Canonical extension in first-order logic and Makkai's topos of types

Dion Coumans

Radboud University Nijmegen

Tbilisi, July 2012

Outline

1. Duality theory and canonical extension for propositional logic
2. Semantics for coherent first order logic ($\wedge$, $\vee$, $\bot$, $\top$, $\exists$):
   - Coherent hyperdoctrines
   - Coherent categories
3. Canonical extension in the categorical setting
4. Relation to Makkai's topos of types

Stone duality

Boolean algebras: structures $(B, \wedge, \vee, \neg, 0, 1)$.

Boolean spaces: compact, totally disconnected, Hausdorff spaces.

Boolean algebras	$\leftrightarrows$	**Boolean spaces**
$Cl(X)$	$\hookleftarrow$	X
B	$\mapsto$	$(PrFlt(B), \tau_B)$

Stone duality

Boolean algebras: structures $(B, \wedge, \vee, \neg, 0, 1)$.

Boolean spaces: compact, totally disconnected, Hausdorff spaces.

$$\begin{array}{ccc} \textbf{Boolean algebras} & \leftrightarrows & \textbf{Boolean spaces} \\ Cl(X) & \leftarrow & X \\ B & \mapsto & (PrFlt(B), \tau_B) \end{array}$$

Stone Representation Theorem: every Boolean algebra is embeddable in a powerset algebra.

Proof: for a Boolean algebra B,

$$B \cong Cl(PrFlt(B)) \hookrightarrow \mathcal{P}(PrFlt(B)).$$

Stone duality and canonical extension

Canonical extension: algebraic description of topological duality.

Study $B \cong Cl(PrFlt(B)) \hookrightarrow \mathcal{P}(PrFlt(B)) = B^\delta$.

$$\begin{tikzcd}
\mathbf{CABA} \arrow[r, shift left, "At"] \arrow[d, shift left, "U"] & \text{Sets} \arrow[l, shift left, "\mathcal{P}"] \\
\text{Boolean algebras} \arrow[u, shift left, "(_)^\delta"] \arrow[r, shift left, "PrFlt"] & \text{Boolean spaces} \arrow[l, shift left, "Cl"] \arrow[u, "U"']
\end{tikzcd}$$

$\mathbf{CABA}$	$=$	complete and atomic Boolean algebras.
Boolean spaces	$=$	compact, totally disconnected Hausdorff spaces.

Canonical extension for distributive lattices

Canonical extension: algebraic description of topological duality.

Study $L \cong ClDwn(PrFlt(L)) \hookrightarrow Dwn(PrFlt(L)) = L^{\delta}$.

$$
\begin{tikzcd}
\mathbf{DL}^{+} \arrow[rr, shift left, "\mathcal{J}^{\infty}"] \arrow[d, shift left, "U"] & & \text{posets} \arrow[ll, shift left, "Dwn"] \\
\text{distributive lattices} \arrow[u, shift left, "(\_)^{\delta}"] \arrow[rr, shift left, "PrFlt"] & & \text{Priestley spaces} \arrow[ll, shift left, "ClDwn"] \arrow[u, "U"']
\end{tikzcd}
$$

$\mathbf{DL}^{+}$	$=$	completely distributive algebraic lattices.
Priestley spaces	$=$	compact, totally order-disconnected Hausdorff spaces.

Canonical extension of distributive lattices

$\mathbf{DL}^+$ = completely distributive algebraic lattices.

Canonical extension is left adjoint to $\mathbf{DL}^+ \hookrightarrow \mathbf{DL}$.

Universal characterisation of canonical extension:

$$\begin{tikzcd} L \arrow[r, "e"] \arrow[dr, "f"'] & L^\delta \arrow[d, dotted, "\tilde{f}"] \\ & K \end{tikzcd}$$

where $L \in \mathbf{DL}$ and $K, L^\delta \in \mathbf{DL}^+$.

Interpolation in propositional logic

Let $\mathbb{T}$ be a theory in intuitionistic propositional logic.

Question: does $\mathbb{T}$ have the **interpolation property**, i.e.,

> for all formulas $\phi(p, q)$ and $\psi(p, r)$ with $\phi(p, q) \vdash_{\mathbb{T}} \psi(p, r)$, there exists a formula $\theta(p)$ s.t.
>
> $$\phi(p, q) \vdash_{\mathbb{T}} \theta(p) \quad \text{and} \quad \theta(p) \vdash_{\mathbb{T}} \psi(p, r).$$

Interpolation in propositional logic

Let $\mathbb{T}$ be a theory in intuitionistic propositional logic.

Question: does $\mathbb{T}$ have the **interpolation property**, i.e.,

> for all formulas $\phi(p,q)$ and $\psi(p,r)$ with $\phi(p,q) \vdash_{\mathbb{T}} \psi(p,r)$, there exists a formula $\theta(p)$ s.t.
>
> $$\phi(p,q) \vdash_{\mathbb{T}} \theta(p) \quad \text{and} \quad \theta(p) \vdash_{\mathbb{T}} \psi(p,r).$$

Question: are monomorphisms stable under pushout in $\mathcal{V}_{\mathbb{T}}$?

Interpolation in first-order logic

Let $\mathbb{T}$ be a theory in intuitionic first order logic.

Question: does $\mathbb{T}$ have the **interpolation property**, i.e.,

for all sentences ϕ, ψ with $\phi \vdash_{\mathbb{T}} \psi$, there exists a sentence θ s.t.

1. $\phi \vdash_{\mathbb{T}} \theta$ and $\theta \vdash_{\mathbb{T}} \psi$;
2. every relation and function symbol which occurs in θ occurs in both ϕ and ψ.

Open problem for some first order intuitionistic theories, e.g.,

$$\mathbb{T} = \mathsf{IFOL} + (\phi \to \psi) \vee (\psi \to \phi).$$

Algebraic semantics for coherent logic

We start from

Signature:	$\Sigma = (f_0, \dots, f_{k-1}, R_0, \dots, R_{l-1})$
Set of var's:	$X = \{x_0, x_1, \dots\}$
Equality:	$=$
Connectives:	$\wedge, \vee, \top, \bot, \exists$
Derivability notion:	$\vdash$ (given by axioms and rules)

Question:

What properties does the logic over Σ have?

Algebraic semantics for coherent logic

We start from

Signature:	$\Sigma = (f_0, \ldots, f_{k-1}, R_0, \ldots, R_{l-1})$
Set of var's:	$X = \{x_0, x_1, \ldots\}$
Equality:	$=$
Connectives:	$\wedge, \vee, \top, \bot, \exists$
Derivability notion:	$\vdash$ (given by axioms and rules)

Question:

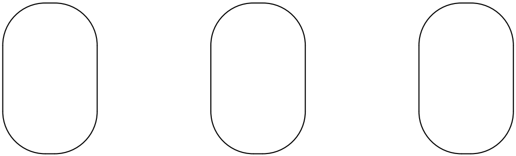
What properties does the logic over Σ have?

First observation:

For each sequence of variables $\vec{x} = \langle x_0, \ldots, x_{n-1} \rangle$,

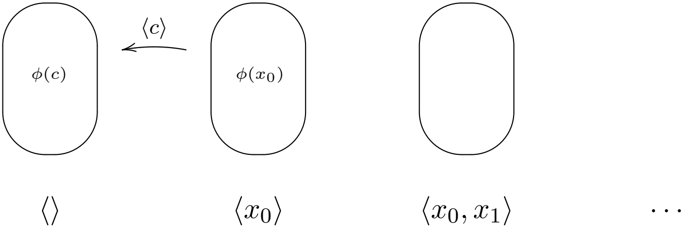
$(Fm(\vec{x})/_{\vdash \cap \dashv}, \vdash)$ is a distributive lattice.

Algebraic semantics for coherent logic



$\langle\rangle$ $\quad$ $\langle x_0 \rangle$ $\quad$ $\langle x_0, x_1 \rangle$ $\quad$ $\cdots$

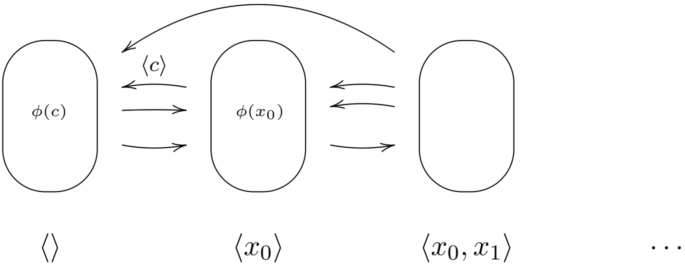
Algebraic semantics for coherent logic



Substitutions:

$$
\begin{array}{ccc}
x_0 & \mapsto & c \\
\phi(x_0) & \mapsto & \phi(c)
\end{array}
$$

Algebraic semantics for coherent logic



Substitutions:

$$
\begin{array}{ccc}
x_0 & \mapsto & c \\
\phi(x_0) & \mapsto & \phi(c)
\end{array}
$$

Algebraic semantics for coherent logic

Contexts and substitutions form a category $\mathbf{B}$:

Objects:	contexts $\vec{x}$
Morphism $\vec{x} \to \vec{y}$:	m-tuple $\langle t_0, \ldots, t_{m-1} \rangle$ s.t. $m = \mathit{length}(\vec{y})$ and $FV(t_i) \subseteq \vec{x}$

$$
\begin{tikzcd}
\langle\rangle \arrow[r, bend left, "\langle c \rangle"] \arrow[rr, bend right, "\langle c, f(c) \rangle"'] & \langle x_0 \rangle \arrow[loop above, "\langle x_0 \rangle"] \arrow[r, bend left, "\langle x_0, f(x_0) \rangle"] & \langle x_0, x_1 \rangle \arrow[l, bend left] & \ldots
\end{tikzcd}
$$

Algebraic semantics for coherent logic

Contexts and substitutions form a category $\mathbf{B}$:

Objects:	contexts $\vec{x}$
Morphism $\vec{x} \to \vec{y}$:	m-tuple $\langle t_0, \ldots, t_{m-1} \rangle$ s.t. $m = length(\vec{y})$ and $FV(t_i) \subseteq \vec{x}$

This category has **finite products**:

$$\begin{tikzcd}
\langle x_0,\, x_1 \rangle & \langle x_0,\, x_1,\, x_2 \rangle \arrow[l, "\langle x_0, x_1 \rangle"'] \arrow[r, "\langle x_2 \rangle"] & \langle x_0 \rangle \\
& \langle \ldots \rangle \arrow[ul, "\langle t_0, t_1 \rangle"] \arrow[u, dotted, "\langle t_0, t_1, s_0 \rangle"'] \arrow[ur, "\langle s_0 \rangle"'] &
\end{tikzcd}$$

Algebraic semantics for coherent logic

Formulas and substitutions: functor $\mathbf{B}^{op} \to \mathbf{DL}$

$$
\begin{array}{rcrcl}
\vec{x} & \mapsto & Fm(\vec{x}) & & \\
\vec{x} \xrightarrow{\langle t_0,\ldots,t_{m-1}\rangle} \vec{y} & \mapsto & Fm(\vec{y}) & \to & Fm(\vec{x}) \\
 & & \phi(\vec{y}) & \mapsto & \phi[\vec{t}/\vec{y}]
\end{array}
$$

$$
\begin{tikzcd}
\phi(c,\ f(c)) & \phi(x_0,\ f(x_0)) \arrow[l] & \phi(x_0,\ x_1) \arrow[l] \arrow[ll, bend left]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\langle\rangle \arrow[r, bend left, "\langle c\rangle"] \arrow[rr, bend right, "\langle c, f(c)\rangle"'] & \langle x_0\rangle \arrow[r, bend left, "\langle x_0, f(x_0)\rangle"] & \langle x_0, x_1\rangle & \cdots
\end{tikzcd}
$$

Algebraic semantics for coherent logic

Existential quantification: related to the inclusion map

$$\begin{tikzcd} \phi(x_0) \arrow[r, bend left, "i"] & \psi(x_0, x_1) \arrow[l, bend left, "\exists_{x_1}"] \end{tikzcd}$$

$$\langle x_0 \rangle \xleftarrow{\langle x_0 \rangle} \langle x_0, x_1 \rangle$$

$$\begin{array}{ccc} \exists_{x_1}(\psi(x_0, x_1)) & \vdash & \phi(x_0) \\ \hline\hline \psi(x_0, x_1) & \vdash & \phi(x_0) \end{array}$$

Existential quantification: related to the inclusion map

$$
\begin{tikzcd}
\phi(x_0) \arrow[r, bend left, "i"] & \psi(x_0, x_1) \arrow[l, bend left, "\exists_{x_1}"]
\end{tikzcd}
$$

$$\langle x_0 \rangle \xleftarrow{\langle x_0 \rangle} \langle x_0, x_1 \rangle$$

$$
\dfrac{\exists_{x_1}(\psi(x_0, x_1)) \quad \vdash_{x_0} \quad \phi(x_0)}{\psi(x_0, x_1) \quad \vdash_{x_0, x_1} \quad i(\phi(x_0))}
$$

Algebraic semantics for coherent logic

A **coherent hyperdoctrine** is a functor $P \colon \mathbf{B}^{\mathbf{op}} \to \mathbf{DL}$ s.t.

1. $\mathbf{B}$ is a category with finite limits;
2. for all $A \xrightarrow{\alpha} B \in \mathbf{B}$, $P(\alpha)$ has a left adjoint $\exists_\alpha$ with
   - **Frobenius reciprocity**, *i.e.*, for all $a \in P(A)$, $b \in P(B)$,

     $$\exists_\alpha(a \wedge P(\alpha)(b)) = \exists_\alpha(a) \wedge b$$

   - **Beck-Chevalley condition**, *i.e.*, for every pullback square

     $$\begin{tikzcd} Q \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & C \end{tikzcd}$$

     in $\mathbf{B}$, $P(\beta) \circ \exists_\alpha = \exists_{\alpha'} \circ P(\beta')$.

Algebraic semantics for coherent logic

Examples of coherent hyperdoctrines:

- **Syntactic hyperdoctrine**

  $\mathbf{B}$ = contexts and substitutions

$$\begin{array}{rccc} \mathcal{F}\colon & \mathbf{B}^{op} & \to & \mathbf{DL} \\ & \vec{x} & \mapsto & Fm(\vec{x})/_{\vdash\cap\dashv} \end{array}$$

- **Powerset hyperdoctrine**

  $\mathbf{B} = \mathbf{Set}$

$$\begin{array}{rccc} \mathcal{P}\colon & \mathbf{B}^{op} & \to & \mathbf{DL} \\ & A & \mapsto & \mathcal{P}(A) \\ & A \xrightarrow{f} B & \mapsto & \mathcal{P}(B) \xrightarrow{f^{-1}} \mathcal{P}(A). \end{array}$$

Coherent hyperdoctrines and coherent categories

A **coherent hyperdoctrine** is a functor $P\colon \mathbf{B}^{op} \to \mathbf{DL}$ s.t.

1. $\mathbf{B}$ has finite limits;
2. for all $A \xrightarrow{\alpha} B$ in $\mathbf{B}$, $P(\alpha)$ has a left adjoint satisfying Frobenius and Beck-Chevalley.

Coherent hyperdoctrines and coherent categories

A **coherent hyperdoctrine** is a functor $P\colon \mathbf{B}^{op} \to \mathbf{DL}$ s.t.

1. $\mathbf{B}$ has finite limits;
2. for all $A \xrightarrow{\alpha} B$ in $\mathbf{B}$, $P(\alpha)$ has a left adjoint satisfying Frobenius and Beck-Chevalley.

A **coherent category** is a category $\mathbf{C}$ satisfying

1. $\mathbf{C}$ has finite limits;
2. $\mathbf{C}$ has stable finite unions;
3. $\mathbf{C}$ has stable images.

Coherent hyperdoctrines and coherent categories

A **coherent hyperdoctrine** is a functor $P \colon \mathbf{B}^{op} \to \mathbf{DL}$ s.t.

1. $\mathbf{B}$ has finite limits;
2. for all $A \xrightarrow{\alpha} B$ in $\mathbf{B}$, $P(\alpha)$ has a left adjoint satisfying Frobenius and Beck-Chevalley.

A **coherent category** is a category $\mathbf{C}$ satisfying

1. $\mathbf{C}$ has finite limits;
2. $\mathbf{C}$ has stable finite unions;
3. $\mathbf{C}$ has stable images.

Remark: for a coherent category $\mathbf{C}$, $Sub_{\mathbf{C}} \colon \mathbf{C}^{op} \to \mathbf{DL}$ is a coherent hyperdoctrine.

Coherent hyperdoctrines and coherent categories

Proposition: there is a 2-categorical adjunction

$$\mathcal{A}\colon \mathbf{CHyp} \leftrightarrows \mathbf{Coh}\colon \mathcal{S},$$

where $\mathcal{A} \dashv \mathcal{S}$ and $\mathcal{A}(\mathcal{S}(\mathbf{C})) \simeq \mathbf{C}$.

Coherent hyperdoctrines and coherent categories

Proposition: there is a 2-categorical adjunction

$$\mathcal{A} \colon \mathbf{CHyp} \leftrightarrows \mathbf{Coh} \colon \mathcal{S},$$

where $\mathcal{A} \dashv \mathcal{S}$ and $\mathcal{A}(\mathcal{S}(\mathbf{C})) \simeq \mathbf{C}$.

For $\mathbf{C} \in \mathbf{Coh}$, $\mathcal{S}(\mathbf{C}) = \mathcal{S}_{\mathbf{C}} \colon \mathbf{C}^{op} \rightarrow \mathbf{DL}$
$$A \mapsto Sub_{\mathbf{C}}(A)$$

Coherent hyperdoctrines and coherent categories

Proposition: there is a 2-categorical adjunction

$$\mathcal{A}\colon \mathbf{CHyp} \leftrightarrows \mathbf{Coh}\colon \mathcal{S},$$

where $\mathcal{A} \dashv \mathcal{S}$ and $\mathcal{A}(\mathcal{S}(\mathbf{C})) \simeq \mathbf{C}$.

For $\mathbf{C} \in \mathbf{Coh}$, $\mathcal{S}(\mathbf{C}) = \mathcal{S}_{\mathbf{C}}\colon \begin{array}{rcl} \mathbf{C}^{op} & \to & \mathbf{DL} \\ A & \mapsto & Sub_{\mathbf{C}}(A) \end{array}$

For $P\colon \mathbf{B}^{op} \to \mathbf{DL}$, $\mathcal{A}(P)$ is the category with:

objects are pairs (A, a), where $A \in \mathbf{B}$, $a \in P(A)$;

a morphism $(A, a) \to (B, b)$ is an element $f \in P(A \times B)$ which is a functional relation $(A, a) \to (B, b)$.

Canonical extension of coherent hyperdoctrines

Recall: canonical extension for DL's is a functor $\mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}^+$.

Definition

For a coh. hyperdoctrine $P\colon \mathbf{B}^{op} \to \mathbf{DL}$ we define:

$$P^\delta\colon \mathbf{B}^{op} \xrightarrow{P} \mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}.$$

Canonical extension of coherent hyperdoctrines

Recall: canonical extension for DL's is a functor $\mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}^+$.

Definition

For a coh. hyperdoctrine $P\colon \mathbf{B}^{op} \to \mathbf{DL}$ we define:

$$P^\delta\colon \mathbf{B}^{op} \xrightarrow{P} \mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}.$$

Proposition

For a coh. hyperdoctrine P, P^δ is again a coh. hyperdoctrine.

Proof: check that, for all $A \xrightarrow{\alpha} B$ in $\mathbf{B}$, $P^\delta(\alpha)$ has a left adjoint satisfying BC and Frobenius.

Canonical extension of coherent categories

We have:

- adjunction $\mathcal{A}\colon \mathbf{CHyp} \leftrightarrows \mathbf{Coh}\colon \mathcal{S}$, $\mathbf{C} \simeq \mathcal{A}(\mathcal{S}_{\mathbf{C}})$;
- for $P \in \mathbf{CHyp}$, $P^{\delta}\colon \mathbf{B}^{op} \xrightarrow{P} \mathbf{DL} \xrightarrow{(\_)^{\delta}} \mathbf{DL}$.

Definition

For a coherent category $\mathbf{C}$ we define:

$$\mathbf{C}^{\delta} = \mathcal{A}(\mathcal{S}^{\delta}_{\mathbf{C}}).$$

Canonical extension of coherent categories

We have:

- adjunction $\mathcal{A}\colon \mathbf{CHyp} \leftrightarrows \mathbf{Coh}\colon \mathcal{S}$, $\mathbf{C} \simeq \mathcal{A}(\mathcal{S}_{\mathbf{C}})$;
- for $P \in \mathbf{CHyp}$, $P^{\delta}\colon \mathbf{B}^{op} \xrightarrow{P} \mathbf{DL} \xrightarrow{(\_)^{\delta}} \mathbf{DL}$.

Definition

For a coherent category $\mathbf{C}$ we define:

$$\mathbf{C}^{\delta} = \mathcal{A}(\mathcal{S}^{\delta}_{\mathbf{C}}).$$

Proposition

For a distributive lattice $\mathbf{L}$, $\mathcal{A}(\mathcal{S}^{\delta}_{\mathbf{L}}) \simeq \mathbf{L}^{\delta}$.

Canonical extension of coherent categories

Properties of $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}^\delta_{\mathbf{C}})$:

1. subobject lattices are in $\mathbf{DL}^+$;
2. pullback morphisms are complete lattice homomorphisms.

$\mathbf{Coh}^+$ = coherent categories satisfying (1) and (2).

Canonical extension of coherent categories

Properties of $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}^\delta_{\mathbf{C}})$:

1. subobject lattices are in $\mathbf{DL}^+$;
2. pullback morphisms are complete lattice homomorphisms.

$\mathbf{Coh}^+$ = coherent categories satisfying (1) and (2).

Universal characterisation:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "M_0"] \arrow[dr, "M"'] & \mathbf{C}^\delta \arrow[d, dotted, "\tilde{M}"] \\ & \mathbf{E} \end{tikzcd}$$

where $\mathbf{C} \in \mathbf{Coh}$, $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{Coh}^+$, M a coherent functor satisfying:

Canonical extension of coherent categories

Properties of $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}^\delta_{\mathbf{C}})$:

1. subobject lattices are in $\mathbf{DL}^+$;
2. pullback morphisms are complete lattice homomorphisms.

$\mathbf{Coh}^+$ = coherent categories satisfying (1) and (2).

Universal characterisation:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "M_0"] \arrow[dr, "M"'] & \mathbf{C}^\delta \arrow[d, dotted, "\tilde{M}"] \\ & \mathbf{E} \end{tikzcd}$$

where $\mathbf{C} \in \mathbf{Coh}$, $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{Coh}^+$, M a coherent functor satisfying:

for all $A \xrightarrow{\alpha} B$ in $\mathbf{C}$, ρ (prime) filter in $\mathcal{S}_C(A)$,

$\exists_{M(\alpha)}(\bigwedge\{M(U) \mid U \in \rho\}) \cong \bigwedge\{\exists_{M(\alpha)}(M(U)) \mid U \in \rho\}$.

Canonical extension of Heyting categories

Heyting categories provide semantics for first order logic.

Canonical extension interacts well with Heyting structure:

- for a coherent category $\mathbf{C}$, $\mathbf{C}^\delta$ is a Heyting category;
- for a Heyting category $\mathbf{C}$, $\mathbf{C} \hookrightarrow \mathbf{C}^\delta$ is a Heyting functor;
- for a morphism of Heyting categories $F\colon \mathbf{C} \to \mathbf{D}$,

$$F^\delta\colon \mathbf{C}^\delta \to \mathbf{D}^\delta$$

  is again Heyting functor.

Topos of types

Topos of types was introduced by Makkai in 1979 as:

- 'a reasonable codification of the 'discrete' (non topological) syntactical structure of types of the theory',
- a tool to prove representation theorems,
- 'conceptual tool meant to enable us to formulate precisely certain natural intuitive questions'.

Some later work by: Magnan & Reyes and Butz.

Topos of types

Topos of types was introduced by Makkai in 1979 as:

- 'a reasonable codification of the 'discrete' (non topological) syntactical structure of types of the theory',
- a tool to prove representation theorems,
- 'conceptual tool meant to enable us to formulate precisely certain natural intuitive questions'.

Some later work by: Magnan & Reyes and Butz.

Alternative construction:

The functor $\mathcal{S}^{\delta}_{\mathbf{C}} \colon \mathbf{C}^{op} \to \mathbf{DL}$ is an internal locale in $Sh(\mathbf{C}, J_{coh})$.

Then $Sh(\mathcal{S}^{\delta}_{\mathbf{C}}) \simeq T(\mathbf{C}) =$ topos of types of $\mathbf{C}$.

Topos of types

Let $\mathcal{L}$ be a coherent logic, $\mathfrak{A} = (A, \ldots)$ a model of $\mathcal{L}$.

For $a \in A$, the **type** of a in $\mathfrak{A}$ is given by:

$$t(a, \mathfrak{A}) := \{\phi(x) \,|\, \mathfrak{A} \vDash \phi[a]\}.$$

This is a **prime filter** in

$$Fm_{\mathcal{L}}(\langle x \rangle)/_{\vdash \cap \dashv} = Sub_{\mathbf{C}_{\mathcal{L}}}(x \,|\, \top)$$

(where $\mathbf{C}_{\mathcal{L}}$ = syntactic category of $\mathcal{L}$).

Idea: study prime filters in subobject lattices.

Topos of types

Makkai defined, for a coherent category $\mathbf{C}$,

$$\mathbf{T(C) = Sh(\tau C, J_p)}.$$

The category $\tau\mathbf{C}$ consists of

Objects:	pairs (A, ρ) where $A \in \mathbf{C}$ and ρ prime filter in $Sub_{\mathbf{C}}(A)$
Morphisms:	local continuous maps

Topology J_p is the topology induced by the coherent topology on the category of filters of $\mathbf{C}$.

Topos of types

Makkai defined, for a coherent category $\mathbf{C}$,

$$\mathbf{T(C) = Sh(\tau C, J_p)}.$$

The category $\tau\mathbf{C}$ consists of

Objects: pairs (A, ρ) where $A \in \mathbf{C}$ and ρ prime filter in $Sub_{\mathbf{C}}(A)$

Morphisms: local continuous maps

Topology J_p is the topology induced by the coherent topology on the category of filters of $\mathbf{C}$.

Theorem: for a coherent category $\mathbf{C}$, $T(\mathbf{C}) \simeq Sh(\mathcal{S}^{\delta}_{\mathbf{C}})$.

Topos of types

Theorem: $T(\mathbf{C}) \simeq Sh(\mathcal{S}^{\delta}_{\mathbf{C}})$.

Proof:

- $T(\mathbf{C}) = Sh(\tau\mathbf{C})$
  $\tau\mathbf{C}$: pairs (A, ρ) with $A \in \mathbf{C}$ and ρ prime filter in $Sub_{\mathbf{C}}(A)$.

- $Sh(\mathcal{S}^{\delta}_{\mathbf{C}}) \simeq Sh(\mathbf{C} \ltimes \mathcal{S}^{\delta}_{\mathbf{C}})$
  $\mathbf{C} \ltimes \mathcal{S}^{\delta}_{\mathbf{C}}$: pairs (A, u) with $A \in \mathbf{C}$ and $u \in \mathcal{S}^{\delta}_{\mathbf{C}}(A)$.

We have:

$$T(\mathbf{C}) = Sh(\tau\mathbf{C}) \simeq Sh(\mathbf{D}) \simeq Sh(\mathbf{C} \ltimes \mathcal{S}^{\delta}_{\mathbf{C}}) \simeq Sh(\mathcal{S}^{\delta}_{\mathbf{C}}).$$

($\mathbf{D}$ = subcategory of $\mathbf{C} \ltimes \mathcal{S}^{\delta}_{\mathbf{C}}$ of pairs (A, x) with $x \in J^{\infty}(\mathcal{S}^{\delta}_{\mathbf{C}}(A))$).

Topos of types and morphisms

Theorem: for a coherent functor $F\colon \mathbf{C} \to \mathbf{D}$,

- if F is conservative, then $T(F)\colon T(\mathbf{D}) \to T(\mathbf{C})$ is a geometric surjection;
- if F is a morphism of Heyting categories, then $T(F)\colon T(\mathbf{D}) \to T(\mathbf{C})$ is open.

Proof: use facts on

- canonical extension of lattice homomorphism
- correspondence between internal locale morphisms and geometric morphisms

Topos of types and the class of models

For a distributive lattice L,

$$\begin{array}{rcl} \text{prime filters of } L & \leftrightarrow & \text{lattice homomorphisms } L \to \mathbf{2} \\ & \leftrightarrow & \text{`models of } L\text{'}. \end{array}$$

$L^\delta = Dwn(Mod(L))$.

Categorical analogue:

$Mod(\mathbf{C})$ = coherent functors $M : \mathbf{C} \to \mathbf{Set}$.

Study: $\mathbf{Set}^{Mod(\mathbf{C})}$.

We have to restrict to an appropriate subcategory $\mathcal{K}$ of $Mod(\mathbf{C})$.

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh(\mathcal{S}_{\mathbf{C}}^{\delta})$?

Topos of types and the class of models

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh(\mathcal{S}^{\delta}_{\mathbf{C}})$?

Evaluation functor

$$\begin{array}{rcl} ev\colon \mathbf{C} & \to & \mathbf{Set}^{\mathcal{K}} \\ A & \mapsto & \begin{array}{rcl} ev(A)\colon \mathcal{K} & \to & \mathbf{Set} \\ M & \mapsto & M(A) \end{array} \end{array}$$

Gives a geometric morphism $\phi_{ev}\colon \mathbf{Set}^{\mathcal{K}} \to Sh(\mathbf{C}, J_{coh})$.

Topos of types and the class of models

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh(\mathcal{S}^{\delta}_{\mathbf{C}})$?

Evaluation functor

$$\begin{aligned} ev\colon \mathbf{C} &\to \mathbf{Set}^{\mathcal{K}} \\ A &\mapsto \begin{array}{rcl} ev(A)\colon \mathcal{K} & \to & \mathbf{Set} \\ M & \mapsto & M(A) \end{array} \end{aligned}$$

Gives a geometric morphism $\phi_{ev}\colon \mathbf{Set}^{\mathcal{K}} \to Sh(\mathbf{C}, J_{coh})$.

Theorem: the topos of types yields the hyper-connected localic factorisation of $\mathbf{Set}^{\mathcal{K}} \xrightarrow{\phi_{ev}} Sh(\mathbf{C}, J_{coh})$:

$$\begin{tikzcd} & T(\mathbf{C}) \arrow[d, two heads] \\ \mathbf{Set}^{\mathcal{K}} \arrow[ur, two heads] \arrow[r, two heads, "\phi_{ev}"'] & Sh(\mathbf{C}, J_{coh}) \end{tikzcd}$$

Topos of types and the class of models

Theorem: the topos of types yields the hyper-connected localic factorisation of $\mathbf{Set}^{\mathcal{K}} \xrightarrow{\phi_{ev}} Sh(\mathbf{C}, J_{coh})$.

Description of the factorisation:

$$\begin{tikzcd}
 & Sh((\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})) \arrow[d] \\
\mathbf{Set}^{\mathcal{K}} \arrow[ur] \arrow[r, "\phi_{ev}"'] & Sh(\mathbf{C}, J_{coh})
\end{tikzcd}$$

$T(\mathbf{C}) = Sh(S^{\delta}_{\mathbf{C}})$

To prove: $S^{\delta}_{\mathbf{C}} \cong (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})$ in $Sh(\mathbf{C}, J_{coh})$.

Topos of types and the class of models

To prove: $S^{\delta}_{\mathbf{C}} \cong (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})$ in $Sh(\mathbf{C}, J_{coh})$

Recall:
$$\begin{tikzcd}
\mathbf{C} \arrow[r, "y"] \arrow[dr, "ev"'] & Sh(\mathbf{C}, J_{coh}) \arrow[d, bend right] \\
& \mathbf{Set}^{\mathcal{K}} \arrow[u, bend right, "\phi_{ev}"']
\end{tikzcd}$$

Hence, for $A \in \mathbf{C}$,

$$\begin{aligned}
(\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})(A) &= Hom_{\mathbf{Set}^{\mathcal{K}}}(ev(A), \Omega_{Set^{\mathcal{K}}}) \\
&= Sub(ev(A)).
\end{aligned}$$

Let $\sigma_A \colon S^{\delta}_{\mathbf{C}}(A) \to (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})(A)$ be the unique map given by:

To prove: $S^{\delta}_{\mathbf{C}} \cong (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})$ in $Sh(\mathbf{C}, J_{coh})$

Recall:
$$\begin{tikzcd} \mathbf{C} \arrow[r, "y"] \arrow[dr, "ev"'] & Sh(\mathbf{C}, J_{coh}) \arrow[d, bend right] \\ & \mathbf{Set}^{\mathcal{K}} \arrow[u, bend right, "\phi_{ev}"'] \end{tikzcd}$$

Hence, for $A \in \mathbf{C}$,

$$\begin{aligned} (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})(A) &= Hom_{\mathbf{Set}^{\mathcal{K}}}(ev(A), \Omega_{Set^{\mathcal{K}}}) \\ &= Sub(ev(A)). \end{aligned}$$

Let $\sigma_A \colon S^{\delta}_{\mathbf{C}}(A) \to (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})(A)$ be the unique map given by:

$$\begin{aligned} Sub_{\mathbf{C}}(A) &\to Sub(ev(A)) \\ U &\mapsto ev(U). \end{aligned}$$

Future work

- Study the following diagram (where $\mathcal{K} \subseteq Mod(\mathbf{C})$):

$$
\begin{tikzcd}
 & T(\mathbf{C}) \arrow[d, two heads] \\
\mathbf{Set}^{\mathcal{K}} \arrow[ur, two heads] \arrow[r, two heads, "\phi_{ev}"'] & Sh(\mathbf{C}, J_{coh})
\end{tikzcd}
$$

- Apply the developed theory in the study of first order logics:
  - study interpolation problems for first order logics, e.g. for IPL $+ (\phi \to \psi) \vee (\psi \to \phi)$;
  - study problems in model theory.

Properties of the class $\mathcal{K}$

The category $\mathcal{K} \hookrightarrow Mod(\mathbf{C})$ should satisfy:

1. for all $M \colon \mathbf{C} \to \mathbf{Set}$ in $\mathcal{K}$, $A \in \mathbf{C}$, ρ prime filter in $Sub_{\mathbf{C}}(A)$,

$$\exists_{M(\alpha)}(\bigwedge\{M(U) \,|\, U \in \rho\}) \cong \bigwedge\{\exists_{M(\alpha)}(M(U)) \,|\, U \in \rho\};$$

2. for all $A \in \mathbf{C}$, ρ prime filter in $Sub_{\mathbf{C}}(A)$, there exist $M \in \mathcal{K}$ and $a \in M(A)$ s.t.

$$\rho = t_A(a, M) = \{U \in Sub_{\mathbf{C}}(A) \,|\, a \in M(U)\};$$

3. for all $A \in \mathbf{C}$, $M, N \in \mathcal{K}$, $a \in M(A)$, $b \in N(A)$, if

$$b \in \bigwedge\{N(U) \,|\, U \in t_A(a, M)\}$$

then there exists a morphism $h \colon M \to N$ s.t. $b = h_A(a)$.