HOCHAS AND MINIMAL TOPOSES

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TBILISI, 25 JULY 2012

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Minimal Toposes

Log = 2-category of (locally small) toposes, logical functors and natural isomorphisms.

Any $F : \mathcal{E} \longrightarrow \mathcal{F}$ in **Log** factors uniquely (up to equivalence) as $\mathcal{E} \xrightarrow{F'} \mathcal{F}' \xrightarrow{G} \mathcal{F}$ where G is full & faithful and F' is covering, i.e. every object of $\mathcal{F}'$ admits a monomorphism to some $F'A$.

Say $\mathcal{E}$ is <u>minimal</u> if the unique logical functor $\mathcal{T} \to \mathcal{E}$ is covering, where $\mathcal{T}$ is the free topos (the initial object of **Log**).

Equivalently, every object of $\mathcal{E}$ admits a mono to a pure type, where the pure types are defined by

- 1 is a pure type
- T a pure type $\Rightarrow$ PT a(n indecomposable) pure type
- $T_1, T_2, \ldots, T_n$ indecomposable pure types $\Rightarrow$ $(T_1 \times T_2 \times \cdots \times T_n)$ a pure type.

N.B.: a (locally small) minimal topos is essentially small.

Write **MLog** for the full sub-2-category of minimal toposes.

Note that **MLog** is coreflective in **Log**; also that (any pullback functor $\mathcal{E} \longrightarrow \mathcal{E}/A$ is covering, and hence) any slice category $\mathcal{T}/A$ is minimal.

<u>Remark</u> If $\mathcal{E}$ is minimal and $F, G : \mathcal{E} \rightrightarrows \mathcal{F}$ are logical, there's at most one isomorphism $F \to G$. Hence **MLog** is equivalent to a 1-dimensional category (i.e. a locally discrete 2-category).

Toposes vs. Theories

For any higher-order theory $\mathbb{T}$, we can build a (small) topos $\mathcal{E}_{\mathbb{T}}$ which 'represents' $\mathbb{T}$ in the sense that for any $\mathcal{F}$ we have $\mathbf{Log}(\mathcal{E}_{\mathbb{T}}, \mathcal{F}) \simeq \mathbb{T}\text{-}\mathbf{Mod}(\mathcal{F})$.

Conversely, any small topos is equivalent to some $\mathcal{E}_{\mathbb{T}}$.

Say $\mathbb{T}$ is sortless if its presentation involves no primitive sorts.

Obvious Fact: minimal toposes are (up to equivalence) exactly the $\mathcal{E}_{\mathbb{T}}$ for sortless theories $\mathbb{T}$.

Another obvious fact: any theory is a directed union of finitely-presented theories. Hence any small topos is a filtered colimit (in $\mathbf{Log}$) of finitely-presentable toposes.

Lemma **(P. Freyd)** Any finitely-presented sortless theory is Morita-equivalent to one having a single constant and a single axiom. Hence any finitely-presentable minimal topos is equivalent to a slice category τ/A.

Moreover, any logical functor $\tau/A \longrightarrow \tau/B$ is isomorphic to f^* for some $f: B \longrightarrow A$ in τ.

Putting it all together, we get:

Theorem For a topos $\mathcal{E}$, the following are equivalent:

(1) $\mathcal{E}$ is minimal.

(2) $\mathcal{E} \simeq \mathcal{E}_{\mathbb{T}}$ where $\mathbb{T}$ is a sortless higher-order theory.

(3) $\mathcal{E}$ is a filtered (pseudo-) colimit of toposes of the form $\mathcal{T}/A$.

(4) $\mathcal{E}$ is a limit slice of $\mathcal{T}$: i.e. there exists a filtered category $\mathcal{I}$ and a functor $\mathcal{I} \to \mathcal{T}^{op}$ such that $\mathcal{E}$ is a pseudo-colimit of the composite $\mathcal{I} \to \mathcal{T}^{op} \xrightarrow{\mathcal{T}/(-)} \mathbf{Log}$.

Corollary $\mathbf{MLog}$ is equivalent to the ind-completion $\text{Ind-}\mathcal{T}^{op}$ (the latter regarded as a locally discrete 2-category)

Remark A minimal subtopos can have a natural number object: for example, in the functor category $[\mathbb{N}, \mathbf{Set}]$ (in either sense!) there's a monomorphism $N \rightarrowtail PP1$, so its minimal subtopos contains its natural number object.

Hence the Theorem above contains Freyd's observation that there are nontrivial slices of $\mathcal{T}$ with natural number objects.

Hochas

I = set of indecomposable types (= power types)

$V = I \times \mathbb{N}$ set of variables $\quad x : T$ means 'x is a variable of type T'

For a product type $T = T_1 \times \cdots \times T_n$, $t : T$ means
$t = \langle x_1, \ldots, x_n \rangle$ where $x_i : T_i$

$\langle\,\rangle$ is the unique term of type 1.

Definition A hocha (higer-order cylindric Heyting algebra) is a Heyting algebra $\mathcal{H}$ equipped with unary operators $(\forall x), (\exists x) : \mathcal{H} \longrightarrow \mathcal{H}$ for each $x \in V$, plus constants $x(t) \in \mathcal{H}$ for all $x : PT$ and $t : T$, satisfying

(1) $(\forall x)(\forall y)\alpha = (\forall y)(\forall x)\alpha$
$(\exists x)(\exists y)\alpha = (\exists y)(\exists x)\alpha$ for all $x, y \in V$

(if $t = \langle x_1, \ldots, x_n \rangle$, write $\begin{matrix}(\forall t)\\(\exists t)\end{matrix}$ for $\begin{matrix}(\forall x_1)\cdots(\forall x_n)\\(\exists x_1)\cdots(\exists x_n)\end{matrix}$)

(2) $(\forall x)(\forall x)\alpha = (\forall x)\alpha = (\exists x)(\forall x)\alpha$
$(\exists x)(\exists x)\alpha = (\exists x)\alpha = (\forall x)(\exists x)\alpha$ for all $x \in V$

(write $\mathcal{H}^x$ for $\{\alpha \in H \mid (\forall x)\alpha = \alpha\} = \{\alpha \in \mathcal{H} \mid (\exists x)\alpha = \alpha\}$
and $\mathcal{H}_0$ for $\bigcap_{x \in V} \mathcal{H}^x$)

(3) $(\forall x)(\alpha \wedge \beta) = (\forall x)\alpha \wedge (\forall x)\beta$

$(\exists x)(\alpha \vee \beta) = (\exists x)\alpha \vee (\exists x)\beta$

$(\forall x)\alpha \leq \alpha \leq (\exists x)\alpha$

(so $(\exists x)$, $(\forall x)$ are left & right adjoints to the inclusion $\mathcal{H}^x \longrightarrow \mathcal{H}$)

(4) $(\forall x)((\forall x)\alpha \Rightarrow (\forall x)\beta) = ((\forall x)\alpha \Rightarrow (\forall x)\beta)$

(so $\mathcal{H}^x$ is a sub-Heyting-algebra)

(5) $(\forall z)\, x(t) = x(t)$ if $t = \langle y_1 \cdots y_n \rangle$ and $z \notin \{x, y_1, \ldots, y_n\}$

(6) $(\exists x)(\varepsilon_t(x, x') \wedge \alpha) = (\forall x)(\varepsilon_t(x, x') \Rightarrow \alpha)$ $\quad (*)$

for all $x, x' : PT$ and any $t : T$ with no repeated variables, where $\varepsilon_t(x, x') = (\forall t)(x(t) \Leftrightarrow x'(t))$

(think of $\varepsilon_t(x, x')$ as 'the element measuring equality of x and x''; write $\alpha[x'/x]_t$ for the common value of the two sides of $(*)$. Will eventually show these are independent of choice of t.)

(7) (extensionality) $(\varepsilon_{t_1}(x_1, x_1') \wedge \cdots \wedge \varepsilon_{t_n}(x_n, x_n')) \leq (y(x_1, \ldots, x_n) \Leftrightarrow y(x_1', \ldots, x_n'))$

for any $y : P(T_1 \times \cdots \times T_n)$, $x_i, x_i' : T_i$ and suitable terms t_i.

(8) (comprehension) $(\exists x)(\forall t)(x(t) \Leftrightarrow (\forall x)\alpha) = \top$

for any $x : PT$ and any $t : T$ without repeated variables.

Say $\alpha \in \mathcal{H}$ is finitary if it belongs to $\mathcal{H}^x$ for all but finitely many x (note that the $x(t)$ are all finitary), and that $\mathcal{H}$ is finitary if all its elements are.

The theory of finitary hochas isn't (single-sorted) algebraic, but it can be presented as a many-sorted algebraic theory, with one sort for each finite subset of V and additional unary operations (inclusions) $A \to B$ whenever $A \subseteq B$.

Hence the category **Hocha**$_f$ of finitary hochas, like **Hocha**, is locally finitely presentable and effective regular; it's also coreflective in **Hocha**.

Properties of hochas

Lemma 1 If $\alpha \in \mathcal{H}^x$, then

$$(\exists x)(\alpha \wedge \beta) = (\alpha \wedge (\exists x)\beta)$$
$$(\forall x)(\alpha \Rightarrow \beta) = (\alpha \Rightarrow (\forall x)\beta)$$
$$(\forall x)(\beta \Rightarrow \alpha) = ((\exists x)\beta \Rightarrow \alpha).$$

Lemma 2 $(\exists x)\, x(t) = \top$, $(\forall x)\, x(t) = \bot$.

Lemma 3 If x, x', x'' are distinct variables of the same type, then

$$\varepsilon_t(x, x')[x''/x']_t = \varepsilon_t(x, x'').$$

Lemma 4 If $\alpha \in \mathcal{H}^{x'}$, then

$$\begin{aligned} \alpha[x'/x]_t[x''/x']_t &= \alpha[x''/x]_t && \text{if } x \neq x'' \\ &= \alpha && \text{if } x = x''. \end{aligned}$$

Lemma 5 If $x \neq x'$, $y \neq y'$, $x \neq y$, $x \neq y'$, x' [illegible] then

$$\alpha[x'/x]_t[y'/y] = \alpha[y'/y]_u[x \text{ [illegible]}]$$

Lemma 6 $(\alpha * \beta)[x'/x]_t = (\alpha[x'/x]_t * \beta[x'/x]_t)$

where $*$ is any of $\wedge, \vee, \Rightarrow$

Lemma 7 (i) If $y \neq x$ and $y \neq x'$, then

$((\forall y)\alpha)[x'/x]_t = (\forall y)(\alpha[x'/x]_t)$ & similarly for $\exists$

(ii) If $\alpha \in \mathcal{H}^{x'}$, then

$(\forall x')(\alpha[x'/x]_t) = (\forall x)\alpha$ " " " ".

(iii) $\alpha \in \mathcal{H}^{x}$ iff $\alpha[x'/x]_t = \alpha$.

Lemma 8 (i) $y(x_1, \dots, x_n)[x_1'/x_1]_{t_1}[x_2'/x_2]_{t_2} \cdots [x_n'/x_n]_{t_n}$

$$= y(x_1', \dots, x_n')$$

provided no accidental substitutions occur.

(ii) $\varepsilon_t(x, x') = \varepsilon_{t'}(x, x')$

(so we can finally drop the subscripts!)

(iii) $y(x_1, \dots, x_n)[y'/y] = y'(x_1, \dots, x_n)$.

Hocha to Topos

We go by way of an allegory $\mathcal{A}(\mathcal{H})$.

- Objects of $\mathcal{A}(\mathcal{H})$ are pure types.
- Morphisms $T \looparrowright U$ are named by triples (t, α, u) where $t : T$ and $u : U$ are terms such that $\langle t, u \rangle$ contains no repeated variables, and $\alpha \in \mathcal{H}^z$ for all z not occurring in $\langle t, u \rangle$.

 I dentify (t, α, u) with (t', α', u') if α' is obtained from α by (capture-avoiding) substitution of the variables in $\langle t', u' \rangle$ for those in $\langle t, u \rangle$.

- The composite of $T \xrightarrow{[t,\alpha,u]} U \xrightarrow{[u,\beta,v]} V$ is

$$T \xrightarrow{[t, (\exists u)(\alpha \wedge \beta), v]} V,$$

 and the identity on T is $[t, \varepsilon(t,t'), t']$.

- Ordering on hom-sets induced by that of $\mathcal{H}$:

$$[t,\alpha,u] \wedge [t,\beta,u] = [t, \alpha\wedge\beta, u]$$

 and the anti-involution is obvious: $[t,\alpha,u]^{o} = [u,\alpha,t]$.

Theorem $\mathcal{A}(\mathcal{H})$ is a pre-tabular power allegory, and its hom-posets have bottom elements.

Hence (Elephant, A3·4·9) we get a topos $\mathcal{E}(\mathcal{H})$ by splitting symmetric idempotents in $\mathcal{A}(\mathcal{H})$ and cutting down to the category of maps.

Remark $\mathrm{Sub}_{\mathcal{E}(\mathcal{H})}(1) \cong \mathcal{H}_0$.

Topos to Hocha

Choose a total ordering of V.

Let $\mathbf{Ctxt}$ denote the (directed) poset of contexts (= finite subsets of V) under inclusion.

For any topos $\mathcal{E}$, we have a functor $\mathrm{Type}_{\mathcal{E}} : \mathbf{Ctxt} \longrightarrow \mathcal{E}^{op}$ sending a context $\langle x_1, x_2, \dots, x_n \rangle$ (listed in order) to (the interpretation in $\mathcal{E}$ of) $T_1 \times T_2 \times \cdots \times T_n$, where $x_i : T_i$, and an inclusion of contexts to the appropriate product projection.

Define $\mathcal{H}(\mathcal{E})$ to be the colimit in **Heyt** of the composite

$$\mathbf{Ctxt} \xrightarrow{\mathrm{Type}_{\mathcal{E}}} \mathcal{E}^{op} \xrightarrow{\mathrm{Sub}_{\mathcal{E}}(-)} \mathbf{Heyt}.$$

(Note that a logical functor $F : \mathcal{E} \longrightarrow \mathcal{F}$ gives rise to

$$\begin{tikzcd}
& \mathcal{E}^{op} \arrow[dd, "F"] \arrow[dr, "\mathrm{Sub}_{\mathcal{E}}(-)"] & \\
\mathbf{Ctxt} \arrow[ur, "\mathrm{Type}_{\mathcal{E}}"] \arrow[dr, "\mathrm{Type}_{\mathcal{F}}"'] \arrow[r, phantom, "\cong"] & {} \arrow[r, phantom, "\Downarrow"] & \mathbf{Heyt} \\
& \mathcal{F}^{op} \arrow[ur, "\mathrm{Sub}_{\mathcal{F}}(-)"'] &
\end{tikzcd}$$

and hence to a morphism $\mathcal{H}(\mathcal{E}) \longrightarrow \mathcal{H}(\mathcal{F})$ in **Heyt**.)

<u>Claim</u>: $\mathcal{H}(\mathcal{E})$ can be given the structure of a (finitary) hocha.

$(\forall x)\,\alpha$ and $(\exists x)\,\alpha$ defined by choosing a representative $A \rightarrowtail T \times U_1 \times \cdots \times U_n$ for α in a context which includes x, and applying the right/left adjoint to pullback along $T \times U_1 \times \cdots \times U_n \longrightarrow U_1 \times \cdots \times U_n$ (where T is the type of x).

If $\langle x, y_1, \dots, y_n \rangle$ is a context (in canonical order), define $x(y_1, \dots, y_n)$ to be $\in_{T_1 \times \cdots \times T_n} \rightarrowtail P(T_1 \times \cdots \times T_n) \times T_1 \times \cdots \times T_n$ (where $y_i : T_i$).
(Otherwise, take the appropriate pullback of this subobject.)

Toposes vs. Hochas

Proposition For any (locally small) topos $\mathcal{E}$, $\mathcal{E}(\mathcal{H}(\mathcal{E}))$ is equivalent to the minimal subtopos of $\mathcal{E}$.

Proposition For any Rocha $\mathcal{H}$, $\mathcal{H}(\mathcal{E}(\mathcal{H}))$ is isomorphic to the finitary part of $\mathcal{H}$.

Theorem The constructions $\mathcal{H}(-)$ and $\mathcal{E}(-)$ induce an equivalence $\mathbf{MLog} \simeq \mathbf{Hocha}_f$
(where the category on the right is regarded as a locally discrete 2-category).

Corollary $\mathbf{Hocha}_f$ is equivalent to $\mathrm{Ind}\text{-}\mathcal{T}^{op}$.

Problem We have a good notion of 'N-minimal topos', such that N-minimal toposes are (up to equivalence) just the limit-slices of the free topos-with-NNO $\mathcal{T}^N$.
Is there a corresponding notion of 'Rocha with NNO'?