Two papers: a scattered tale

T. Litak, U. of Leicester
(still, but not for much more)
Tbilisi, July 2012

...or how I learned to stop worrying
and (reluctantly began to)
love topos theory

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Scattered toposes ☆

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Abstract

A class of toposes is introduced and studied, suitable for semantical analysis of an extension of the Heyting predicate calculus admitting Gödel's provability interpretation.

MSC: 03G30; 18B25

Keywords: Topos; Proof-intuitionistic logic; Heyting algebra; Fixed point

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First steps in synthetic guarded domain theory: step-indexing in the topos of trees

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Abstract

We present the topos $\mathcal{S}$ of trees as a model of guarded recursion. We study the internal dependently-typed higher-order logic of $\mathcal{S}$ and show that $\mathcal{S}$ models two modal operators, on predicates and types, which serve as guards in recursive definitions of terms, predicates, and types. In particular, we show how to solve recursive type equations involving dependent *types. We propose that the internal logic of $\mathcal{S}$ provides the right setting for the synthetic construction of abstract versions of step-indexed models of programming languages and program logics. As an example, we show how to construct a model of a programming language with higher-order store and recursive types entirely inside the internal logic of $\mathcal{S}$.*

The internal logic of $\mathcal{S}$ is a standard many-sorted higher-order logic extended with modal operators on both types and terms. (Recall that terms in higher-order logic include both functions and relations, as the latter are simply Prop-valued functions.) This internal logic can then be used as a language to describe semantic models of programming languages with the features mentioned above. As an example which uses both recursively defined types and recursively defined relations in the $\mathcal{S}$-logic, we present a model of $\mathsf{F}_{\mu,\mathsf{ref}}$, a call-by-value programming language with impredicative polymorphism, recursive types, and general ML-like references.

To situate our work in relation to earlier work, we now give a quick overview of the technical development of the present paper followed by a comparison to related work. We end the introduction with a summary of our contributions.

Two papers from different places...

...with different origins
and motivation

Esakia, Jibladze, Pataraia

- intuitionistic Kuroda principle and related laws
- Kuznetsov-Muravitsky proof-intutionistic logic KM
- fixed point theorem for the Löb logic GL
- Simmons' point-free version of the Cantor-Bendixson derivative
- Johnstone's pioneering analysis of th… theoretic meaning of certain superintuitionistic principles (e.g.

  topos correspondence theory?
- papers of Blass and Freyd on well-orderings in toposes...

Birkedal, Mogelberg et al.

An abstract setting for...

- guarded recursive definitions (of predicates, dependent types...)
- step-indexed models of programming languages (with higher-order store, impredicative polymorphisms...)
- synthetic domain theory
- Banach-style, ultrametric approach to fixpoints (unique rather than least)
- ...

A lot of recent work in Theoretical CS...

- Escardo's metric model of PCF
- Nakano (LiCS'00, TACS'01)
- Di Gianantonio, Miculan (FoSSaCS'04)
- Appel et al. (POPL'07)
- Krishnaswami and Benton (LiCS'11, ICFP'11, POPL'12) and Tabareau, Jabber et al. (TLDI'09, LiCS'12)
- Birkedal, Mogelberg et al. (LiCS'11, POPL'11, FiCS'10)...
  (all these papers use Löb-like modalities!)

The connection

- Copenhagen's "topos of trees" is a paradigm example of a scattered topos (even though, sadly, the APAL paper is not even quoted - seems almost completely unknown)
- Their "internal Banach fixed point theorem" strengthens that of Dito et al. - the main result in the last section of the APAL paper - for the special case of topos of trees...
- ... and this is just one of many other results they prove....

- of course, they can build on an additional decade of developments
- furthermore, they can also use additional facts true in the topos of trees but not in an arbitrary scattered topos
- they also fruitfully employ not just internal perspective on the modality (an operator on omega), but also external (an endofunctor on the topos)

missing in the Esakia et al. APAL paper

- A sad aspect of the story:
  "road not taken" in the Tbilisi school

- Still, the APAL paper unjustly overlooked despite its pioneering character

- And still a lot of research which can be done!

Now for the details

	Kripke frames	top. spaces	algebras
int. prop. logic	$F = (W,\leq)$ $Up(F)$	$T = (W, \tau)$	Heyting algebras
int. normal modal logic	$F = (W,\leq,R)$ additional conditions needed to ensure $[R]A \in UpF$	$T = (W, \tau, \delta)$ $\delta: W \to PP(W)$ $[\delta]A:=\{w \mid A\in\delta w\}$ $A\in T \Rightarrow [\delta]A\in T$	Heyting algebras with operators

Int. modal frames

- Bozic & Dosen (SL'84)
  $[R]A \in Up(F)$ for any $A \in UpF$
  iff
  $\leq ; R \subseteq R; \leq$
- most common - a stronger condition
  $\leq ; R ; \leq = R$
- arises naturality via the Jonsson-Tarski construction
- harmless for validity

- the latter also makes correspondence theory more natural
- e.g., what is the counterpart of GL := [](([]A -> A) -> []A?
- just like in the classical case! transitivity + Noetheranity for R (converse well-foundedness)

A very special case

- R just the irreflexive (strict) part of ≤, i.e., R = ≤ - Δ
- This makes the derivative
  []A -> (B -> A) v B
  or - equivalently -
  []A -> (((B -> A) -> B) -> B)

A classical tautology

- In classical modal logic, always holds:
  corresponds to containment ≤ - Δ ⊆ R

- But this way we also get the converse inclusion: R ⊆ ≤

- This makes A -> []A valid

- This axiom is very exotic classically much more harmless intuitionistically

- side note:
  A -> []A & [](([]A -> A) -> []A
  equivalent normally to
  ([]A -> A) -> A
  equivalent normally to
  ([]A -> A) -> []A

- In this way, we reconstructed the axioms for logic mHC:

- [](A -> B) -> ([]A -> []B)

- []A -> (((B -> A) -> B) -> B)

- A -> []A

(the logic of all frames (W, ≤) with [] interpreted wrt <, i.e., the strict intuitionistic ordes)

- ... but also for the logic KM:

- [](A -> B) -> ([]A -> []B)

- []A -> (((B -> A) -> B) -> B)

- A -> []A

- []([]A -> A) -> []A

furthermore, we know we can merge these two

(the logic of all Noetherian frames (W, ≤) with [] interpreted wrt <, i.e., the strict intuitionistic ordes)

- ultimate reaxiomatization of KM:
- [](A -> B) -> ([]A -> []B)
- []A -> (((B -> A) -> B) -> B)
- ([]A -> A) -> A

(the logic of all Noetherian frames (W, ≤) with [] interpreted wrt <, i.e., the strict intuitionistic ordes)

Now for topological spaces...

- [<] is easily seen to be induced by the Cantor-Bendixson (co-)derviative
- $\delta x := \{ A \in P(W) \mid \exists B \in \tau x.\ B - \{x\} \subseteq A \}$
- In the same way then mHC can be defined as the logic of all topological spaces, with [] being interpreted as the co-derivative...
- ...and KM is the logic of all scattered ones (every non-empty subset contains an isolated point)

Can we be any more general than this?


YES WE CAN!

- ... not in the context of arbitrary Heyting algebras...
- ...but in arbitrary lattice-complete ones...
(also frames or locales depending what you want your morphisms to preserve and in what direction they should go)
- point-free co-derivative: an idea of H. Simmons from early 1980's

- Let h, i be elements of a Heyting algebra H s.t. i ≤ h
- h is i-dense (or dense in [i,T])
  iff for all j ∈ H, h ∧ j = i implies j = i (Simmons)
  iff exists j ∈ H s.t. h = (j -> i) v j (EJP)
- point-free co-derivative
  ▷i := ⋀{h ∈ H | i ≤ h and h is i-dense} (Simmons)

  ▷i := ⋀{(j -> i) v j | j ∈ H} (EJP)

remember the mHC axiom?
[]A -> (B -> A) v B

- Simmons: for T0 spaces, coincides with the standard definition

If S is not T_0 then the constructed operations differ slightly from $(\cdot)^i$, π. The reason for this discrepancy is that for non-T_0-spaces the usual definition of isolated point does not quite capture the intended notion.

Can we push abstraction one more step further?


YES WE CAN!

Propositional quantification!

- ideas developed first by Leo in "Quantification in intuitionistic logic with provability smack"
- Define an operator in QHC
(Heyting calculus with propositional quantifiers- C-H counterpart of Girard's system F):
$\triangleright A := \forall P. (P \to A) \vee P$ (P fresh for A)
- Always a normal modality satisfying the mHC laws
- Which laws needed to ensure it is a KM modality?
(i.e., that the Löb law holds as well for $\triangleright$)
- (Casari scheme) $\forall P. ((A \to \forall P.A) \to \forall P.A) \to \forall P.A$

- also called "the relativized Kuroda principle"

- $(\forall P. \sim\sim A \to \sim\sim\forall P.A)$
  $\leftrightarrow ((\sim \forall P.A) \to [\forall P. ((A \to \forall P.A) \to \forall P.A) \to \forall P.A])$
  is a theorem of QHC

- $\forall P. \sim\sim A \to \sim\sim\forall P.A$ is the Kuroda principle

Recall the remark of Heyting [4, p.104] in connection with the formula $\neg\neg\forall x p(x) \to \forall x \neg\neg p(x)$: "It is one of the most striking features of intuitionistic logic that the inverse implication does not hold, especially because the formula of the propositional calculus which results if we restrict x to a finite set, is true". And further: "It has been conjectured [5, p.46] that the formula $\forall x \neg\neg p(x) \to \neg\neg\forall x p(x)$ is always true if x ranges over a denumerable infinite species, but no way of proving the conjecture presents itself at present".

Using figurative style, we note that there is a curious analogy with the state of affairs in Modal logics above S4. Namely, the formula (Kur) plays the role of the McKisey formula (McK), the well-known formula of

Dummett (Dum) behaves like the Minari formula

$$(Min) \qquad \neg\forall x p \vee \{\forall x[(p \to \forall x p) \to \forall x p] \to \forall x p\},$$

whereas the known connection Grz = Dum + McK "transforms" into Cas = Min + Kur.

- one place where QHC certainly lives in:
  Kripke-Joyal(-Beth) semantics
  of the Mitchell-Bénabou language in a topos

- (a thought in passing:
  not the only place though
  any sound semantics for system F and extensions would do
  see, e.g., Girard's "Proofs and Types" for more)

Theorem 2. *The following conditions on an elementary topos* **X** *are equivalent:*

(i) *the relativized Kuroda principle*

$$\forall_x[(p(x)\Rightarrow\forall_y p(y))\Rightarrow\forall_y p(y)]\Rightarrow\forall_x p(x) \qquad \text{(rKP)}$$

holds in **X**;

(i′) *for any* φ *and* $p(x)$ *with* $\varphi\Rightarrow\forall_x p(x)$ *one has*

$$[\forall_x((p(x)\Rightarrow\varphi)\Rightarrow\varphi)]\Rightarrow[((\forall_x p(x))\Rightarrow\varphi)\Rightarrow\varphi];$$

(ii) *every closed subtopos of* **X** *is* $\perp$*-scattered;*

(iii) *the Löb principle*

$$(\rhd\varphi\Rightarrow\varphi)\Rightarrow\varphi \qquad \text{(LP)}$$

holds in **X**;

(iv) *the principle*

$$[(\psi\Rightarrow\varphi)\Rightarrow\varphi]\Leftrightarrow[\rhd\varphi\Rightarrow(\varphi\vee\psi)]$$

holds in **X**.

right-to-left is always the strong Löb, regardless of the actual definition of $\rhd$

...these are mostly syntactic derivations...

The Kuroda axiom easier to characterize

2. Scatteredness

Theorem 1. *The following conditions on an elementary topos* **X** *are equivalent:*

(i) *the Kuroda principle*

$$(\forall_x \neg\neg p(x)) \Rightarrow \neg\neg\forall_x p(x) \tag{KP}$$

holds in **X**;

(ii) *the smallest dense subtopos* $sh_{\neg\neg}(\mathbf{X})$ *is open in* **X**;

(iii) **X** *has a Boolean open dense subtopos*;

(iv) *interior of a dense subtopos of* **X** *is dense*;

(v) $\neg\neg(Boo)$ *holds in* **X**.

Two proposed characterizations for scatteredness

Proposition 1. *A topos is scattered if and only if the internal lattice $\mathcal{N}$ of its nuclei is a Boolean algebra.*

Proposition 2. *A topos of sheaves on a topological space X is scattered if and only if each of its nondegenerate subtoposes has a point.*

Important examples

- Presheaves over well-founded orders
- Sheaves over scattered spaces

The main result in the last section of EJP APAL...

- ... another Dito's fixpoint theorem
- in fact, it does not even mention the modality
- ...or scatteredness, for that matter

The notion of an unchanging map

Let us call a map $f: X \to X'$ *unchanging* if the equivalence relation

$$R = \{(x, y) \in X \times X \mid f(x) = f(y)\}$$

is a dense element of the lattice [diagonal(X), $X \times X$]. (One might call equivalence relations R with this property *undistinguishing*.) In other words, f is unchanging if one has

$$\forall_{x,y \in X}(fx = fy \Rightarrow x = y) \Rightarrow x = y.$$

Clearly any constant map, i.e. one which factors through a subterminal object, is unchanging. In a Boolean topos the converse also holds, however this is far from

Theorem 3. *Let f be any unchanging endomorphism of an object X. Then the subobject* $\mathrm{Fix}(f)$ *of fixed points of f is a maximal subterminal subobject of X; moreover its support is dense in the support of X.*

- It is certainly unique (subterminal)...
- ...and pretty close to existing (max subterminal)...
- if, e.g., X is injective, you're lucky: Fix(f) is actually a global element
- ...but not always exactly exists

Connection with scatteredness and the Löb law

llary. *Let $f : X \to X$ be an endomorphism in a scattered topos satisfying*

$$\forall_{x,y \in X} \Box(x = y) \Rightarrow f(x) = f(y).$$

- (call it "internally contractive"?)
- internal contractiveness in a scattered topos implies being unchanging

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(iii) *the Löb principle*

$$(\Box\varphi \Rightarrow \varphi) \Rightarrow \varphi \qquad \text{(LP)}$$

holds in $\mathbf{X}$;

(iv) *the principle*

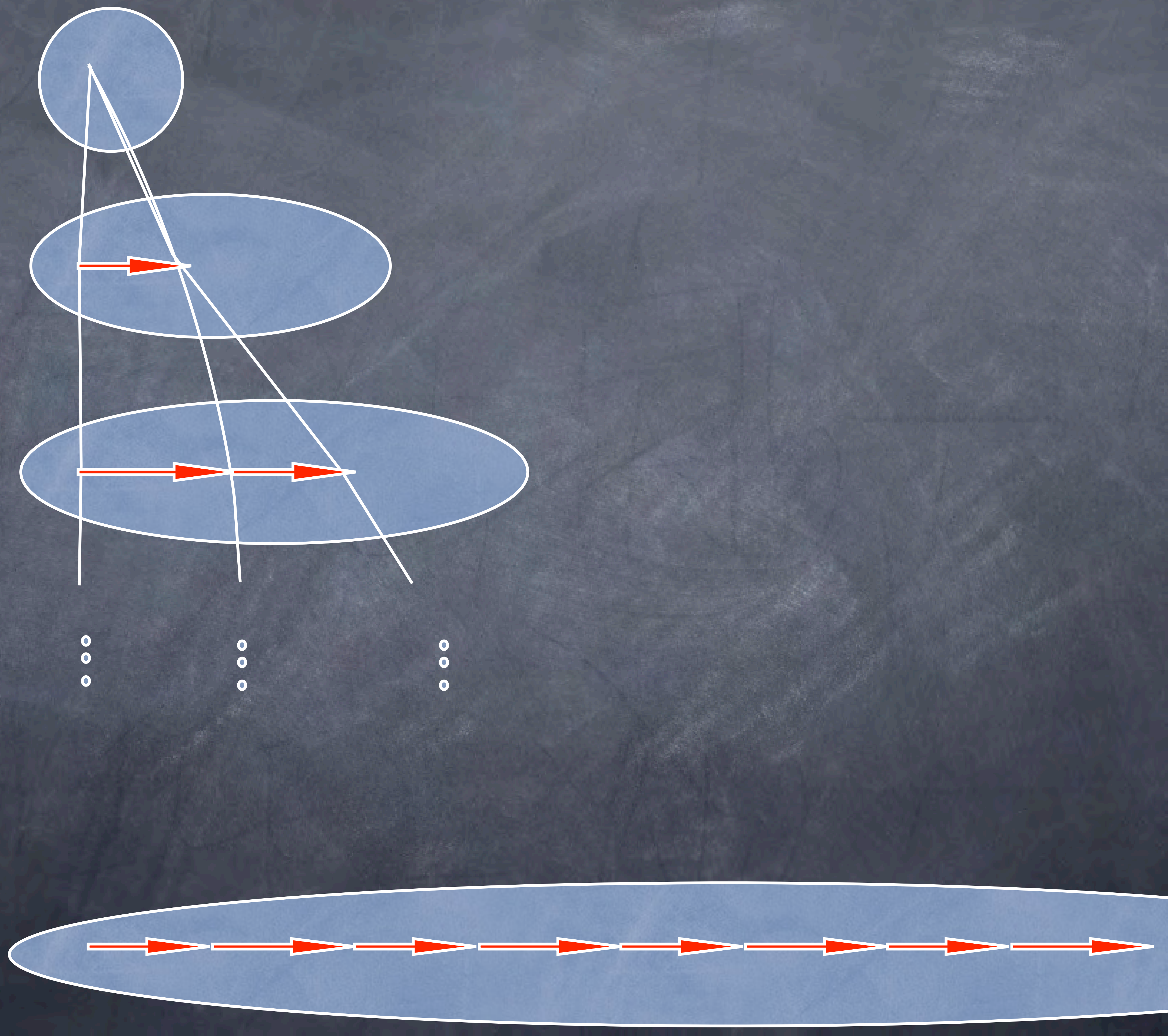
$$[(\psi \Rightarrow \varphi) \Rightarrow \varphi] \Leftrightarrow [\Box\varphi \Rightarrow (\varphi \vee \psi)]$$

holds in $\mathbf{X}$.

right-to-left is always the
strong Löb, regardless of the
actual definition of $\Box$

...now let φ be x=y and let ψ be fx = fy...

- Corollary: any []: Ω -> Ω satisfying the strong Löb law would do
- but even being unchanging via being $\triangleright$-internally contractive does not guarantee that the fixpoint is a global element
- a beautiful counterexample in the topos of presheaves on $\omega + 1$



...fast forward to Copenhagen...

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The internal logic of $\mathcal{S}$ is a standard many-sorted higher-order logic extended with modal operators on both types and terms. (Recall that terms in higher-order logic include both functions and relations, as the latter are simply Prop-valued functions.) This internal logic can then be used as a language to describe semantic models of programming languages with the features mentioned above. As an example which uses both recursively defined types and recursively defined relations in the $\mathcal{S}$-logic, we present a model of $\mathsf{F}_{\mu,\mathsf{ref}}$, a call-by-value programming language with impredicative polymorphism, recursive types, and general ML-like references.

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Birkedal, Mogelberg et al.

An abstract setting for...

- guarded recursive predicates
- guarded recursive dependent types
- step-indexed models of programming languages
- synthetic domain theory
- Banach-style, ultrametric approach to fixpoints (unique rather than least)
- ...

A lot of recent work in Theoretical CS

- Escardo's metric model of PCF
- Nakano (LiCS'00, TACS'01)
- Di Gianantonio, Miculan (FoSSaCS'04)
- Appel et al. (POPL'07)
- Krishnaswami and Benton (LiCS'11, ICFP'11, POPL'12) and Tabareau, Jabber et al. (TLDI'09, LiCS'12)
- Birkedal, Mogelberg et al. (LiCS'11, POPL'11, FiCS'10)...

- Topos of trees:
  topos of presheaves on omega
  (ω starts at 1 for them - they want numbers to encode downsets too)

- Obviously scattered!

- It's possible to describe things pretty concretely

$$
\begin{tikzcd}
X(1) \arrow[d, "f_1"'] & X(2) \arrow[l] \arrow[d, "f_2"'] & X(3) \arrow[l] \arrow[d, "f_3"'] & \dots \arrow[l] \\
Y(1) & Y(2) \arrow[l] & Y(3) \arrow[l] & \dots \arrow[l]
\end{tikzcd}
$$

A subobject A of X is a family of subsets $A(n) \subseteq X(n)$ such that $r_n(A(n+1)) \subseteq A(n)$. The subobject classifier has $\Omega(n) = \{0, \dots, n\}$ and restriction maps $r_n(x) = \min(n, x)$. The characteristic morphism $\chi_A \colon X \to \Omega$ maps $x \in X(n)$ to the maximal m such that $x|_m \in A(m)$ if such an m exists and 0 otherwise.

The natural numbers object N in $\mathcal{S}$ is the constant set of natural numbers.

An operator on predicates. There is a morphism $\triangleright \colon \Omega \to \Omega$ mapping $n \in \Omega(m)$ to $\min(m, n+1)$. By setting $\chi_{\triangleright A} = \triangleright \circ \chi_A$ there is an induced operation on subobjects, again denoted $\triangleright$. This operation, which we also call

- As they work in a very specific setting, they can afford a stronger "internal Banach theorem"

Definition 2.6. *The predicate* Contr *on* Y^X *is defined in the internal logic by*

$$\mathrm{Contr}(f) \overset{\text{def}}{\iff} \forall x, x' : X. \triangleright(x = x') \to f(x) = f(x').$$

Theorem 2.7 (Internal Banach Fixed-Point Theorem). *The following holds in* $\mathcal{S}$*:*

$$(\exists x : X. \top) \wedge \mathrm{Contr}(f) \to \exists! x : X.\, f(x) = x.$$

Lemma 2.8. *The following holds in* $\mathcal{S}$*:*

$$\mathrm{Contr}(f) \to \exists n : N. \forall x, x' : X.\, f^n(x) = f^n(x').$$

- But the Copenhagen paper employs also an external perspective on modality (Curry-Howard-Lambek semantics for extensions of strong Löb)

The $\blacktriangleright$ endofunctor. Define the functor $\blacktriangleright : \mathcal{S} \to \mathcal{S}$ by $\blacktriangleright X(1) = \{\star\}$ and $\blacktriangleright X(n+1) = X(n)$. This functor, called *later*, has a left adjoint (so $\blacktriangleright$ preserves all limits) given by $\blacktriangleleft X(n) = X(n+1)$. Since limits are computed

There is a natural transformation $\mathrm{next}_X : X \to \blacktriangleright X$ whose 1st component is the unique map into $\{\star\}$ and whose $(n+1)$st component is r_n.

Since $\blacktriangleright$ preserves finite limits, there is always a morphism

$$J : \blacktriangleright(X \to Y) \to (\blacktriangleright X \to \blacktriangleright Y). \qquad (1)$$

- The internal modality induced by the external one

later, is connected to the $\blacktriangleright$ functor, since there is a pullback diagram

$$
\begin{tikzcd}
\rhd m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \blacktriangleright A \arrow[d, "\blacktriangleright m"] \\
X \arrow[r, "\mathrm{next}_X"] & \blacktriangleright X
\end{tikzcd}
$$

for any subobject $m \colon A \to X$.

- plays the same role as the "shrinking" delay endofunctor on complete ultrametric spaces
- in fact, bisected complete bounded ultrametric spaces live inside the topos of trees as "total objects" – those whose restriction maps are all surjective

- allows guarded recursive dependent types
- also, allows not only (external) notion of contractivity for morphisms but also for functors (locally contractive ones)
- in the topos of trees, we can solve domain equations for locally contractive functors

- thus, two notions of contractivity:
  one in terms of the internal modality
  another analogous to ultrametric spaces
  (factoring through the delay endofunctor)

- the second stronger than the first

- equivalent for total and inhabited objects
  not in general

- More general approaches:
  (sketch) for presheaves over well-founded frames: take the limit of the diagram induced by all proper successors?

- Birkedal et al. choose the approach of Di Gianantonio, Miculan: sheaves over well-founded Heyting algebras (using the notion of a well-founded base)

- even more abstract approach in the journal version of the LiCS paper

Definition 6.1. A model of guarded recursive terms is a category $\mathcal{E}$ with finite products together with an endofunctor $\blacktriangleright : \mathcal{E} \to \mathcal{E}$ and a natural transformation $\mathrm{next} : \mathrm{id} \to \blacktriangleright$ such that

- for every morphism $f : \blacktriangleright X \to X$ there exists a unique morphism $h : 1 \to X$ such that $f \circ \mathrm{next} \circ h = h$.
- $\blacktriangleright$ preserves finite limits

a small difference with C-H for strong Löb:
limits, not just products
(because of dependent types)

To return to topos of trees and related ones

- other propositional and modal principles inherent in intended models of guarded recursion...
- ...and their computational interpretation?

...implicitly present already in Nakano's (LICS 2000) subtyped polymorphic λ-calculus...

$$\frac{}{\gamma \cup \{X \preceq Y\} \vdash X \preceq Y}\ (\preceq\text{-assump}) \qquad \frac{}{\gamma \vdash A \preceq \top}\ (\preceq\text{-}\top)$$

$$\frac{}{\gamma \vdash A \preceq A'}\ (\preceq\text{-reflex})\quad (A \simeq A') \qquad \frac{\gamma_1 \vdash A \preceq B \quad \gamma_2 \vdash B \preceq C}{\gamma_1 \cup \gamma_2 \vdash A \preceq C}\ (\preceq\text{-trans})$$

$$\frac{\gamma \vdash A \preceq B}{\gamma \vdash \bullet A \preceq \bullet B}\ (\preceq\text{-}\bullet) \qquad \frac{\gamma_1 \vdash A' \preceq A \quad \gamma_2 \vdash B \preceq B'}{\gamma_1 \cup \gamma_2 \vdash A \to B \preceq A' \to B'}\ (\preceq\text{-}\to)$$

$$\frac{}{\gamma \vdash A \preceq \bullet A}\ (\preceq\text{-approx}) \qquad \frac{}{\gamma \vdash A \to B \preceq \bullet A \to \bullet B}\ (\preceq\text{-}\to\bullet) \qquad \frac{}{\gamma \vdash \bullet A \to \bullet B \preceq \bullet(A \to B)}\ (\preceq\text{-}\bullet\to)$$

??

$$\frac{\gamma \cup \{X \preceq Y\} \vdash A \preceq B}{\gamma \vdash \mu X.A \preceq \mu Y.B}\ (\preceq\text{-}\mu) \quad \left(\begin{array}{l} X \notin FTV(\gamma) \cup FTV(B), Y \notin FTV(\gamma) \cup FTV(A), \\ \text{and } A \text{ and } B \text{ are proper in } X \text{ and } Y\text{, respectively} \end{array}\right)$$

Figure 2. The subtyping rules of $\lambda\bullet\mu$

- usually, our models (like the topos of trees) are not only well-founded/Noetherian...

- but also linearly ordered...
  (A -> B) v (B ->A) (GD - Gödel-Dummet)

- ... and the modality is interpreted as the strict part of intuitionistic poset order
  ▷A -> (B -> A) v B (CB - Cantor-Bendixson)

  ▷A -> (((B -> A) ->B) -> B)

(valid internally, but are they valid externally?)

- The combination of SL, GD and CB allows to derive that fishy Nakano thing
- The computational meaning of GD:
  - Danos-Krivine '03, "Disjunctive tautologies as synchronization schemes"
  - a recent work of Hirai: Curry-Howard for GD in terms of wait-free synchronization

Computational meaning of CB?

- only my own preliminary work
- restricted form of catch-and-throw calculus of Crolard
- restricted (delimited??) form of continuations?

4.2 *Typing the* catch *and* throw *operators*

Let us now use the naming rules to derive type judgments for the $\lambda\mu ct$-terms *throw* α t and *catch* α t.

- We recall that *catch* $\alpha\ t = \mu\alpha[\alpha]t$:

$$\dfrac{\dfrac{t : \Gamma \vdash \Delta, A^\alpha; A}{[\alpha]t : \Gamma \vdash \Delta, A^\alpha;}}{\mu\alpha[\alpha]t : \Gamma \vdash \Delta; A}$$

- We recall that *throw* $\alpha\ t = \mu\beta[\alpha]t$ where β does not occur free in t:

$$\dfrac{\dfrac{\dfrac{t : \Gamma \vdash \Delta; A}{[\alpha]t : \Gamma \vdash \Delta, A^\alpha;}}{[\alpha]t : \Gamma \vdash \Delta, A^\alpha, B^\beta;}}{\mu\beta[\alpha]t : \Gamma \vdash \Delta, A^\alpha; B}$$

Hence, we are now able to type the native **throw** and **catch** operators.

The catch rule

$$\dfrac{t : \Gamma \vdash \Delta, A^\alpha; A}{\mathbf{catch}\ \alpha\ t : \Gamma \vdash \Delta; A}$$

leave catch as it is..

The throw rule

$$\dfrac{t : \Gamma \vdash \Delta; A}{\mathbf{throw}\ \alpha\ t : \Gamma \vdash \Delta, A^\alpha; B}$$

...but add guardedness condition to throw

- what is the programming power/importance of the language incorporating all these constructs? (P-A Melliès remark: for principles that are valid only internally, it seems you should be looking at properties of program testing rather than the programming language itself)
- Can we prove some additional important predicate principles valid in the topos of trees - or related ones??

- case study - one needs to move from the original topos of trees to a more complex topos of sheaves to tackle countable non-determinism.
- Lars' question: what is the statement of M-B language valid in sheaves on omega_1 but not on omega which makes things go through?
- my question: would it have any computational interpretation? See, e.g., Krivine's "Dependent choice, quote and clock..."

To hear more about the topos of trees...

- AiML 2012 invited talk of Lars
- Rasmus' talk at the farewell Wessex seminar for Dirk, Nick and myself Imperial College Aug 29

The importance of being scattered

- We can agree Georgia is the best place in the world to understand it
- Just wait for the morning after the final banquet...