Definition 2.6.** The predicate $\operatorname{Contr}$ on $Y^X$ is defined in the internal logic by

$$\operatorname{Contr}(f) \overset{\text{def}}{\iff} \forall x, x' : X. \vdash (x = x') \rightarrow f(x) = f(x').$$

**Theorem 2.7** (Internal Banach Fixed-Point Theorem). The following holds in $S$:

$$(\exists x : X. \top) \land \operatorname{Contr}(f) \rightarrow \exists！x : X. f(x) = x.$$  

**Lemma 2.8.** The following holds in $S$:

$$\operatorname{Contr}(f) \rightarrow \exists n : N. \forall x, x' : X. f^n(x) = f^n(x').$$
But the Copenhagen paper employs also an external perspective on modality (Curry-Howard-Lambek semantics for extensions of strong Löb)

The ▶ endofunctor. Define the functor ▶ : S → S by ▶ X(1) = {★} and ▶ X(n + 1) = X(n). This functor, called later, has a left adjoint (so ▶ preserves all limits) given by ◁ X(n) = X(n + 1). Since limits are computed

There is a natural transformation next_X : X → ▶ X whose 1st component is the unique map into {★} and whose (n + 1)st component is r_n.

Since ▶ preserves finite limits, there is always a morphism

\[ J : ▶(X → Y) → (▶ X → ▶ Y). \]
The internal modality induced by the external one later is connected to the functor, since there is a pullback diagram.

\[
\begin{array}{ccc}
\triangleright m & \rightarrow & \triangleright A \\
\downarrow & & \downarrow m \\
X & \xrightarrow{\text{next}_X} & X
\end{array}
\]

for any subobject \( m: A \rightarrow X \).
plays the same role as the “shrinking” delay endofunctor on complete ultrametric spaces

in fact, bisected complete bounded ultrametric spaces live inside the topos of trees as “total objects” - those whose restriction maps are all surjective
allows guarded recursive dependent types

also, allows not only (external) notion of contractivity for morphisms but also for functors (locally contractive ones)

in the topos of trees, we can solve domain equations for locally contractive functors
thus, two notions of contractivity:
  one in terms of the internal modality
  another analogous to ultrametric spaces
  (factoring through the delay endofunctor)

the second stronger than the first

equivalent for total and inhabited objects
  not in general
More general approaches:
(sketch) for presheaves over well-founded frames:
take the limit of the diagram induced by all proper successors?

Birkedal et al. choose the approach of Di Gianantonio, Miculan: sheaves over well-founded Heyting algebras (using the notion of a well-founded base)
even more abstract approach in the journal version of the LiCS paper

Definition 6.1. A model of guarded recursive terms is a category $\mathcal{E}$ with finite products together with an endofunctor $\triangleright: \mathcal{E} \to \mathcal{E}$ and a natural transformation $\text{next}: \text{id} \to \triangleright$ such that

- for every morphism $f: \triangleright X \to X$ there exists a unique morphism $h: 1 \to X$ such that $f \circ \text{next} \circ h = h$.
- $\triangleright$ preserves finite limits

a small difference with C-H for strong Löb: limits, not just products (because of dependent types)
To return to topos of trees and related ones

- other propositional and modal principles inherent in intended models of guarded recursion...
- ...and their computational interpretation?
...implicitly present already in Nakano's (LICS 2000) subtyped polymorphic λ-calculus...

Figure 2. The subtyping rules of $\lambda\cdot\mu$
usually, our models (like the topos of trees) are not only well-founded/Noetherian...

but also linearly ordered...
\[(A \rightarrow B) \lor (B \rightarrow A)\]  (GD - Gödel-Dummet)

... and the modality is interpreted as the strict part of intuitionistic poset order
\[\triangleright A \rightarrow (B \rightarrow A) \lor B\]  (CB - Cantor-Bendixson)
\[\triangleright A \rightarrow (((B \rightarrow A) \rightarrow B) \rightarrow B)\]

(valid internally, but are they valid externally?)
The combination of SL, GD and CB allows to derive that fishy Nakano thing.

The computational meaning of GD:

- Danos-Krivine ‘03, “Disjunctive tautologies as synchronization schemes”
- A recent work of Hirai: Curry-Howard for GD in terms of wait-free synchronization.
Computational meaning of CB?

- only my own preliminary work
- restricted form of catch-and-throw calculus of Crolard
- restricted (delimited??) form of continuations?
4.2 Typing the catch and throw operators

Let us now use the naming rules to derive type judgments for the $\lambda\mu\nu\tau$-terms

\textit{throw} $\alpha \ t$ and \textit{catch} $\alpha \ t$.

- We recall that \textit{catch} $\alpha \ t = \mu\alpha[\alpha]t$:

\[
\begin{align*}
t &: \Gamma \vdash \Delta, A^\alpha; A \\
[\alpha]t &: \Gamma \vdash \Delta, A^\alpha; A \\
\mu\alpha[\alpha]t &: \Gamma \vdash \Delta; A
\end{align*}
\]

- We recall that \textit{throw} $\alpha \ t = \mu\beta[\alpha]t$ where $\beta$ does not occur free in $t$:

\[
\begin{align*}
t &: \Gamma \vdash \Delta; A \\
[\alpha]t &: \Gamma \vdash \Delta, A^\alpha; A \\
[\alpha]t &: \Gamma \vdash \Delta, A^\alpha, B^\beta; A \\
\mu\beta[\alpha]t &: \Gamma \vdash \Delta, A^\alpha; B
\end{align*}
\]

Hence, we are now able to type the native \textit{throw} and \textit{catch} operators.

\textit{The catch rule}

\[
\begin{align*}
t &: \Gamma \vdash \Delta, A^\alpha; A \\
\text{catch} \ \alpha \ t &: \Gamma \vdash \Delta; A
\end{align*}
\]

\textit{The throw rule}

\[
\begin{align*}
t &: \Gamma \vdash \Delta; A \\
\text{throw} \ \alpha \ t &: \Gamma \vdash \Delta, A^\alpha; B
\end{align*}
\]

leave catch as it is..

...but add guardedness condition to throw
what is the programming power/importance of the language incorporating all these constructs? (P-A Melliès remark: for principles that are valid only internally, it seems you should be looking at properties of program testing rather than the programming language itself)

Can we prove some additional important predicate principles valid in the topos of trees - or related ones??
case study - one needs to move from the original topos of trees to a more complex topos of sheaves to tackle countable non-determinism.

Lars’ question: what is the statement of M-B language valid in sheaves on omega_1 but not on omega which makes things go through?

my question: would it have any computational interpretation? See, e.g., Krivine’s “Dependent choice, quote and clock...”
To hear more about the topos of trees...

- AiML 2012 invited talk of Lars
- Rasmus’ talk at the farewell Wessex seminar for Dirk, Nick and myself
- Imperial College Aug 29
The importance of being scattered

- We can agree Georgia is the best place in the world to understand it.

- Just wait for the morning after the final banquet...