International Workshop on Topological Methods in Logic III

Dedicated to the memory of Dito Pataraia
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On finitely generated free and projective monadic Gödel algebras

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Monadic Intuitionistic Logic

Modal Intuitionistic Propositional Calculus (*MIPC*) is the well-known modal intuitionistic propositional calculus introduced by Prior

[A. Prior, *Time and Modality,* Clarendon Press, Oxford, 1957]

in order to give a satisfactory axiomatization of the one-variable fragment of Intuitionistic Predicate Logic.

Monadic Intuitionistic Logic

On the other hand, *MIPC* is considered as one of the most acceptable intuitionistic formalizations of *S5.*

Monadic Intuitionistic Logic

The language of the logic *MIPC*

$\rightarrow, \vee, \wedge, \bot, \Box, \Diamond$

Monadic Intuitionistic Logic

Axioms of the logic *MIPC* are the following:

(MIL1) $\alpha \to (\beta \to \alpha)$,
(MIL2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$,
(MIL3) $(\alpha \wedge \beta) \to \alpha$,
(MIL4) $(\alpha \wedge \beta) \to \beta$,
(MIL5) $\alpha \to (\beta \to (\alpha \wedge \beta))$,
(MIL6) $\alpha \to (\alpha \vee \beta)$,
(MIL7) $\alpha \to (\beta \vee \alpha)$,
(MIL8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$,
(MIL9) $\perp \to \alpha$

Monadic Intuitionistic Logic

(A1) $\Box\alpha \to \alpha$,

$\alpha \to \Diamond\alpha$

(A2) $(\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)$,

$\Diamond(\alpha \vee \beta) \to (\Diamond\alpha \vee \Diamond\beta)$

(A3) $\Diamond\alpha \to \Box\Diamond\alpha$,

$\Diamond\Box\alpha \to \Box\alpha$

(A4) $\Box(\alpha \to \beta) \to (\Diamond\alpha \to \Diamond\beta)$

Inference rules:

Modus Ponens: $\alpha, \alpha \rightarrow \beta \Rightarrow \beta$

Necessitation Rule: $\alpha \Rightarrow \Box\, \alpha$

Monadic Gödel Logic

(MG1) $\alpha \to (\beta \to \alpha)$,

(MG2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$,

(MG3) $(\alpha \wedge \beta) \to \alpha$,

(MG4) $(\alpha \wedge \beta) \to \beta$,

(MG5) $\alpha \to (\beta \to (\alpha \wedge \beta))$,

(MG6) $\alpha \to (\alpha \vee \beta)$,

(MG7) $\alpha \to (\beta \vee \alpha)$,

(MG8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$,

(MG9) $\perp \to \alpha$

(MG10) $(\alpha \to \beta) \vee (\beta \to \alpha)$

Monadic Gödel Logic

(A1) $\Box\alpha \to \alpha$,

$\alpha \to \Diamond\alpha$

(A2) $(\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)$,

$\Diamond(\alpha \vee \beta) \to (\Diamond\alpha \vee \Diamond\beta)$

(A3) $\Diamond\alpha \to \Box\Diamond\alpha$,

$\Diamond\Box\alpha \to \Box\alpha$

(A4) $\Box(\alpha \to \beta) \to (\Diamond\alpha \to \Diamond\beta)$

Monadic Gödel Logic

Let L be a first-order language based on

$$\rightarrow, \vee, \wedge, \perp, \forall, \exists$$

and L_m *monadic propositional lang*auage based on

$$\rightarrow, \vee, \wedge, \perp, \square, \lozenge,$$

and *Form*(L) and *Form*(L_m) – the set of formulas of L and L_m, respectively.

Monadic Gödel Logic

We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p^*(x)$ in L and define by induction a translation

$$\Psi : Form(L_m) \rightarrow Form(L)$$

by putting:

Monadic Gödel Logic

- $\Psi(p) = p^*(x)$ if p is propositional variable,
- $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \rightarrow, \vee, \wedge$,
- $\Psi(\Box\alpha) = \forall x\ \Psi(\alpha)$,
- $\Psi(\Diamond\alpha) = \exists x\ \Psi(\alpha)$.

Through this translation Ψ, we can identify the formulas of L_m with monadic formulas of L containing the variable x.

Monadic Gödel Algebras

*A **monadic (Boolean) algebra** is a Boolean algebra A together with an operator ∃ on A (called an existential quantifier, or, simply, a quantifier) such that*

$$\exists 0 = 0, \quad x \leq \exists x, \quad \exists(x \wedge \exists y) = \exists x \wedge \exists y$$

whenever *x* and *y* are in *A*. A systematic study of monadic algebras appears in

[Paul R. Halmos, *Algebraic Logic,* I. *Monadic Boolean Algebras,* Composotio Mathematica, vol. 12 (1955), pp. 217-249.]

Monadic Gödel Algebras

Operators such as $\exists$ had occurred before in, for instance, Lewis' studies of modal logic and Tarski's studies of algebraic logic.

One motivation for studying monadic algebras is the desire to understand certain aspects of mathematical logic; the connection with logic is also the source of much of the terminology and notation used in the theory.

Monadic Heyting Algebras

The full developing of the theory of monadic Heyting Algebras was given by G. Bezhanisvili in his PhD Thesis and following to them the serial of papers. One of them

[G. Bezhanishvili and R. Grigolia, "*Locally tabular extensions of MIPC*", Proceedings of Uppsala Symposium ", Advances in Modal Logic'98" , vol. 2, Csli Publications, Stanford, California, 101-120 (2001)]

Monadic Heyting Algebras

We call an universal algebra

$$(H, \wedge, \vee, \rightarrow, \Box, \Diamond, 0, 1)$$

monadic Heyting algebra, if $(H, \wedge, \vee, \rightarrow, 0, 1)$ is a Heyting algebra and in addition it satisfies the following identities:

- (MH1) $\Box x \leq x$, $x \leq \Diamond x$,
- (MH2) $(\Box x \wedge \Box y) \leq \Box (x \wedge y)$, $\Diamond(x \vee y) \leq (\Diamond x \vee \Diamond y)$,
- (MH3) $\Diamond x \leq \Box \Diamond x$, $\Diamond \Box x \leq \Box x$,
- (MH4) $\Box(x \rightarrow y) \leq (\Diamond x \rightarrow \Diamond y)$.

Monadic Gödel Algebras

We call an universal algebra

$$(H, \wedge, \vee, \to, \Box, \Diamond, 0, 1)$$

monadic Gödel algebra, if it is monadic Heyting algebra and in addition it satisfies the following identities:

$$(x \to y) \vee (y \to x) = 1.$$

MIPC-frames

- A triple (X,R,Q) is an *MIPC-frame* if R is a partial order on $X \neq \emptyset$ and Q a qvasi-order on X such that

  $R \subseteq Q$ and $\forall x,y \in X\ (xQy \Rightarrow \exists z \in X (xRz\ \&\ zE_Q y))$,

  where $zE_Q y$ iff zQy and yQz, i. e. z and y belong to the same Q-cluster.

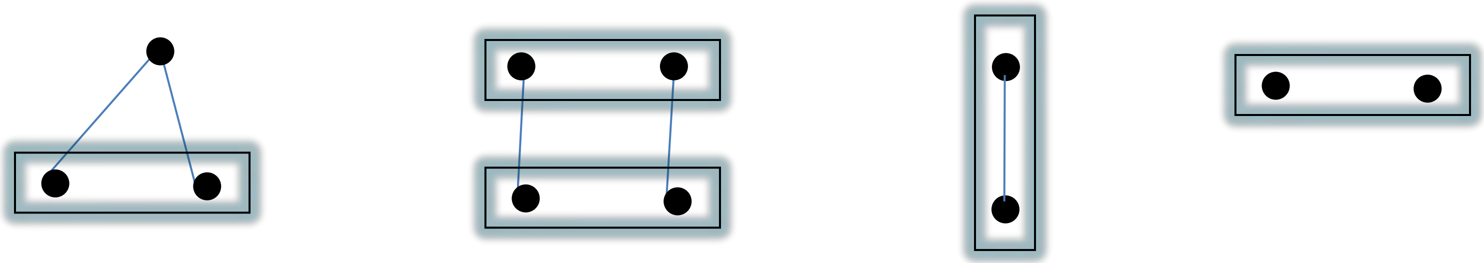
MIPC-frames

A general frame for *MIPC* is a quadruple (X,R,Q,P), where (X,R,Q) is a *MIPC*-frame and P a family of R-cones in X which contains $\varnothing$ and is closed under $\cup$, $\cap$, and the operations:

$Y \to Z = -R^{-1}(X - Y)$, $\Box Y = -Q^{-1} - Y$,
$\Diamond Y = E_Q(Y) = Q(Y)$.

If (X,R) is a root system, then P forms a Gödel algebra.

MIPC-frames



MIPC-frames

- A frame (X,R,Q,P) is *refined* when $\neg(xRy)$ only if there is a $Y \in P$ such that $x \in Y$ and $y \notin Y$, and $\neg(xQy)$ only if there is a Q-cone $Y \in P$ such that $x \in Y$ and $y \notin Y$.

MIPC-frames

A frame (X,R,Q,P) is *compact* if for all $T \subseteq P$ and $S \subseteq \{-Y : Y \in P\}$, if $T \cup S$ has the finite intersection property, then $\bigcap (T \cup S) \neq \varnothing$.

MIPC-frames

- Refined and compact frames are called *descriptive*. It is known from that every logic *L* over *MIPC* is complete with respect to descriptive frames for *L.*

- Let **MG** be the variety of all monadic Gödel algebras.

MIPC-frames

- Notice that finite *MIPC*-frame is descriptive. Recall that a quasi-order (X,R) is of *R-depth n* if it contains an *R*-chain

$$x_1 R \ldots R x_n$$

of *n* points from distinct *R*-clusters but not such chain of greater length.

MIPC-frames

- Let *L* be an extension of the logic *MG*. A logic *L* is said to be of *R-depth n* if there exists an *MIPC*-frame of *R*-depth *n* but there is no *MIPC*-frame of *R*-greater depth. *L* is called a *finite R-depth logic* if *L* is a logic of *R*-depth *n*, for some $n<\omega$.

MIPC-frames

Consider the list of formulas:

P_0: $\perp$;

$P_n : q_n \vee (q_n \rightarrow P_{n-1}),\ n \geq 1.$

Proposition 1. *A logic L is of R-depth n iff* $L \vdash P_n$ *and* $L \nvdash P_{n-1}$.

Analogical way we can define a subvariety **L** of *R-depth n* of the variety **MG**: **L** is a subvariety of *R-depth n* if in **L** hold identities $P_n = 1$ and does not hold $P_{n-1} = 1$.

Proposition 2. *The subvariety* $\mathbf{MG}_n$ *of the variety* **MG** *is a subvariety of R-depth n.*

Finitely generated *MIPC*-frames

The equivalence relation E on the descriptive MIPC-frame is called *correct* **[G. Bezhanishvili and R. Grigolia, "*Locally tabular extensions of MIPC*", Proceedings of Uppsala Symposium ", Advances in Modal Logic'98" , vol. 2, Csli Publications, Stanford, California, 101-120 (2001)]** if

- E-saturation of any cone is a cone;
- If $\neg(xEy)$, then there exists $U \in P$ such that an $E(U) = U$ and either $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$.
- $RE(x) \subset ER(x)$ for every $x \in X$;
- $QE(x) \subset EQ(x)$ for every $x \in X$;
- $EQ(x) \subset QER(x)$ for every $x \in X$;

- The concept of correct partition of Intuitionistic (or $S4$-) frame is introduced in
- [**L. Esakia and R. Grigolia**, *The criterion of Browerian and closure algebras to be finitely generated*, Bull. Sect. Logic, 6, 2, (1973), 46-52.].
- A partition X/E of X is said to be *correct* if the equivalence relation E satisfies the following conditions:

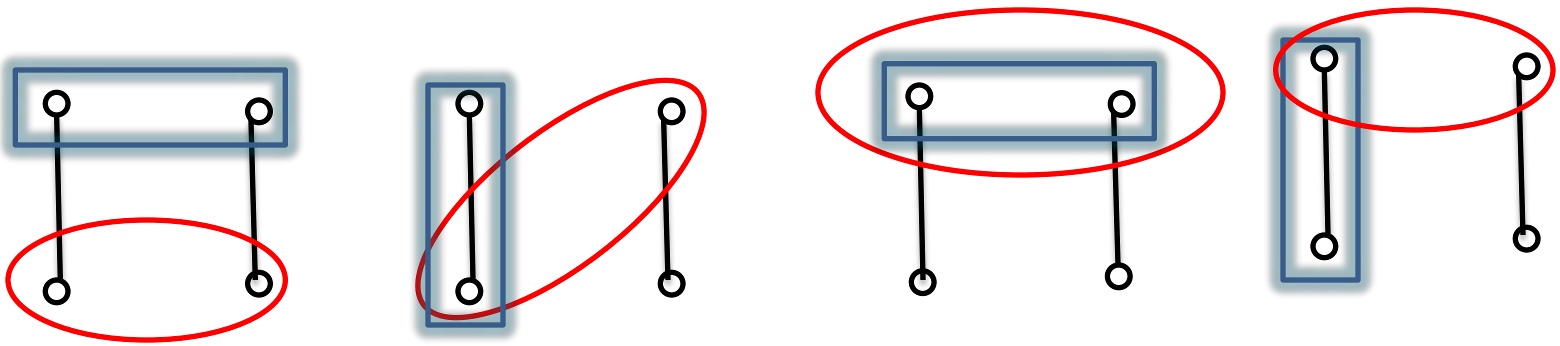
- E *is a closed equivalence relation, i. e.* E*-saturation of any closed* subset is closed;
- E*-saturation of any upper cone is an upper cone;*
- *there is D-frame* (Y,Q) *and a strongly isotone map* $f : X \to Y$ *such that* $Ker f = E$.

Finitely generated *MIPC*-frames

- ***There exists one-to-one correspondence between subalgebras of an MG–algebra A and correct partition of the corresponding to it general frame*** X_A**.**

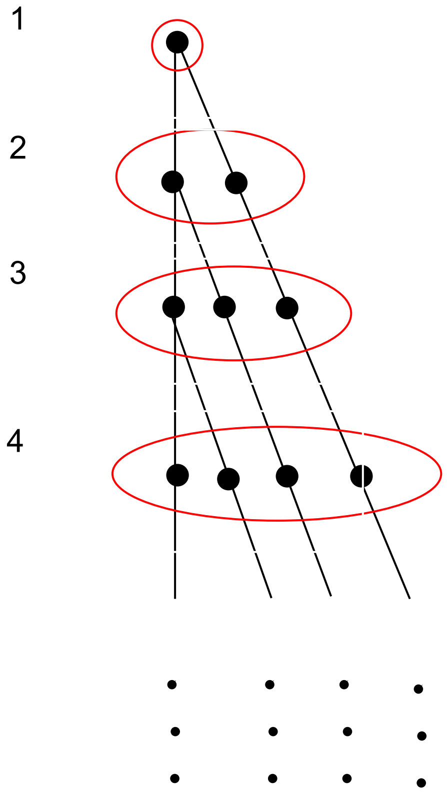
Finitely generated *MIPC*-frames

EXAMPLS



Incorrect **Correct**

MIPC-frames



Finitely generated *MIPC*-frames

- Suppose (X,R,Q,P) is a descriptive *MIPC*-frame and $g_1, \dots, g_n \in P$.

  Denote by $\boldsymbol{n}$ the set $\{1, \dots, n\}$.

  Let $G_p = g_1^{\varepsilon_1} \cap \dots \cap g_n^{\varepsilon_n}$,

  where $\varepsilon_i \in \{0,1\}$, $p = \{ i: \varepsilon_i = 1\}$ and

$$g_i^{\varepsilon_i} = \begin{cases} g_i, & \varepsilon_i = 1 \\ -g_i, & \varepsilon_i = 0 \end{cases}$$

Finitely generated *MIPC*-frames

- Call a frame (X,R,Q,P) *n-generated* if there exist $X_1, \dots, X_n \in P$ such that every element of P is obtained from $X_1, \dots, X_n$ by the operations $\cup$, $\cap$, $\to$, $\Box$, $\Diamond$. A frame (X,R,Q,P) is said to be *finitely generated* if (X,R,Q,P) is n-generated for some n.

Finitely generated *MIPC*-frames

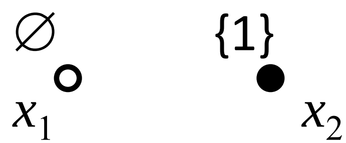
- It is obvious that $\{G_p\}_{p \subseteq \mathbf{n}}$ is a partition of X which we call *the colouring of* X. A given $x \in G_p$ is said to have *colour* p, written $Col(x) = p$.

- **Theorem 3.** *(Colouring Theorem)* (X,R,Q,P) *is n-generated by* $g_1, \ldots, g_n \in P$ *iff for every non-trivial correct partition* E *of* X*, there exists an equivalence class of* E *containing the points of different colours.*

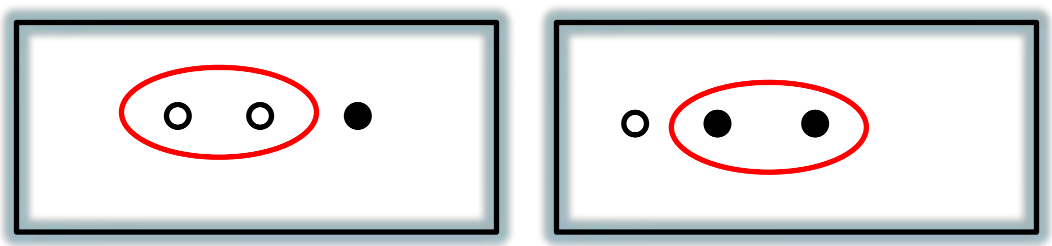
Finitely generated *MIPC*-frames

One generated Boolean algebra

$\mathbf{1} = \{1\}$, $X = \{x_1, x_2\}$, $\mathrm{Col}(x_1) = \varnothing$, $\mathrm{Col}(x_2) = \{1\}$



$F_{\mathbf{B}}(1)$

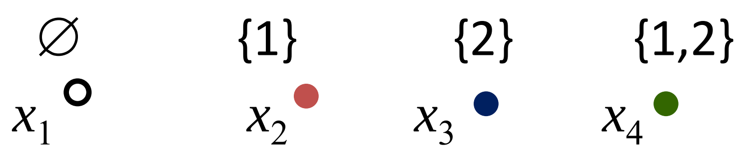


Finitely generated *MIPC*-frames

Two generated Boolean algebra

2 = {1,2}, $X = \{x_1, x_2, x_3, x_4\}$,

$\mathrm{Col}(x_1) = \varnothing$, $\mathrm{Col}(x_2) = \{1\}$, $\mathrm{Col}(x_3) = \{2\}$, $\mathrm{Col}(x_4) = \{1,2\}$



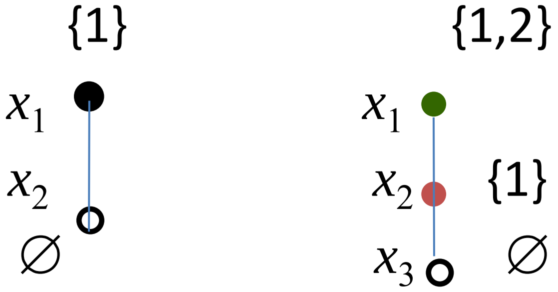
$F_{\mathbf{B}}(2)$

Finitely generated *MIPC*-frames

One and two generated Heyting algebra

1 = {1}, $X_1 = \{x_1, x_2\}$;   ***2*** = {1,2}, $X_2 = \{x_1, x_2, x_3\}$

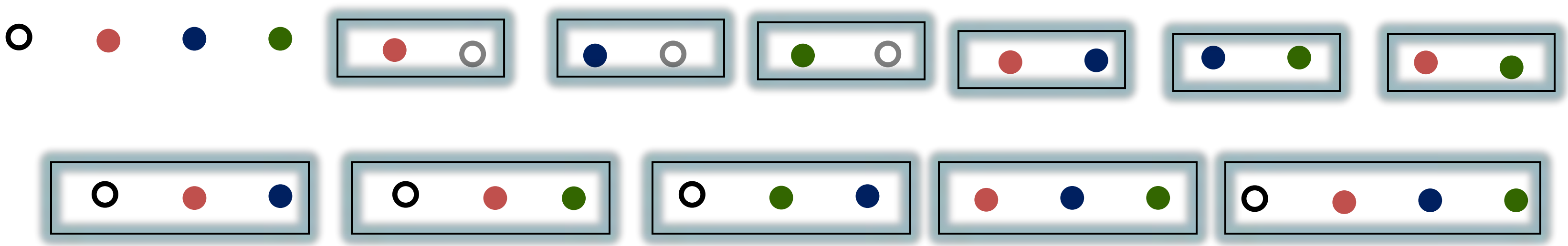
$\mathrm{Col}(x_1) = \{1\}$, $\mathrm{Col}(x_2) = \varnothing$; $\mathrm{Col}(x_1) = \{1,2\}$, $\mathrm{Col}(x_2) = \{1\}$, $\mathrm{Col}(x_2) = \varnothing$



Finitely generated *MIPC*-frames

S5

$F_{S5}(1)$



$F_{S5}(2)$

Finitely generated *MIPC*-frames

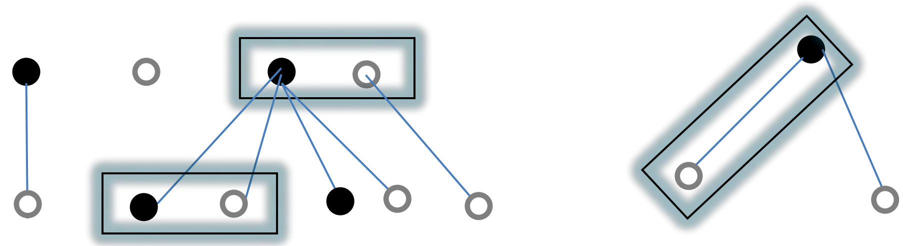
Now we represent one-generated free algebra $F_{\mathbf{MG}}(1)$ by means of colouring general MIPC-frame. We have the following colouring:

$$\mathbf{1} = \{1\},\ \mathrm{Col}(x) = \varnothing \text{ or } \mathrm{Col}(x) = \{1\}.$$

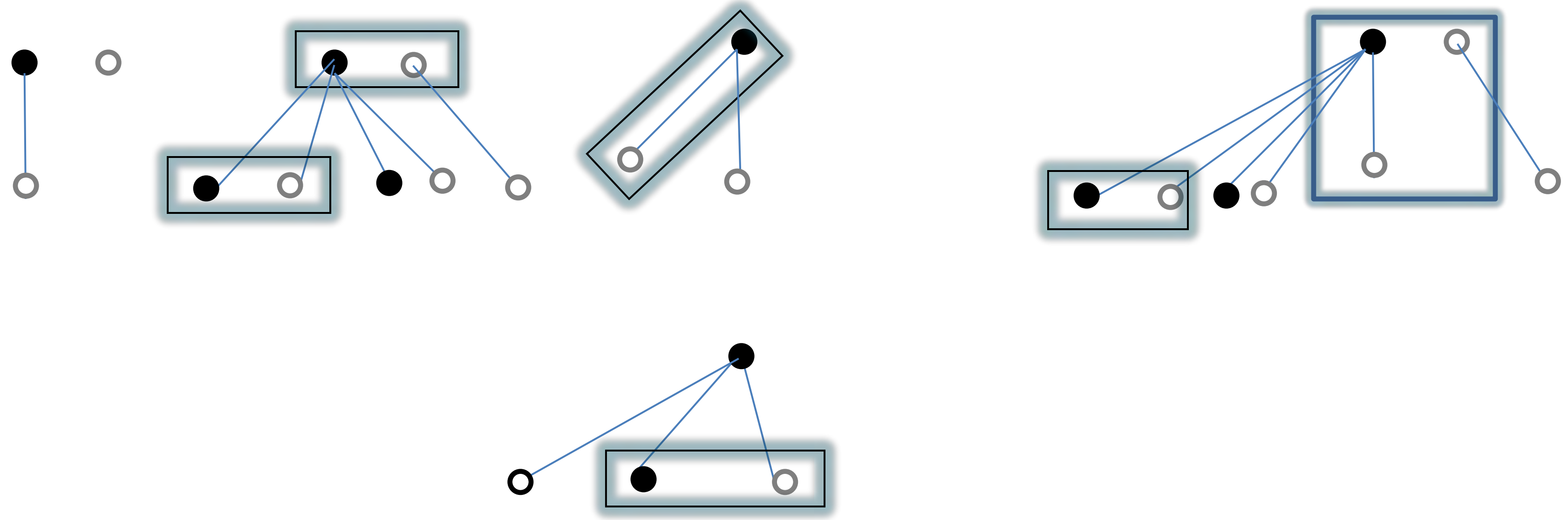
We denote $\varnothing$ colour by $\circ$ and $\{1\}$ colour by $\bullet$.

One generated *MIPC*-frame

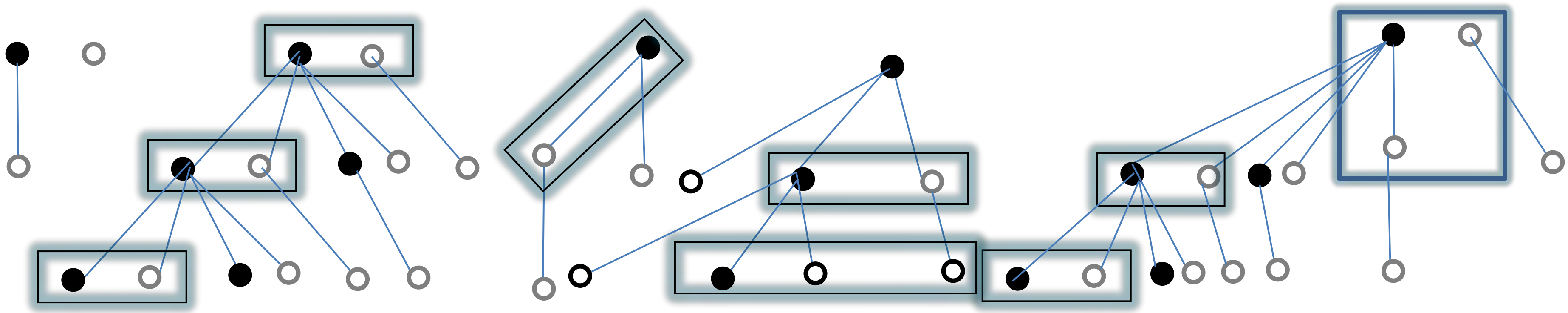
One generated *MIPC*-frame



One generated *MIPC*-frame



One generated *MIPC*-frame



One generated free *MIPC*-frame

Let $\mathbf{K}$ be a variety of algebras. Recall that an algebra $A \in \mathbf{K}$ is said to be *a free algebra* in $\mathbf{K}$, if there exists a set $A_0 \subset$ A such that A_0 generates A and every map f from A_0 to any algebra $B \in \mathbf{K}$ is extended to a homomorphism h from A to B. In this case A_0 is said to be the set of free generators of A. If the set of free generators is finite then A is said to be a free algebra of finitely many generators. The n-generated free algebra in $\mathbf{K}$ is denoted by $F_{\mathbf{K}}(n)$.

One generated free *MIPC*-frame

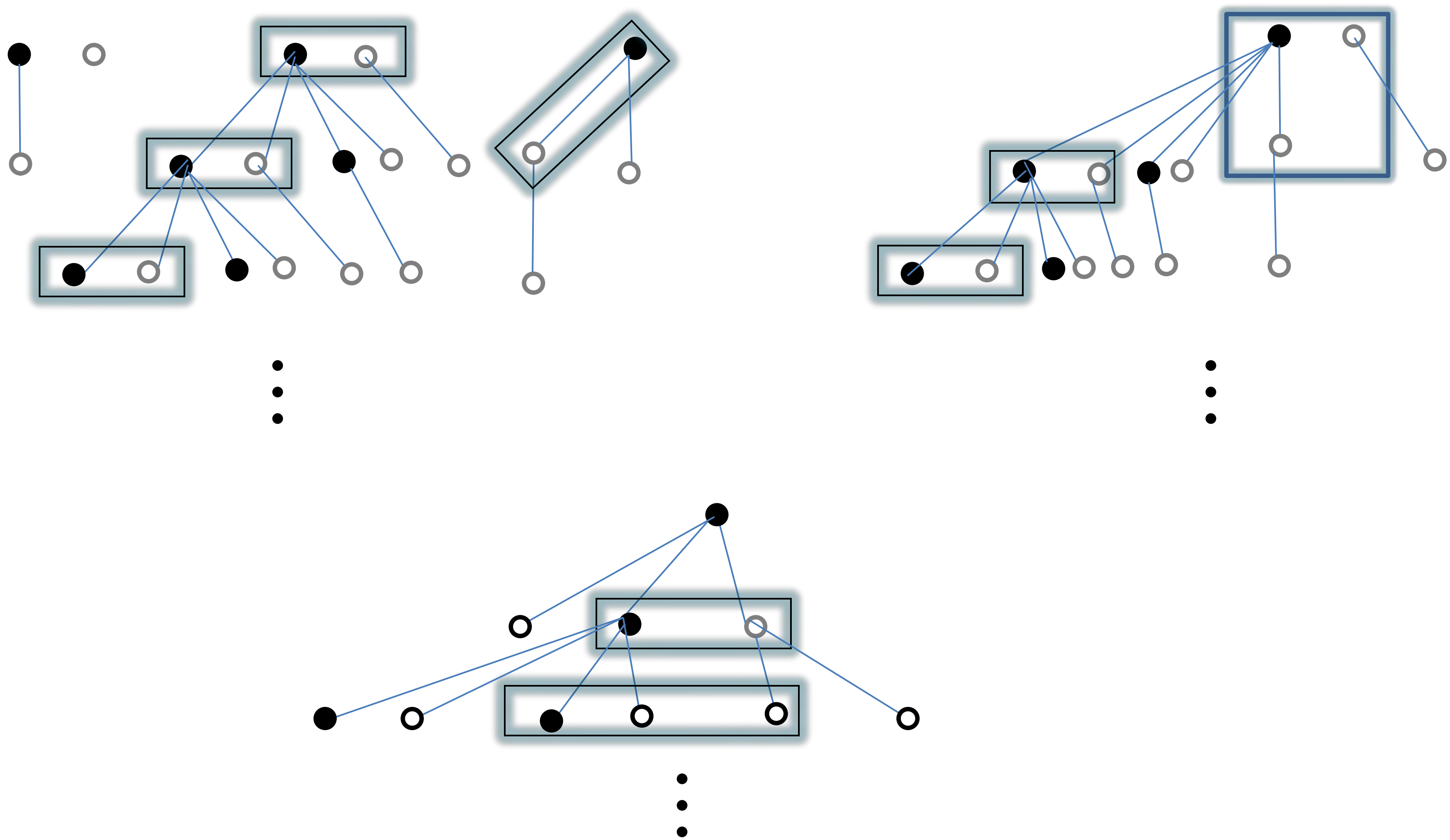
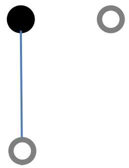


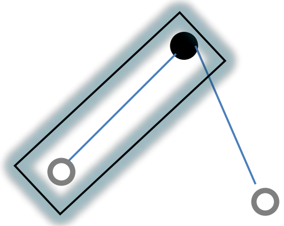
Fig. 1

One generated free *MIPC*-frame



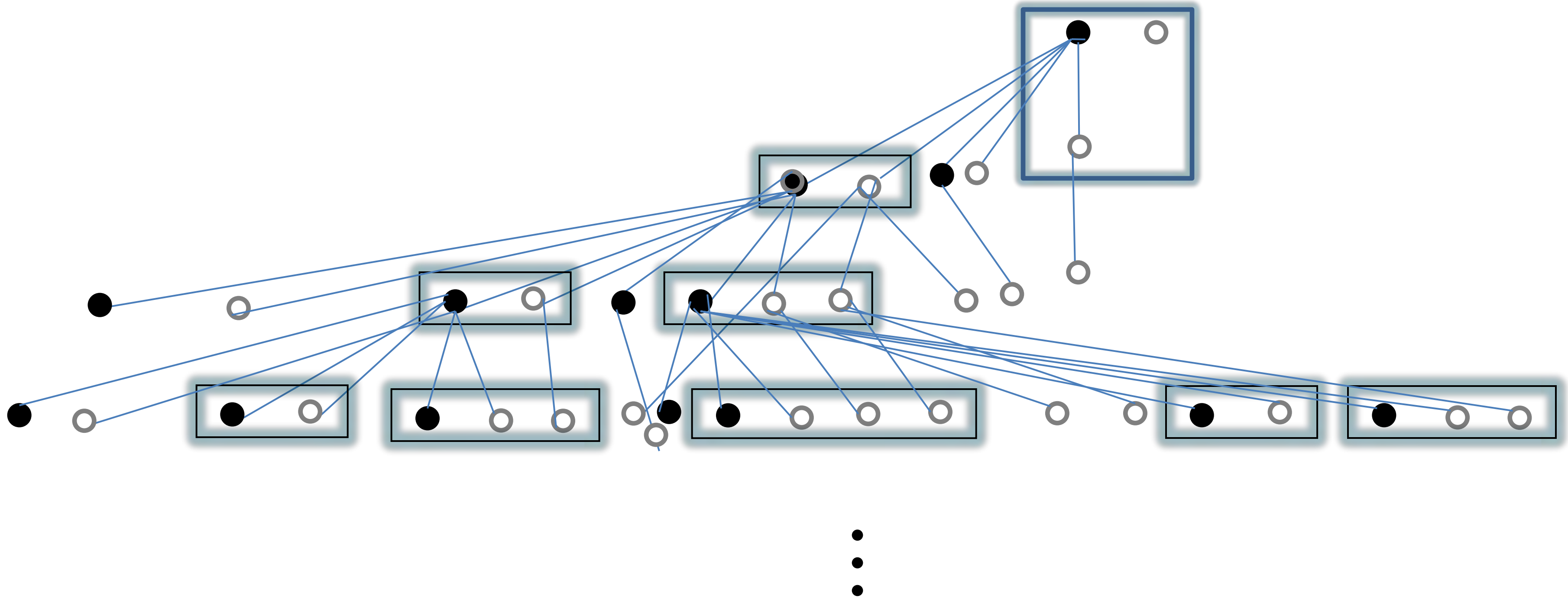
$Y_1(1)$

One generated free *MIPC*-frame



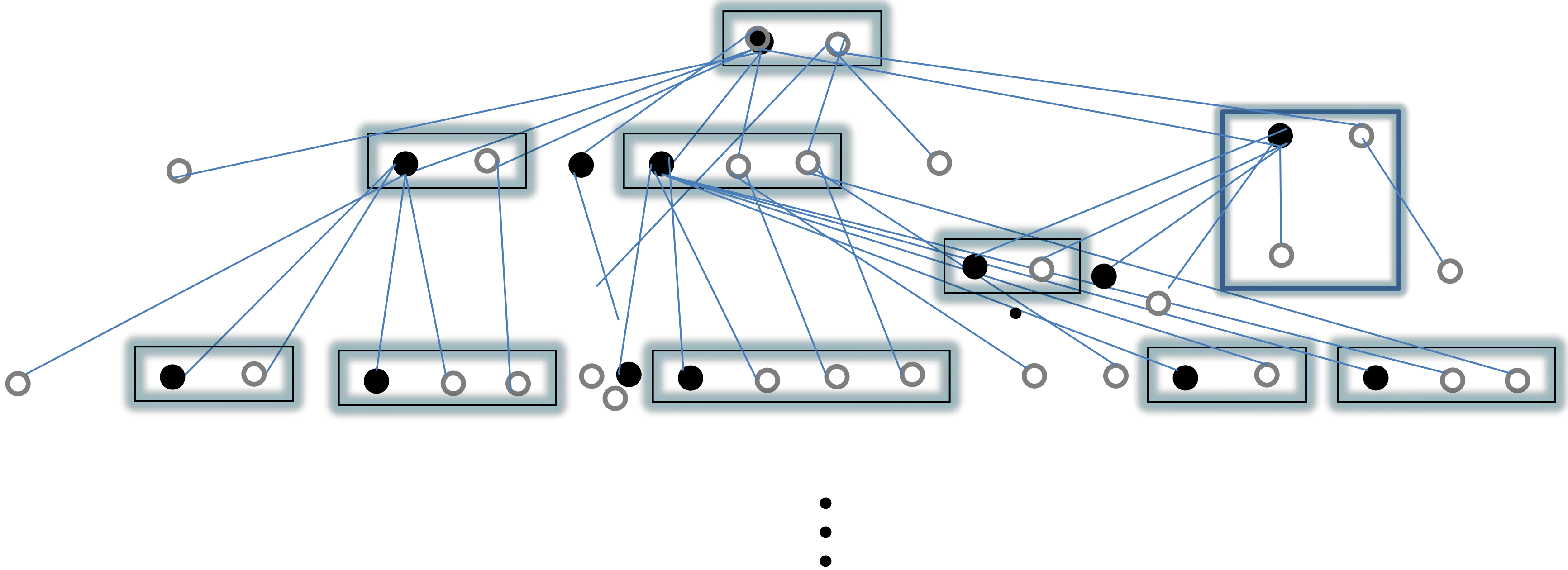
$Y_2(1)$

One generated free *MIPC*-frame



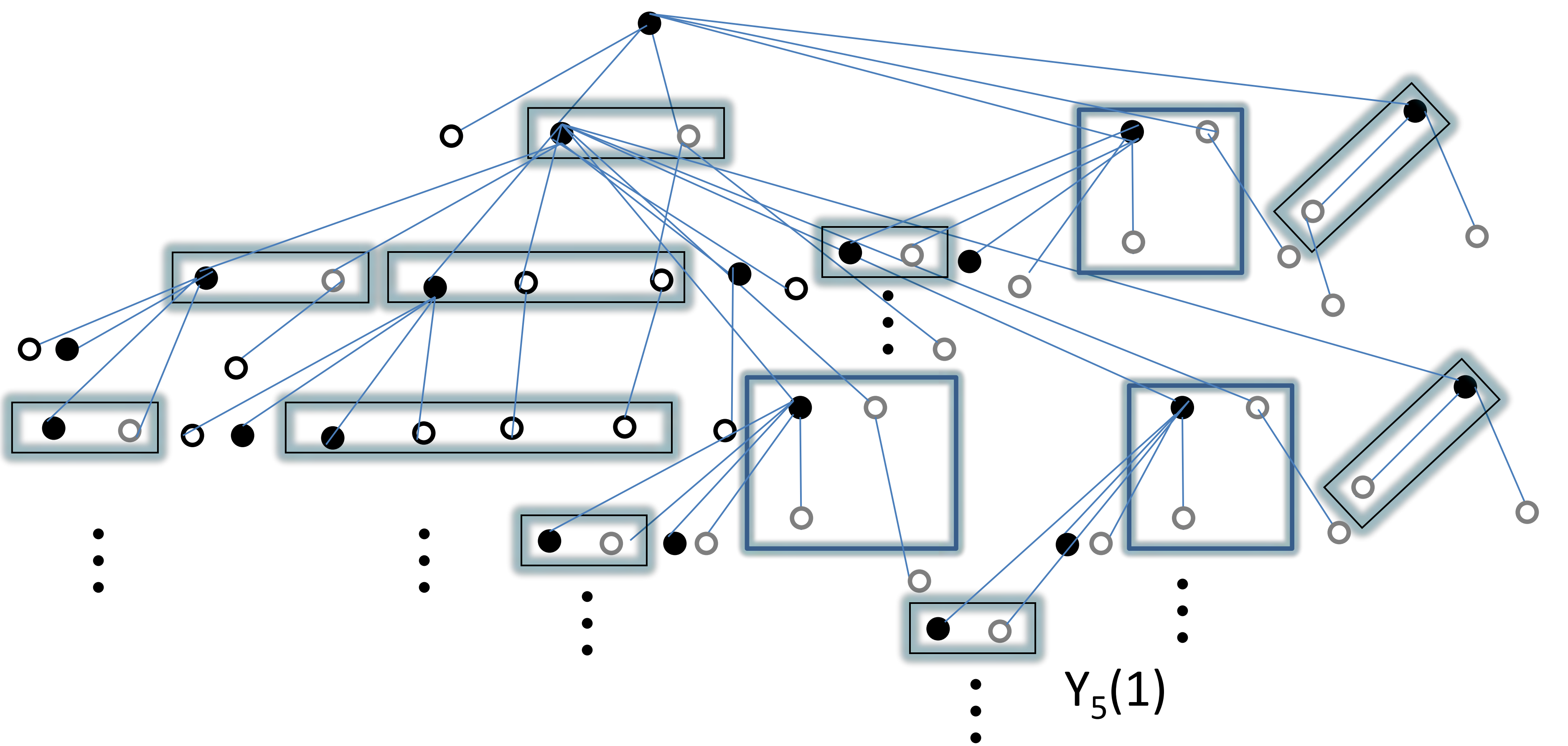
$Y_3(1)$

One generated free *MIPC*-frame



$Y_4(1)$

One generated free *MIPC*-frame



One generated free *MIPC*-frame

Let $Y(1) = Y_1(1) \cup Y_2(1) \cup Y_3(1) \cup Y_4(1) \cup Y_5(1)$ and *G* the set of elements of $Y(1)$ having (black) colour $\{1\}$.

Theorem 4. *Monadic Gödel algebra generated by the element* $G \subseteq Y(1)$ *is the free one-generated monadic Gödel algebra.*

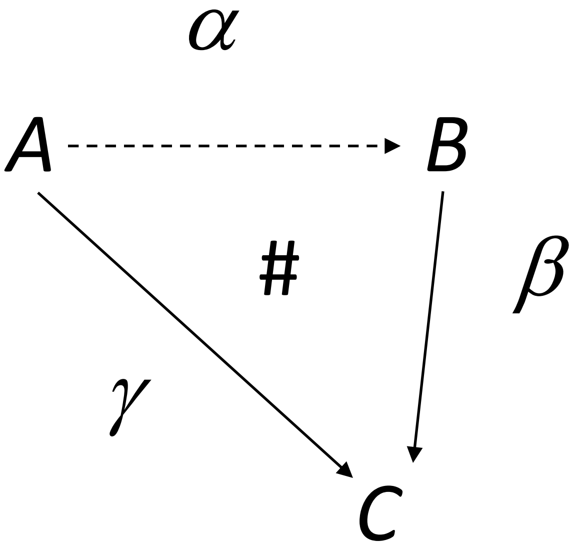
Projective algebras

Let $\mathbf{K}$ be a variety of algebras.

Definition. *An algebra* $A \in \mathbf{K}$ *is called* *projective*, *if for any* $B, C \in \mathbf{K}$, *any epimorphism (that is an onto homomorphism)* $\beta : B \to C$ *and any homomorphism* $\gamma : A \to C$, *there exists a homomorphism* $\alpha : A \to B$ *such that*

$$\beta \alpha = \gamma.$$

Projective algebras



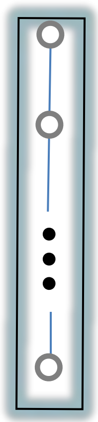
Projective algebras

- In varieties of algebras the projective algebras coincides with retracts of free algebras.

- An algebra A is *a retract* of an algebra B if there exist homomorphisms $\alpha : A \to B$ and $\beta : B \to A$ such that $\beta \alpha = \mathrm{Id}_A$.

m-generated projective *MG*-algebra

Theorem 5. *m-generated MG-algebra A is projective if and only if A contains either an atom $a \in A$ such that $\Diamond a = a$ or an atom such that $\Diamond a$ is isomorphic to K_n, where $1 < n \leq m+1$.*



K_n

m-generated projective *MG*-algebra

Definition. *An algebra A is called* ***finitely presented*** *if A is finitely generated, with the generators $a_1, \dots, a_m \in A$, and there exist a finite number of equations*

$$P_1(x_1,\dots, x_m) = Q_1(x_1,\dots, x_m)\ , \dots ,\ P_n(x_1,\dots, x_m) = Q_n(x_1, \dots , x_m)$$

holding in A on the generators $a_1, \dots, a_m \in A$ such that if there exists an m-generated algebra B, with generators $b_1, \dots, b_m \in B$, such that the equations

$$P_1(x_1,\dots, x_m) = Q_1(x_1,\dots, x_m)\ , \dots ,\ P_n(x_1,\dots, x_m) = Q_n(x_1, \dots , x_m)$$

hold in B on the generators $b_1, \dots, b_m \in B$, then there exists a homomorphism $h : A \to B$ sending a_i to b_i.

m-generated projective *MG*-algebra

- **Theorem 7**. *An MG-algebra* B *is finitely presented iff* $B \cong F_{\mathbf{MG}}(m)/[u)$, *where* $[u)$ *is a principal monadic filter generated by some element* $\square u \in F_{\mathbf{MG}}(m)$.

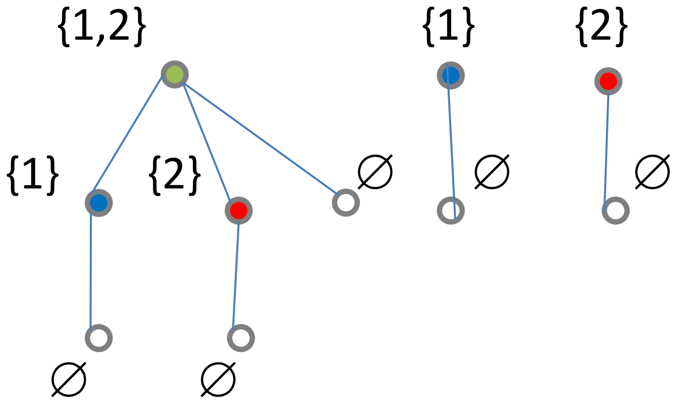
m-generated projective *MG*-algebra

- **Theorem 8**. *If A is m-generated finitely presented algebra, then* $A \times B_2$, $A \times C_n$, $A \times D_n$ *is projective algebra, where* B_2 *is two-element Boolean algebra,* C_n *is n-element chain MG-algebra and* D_n *is MG-algebra corresponding to MIPC-frame* K_n, $1 < n \leq m+1$.

m-generated projective *MG*-algebra

- **Theorem 8**. *If A is m-generated finitely presented algebra, then* $A \times B_2$, $A \times C_n$, $A \times D_n$ *is projective algebra, where* B_2 *is two-element Boolean algebra,* C_n *is n-element chain MG-algebra and* D_n *is MG-algebra corresponding to MIPC-frame* K_n, $1 < n \leq m+1$.

- **Theorem 9.** *Any m-generated subalgebra of m-generated free MG-algebra* $F_{\mathbf{MG}}(m)$ *is projective.*


{1,2}
{1}
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∅
∅
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