Completeness and definability of a modal logic interpreted over iterated strict partial orders

Philippe Balbiani    Levan Uridia

Institut de recherche en informatique de Toulouse
CNRS — Université de Toulouse

Escuela Técnica Superior de Ingeniería Informática
Universidad Rey Juan Carlos
Introduction

Given a topology $\tau$ on a nonempty set $X$

- the $\tau$-derived set $d_\tau(A)$ of a set $A \subseteq X$ of points
  
  \[ d_\tau(A) = \text{the set of all limit points of } A \text{ with respect to } \tau \]

The derivative operator $d_\tau$ possesses interesting properties

- a set $A \subseteq X$ of points is $\tau$-closed
  
  iff

  \[ d_\tau(A) \subseteq A \]

What happens if we iterate the derivative operator $d_\tau$

- considering the sequence $d_\tau$, $d_\tau \circ d_\tau$, $\ldots$ of operators
If \( \tau \) is \( T_D \), then

- each element \( d^\alpha \) of this sequence is a derivative operator

A question arises

- what is the link between the topologies \( \tau^\alpha \) corresponding to the elements \( d^\alpha \) of the sequence

The answer is simple

- the topologies \( \tau^\alpha \) are getting finer when \( \alpha \) increases
The lattice of all $T_D$ topologies on $X$ is complete
  ▶ this iteration process should stop

The Cantor-Bendixson rank of $(X, \tau)$ is defined as
  ▶ the least ordinal $\alpha$ such that $d_\tau(d^\alpha(\tau)(X)) = d^\alpha(\tau)(X)$

A consequence of Tarski’s fixpoint theorem is that
  ▶ there exists an ordinal $\alpha^*$ such that $\alpha \leq \alpha^*$ and $d_\tau \circ d^\alpha(\tau) = d^\alpha(\tau)$
Any strict partial order $R$ on $X$ defines a function $\theta_R$ which associates to each strict partial order $S \subseteq R$ on $X$ the strict partial order $\theta_R(S) = R \circ S$ on $X$.

What happens if we iterate the function $\theta_R$

- considering the sequence $R, \theta_R(R), \ldots$ of partial orders

Simply

- the partial orders $\theta^\alpha_R(R)$ are getting smaller when $\alpha$ increases
The lattice of all strict partial orders on $X$ is complete

- this iteration process should stop

There exists an ordinal $\alpha^*$ such that

- $\theta_R(\theta_R^{\alpha^*}(R)) = \theta_R^{\alpha^*}(R)$

If $R$ is the strict partial order on $X$ corresponding to a given Alexandroff $T_D$ derivative operator $d$, then

- $\theta_R^{\alpha^*}(R)$ is a strict partial order on $X$ corresponding to the derivative operator $d_T^{\alpha^*}$ considered above
1. Introduction
2. Topologies and derivative operators
3. Alexandroff $T_D$ derivative operators and strict partial orders
4. Cantor-Bendixson ranks
5. A modal logic
6. Axiomatization and completeness
7. Definability
8. Notes
A topology on $X$ is a set $\tau$ of subsets of $X$ such that

- $\emptyset \in \tau$
- $X \in \tau$
- if $A, B \in \tau$, then $A \cap B \in \tau$
- if $(A_i)_i$ is a collection of subsets of $X$ such that $A_i \in \tau$ for every $i$, then $\bigcup_i A_i \in \tau$

We shall say that

- $A \subseteq X$ is $\tau$-closed iff $X \setminus A \in \tau$
\[ \tau \] is said to be \( T_D \) iff

\begin{itemize}
  \item for all \( x \in X \), there exists \( A, B \in \tau \) such that \( A \setminus B = \{ x \} \)
\end{itemize}

We shall say that \( \tau \) is **Alexandroff** iff

\begin{itemize}
  \item each intersection of members of \( \tau \) is in \( \tau \)
\end{itemize}
Topologies and derivative operators

Topologies

Let $\leq$ be the binary relation between topologies on $X$ such that
- $\tau \leq \tau'$ iff $\tau \subseteq \tau'$

Remark that for all topologies $\tau, \tau'$ on $X$
- if $\tau \leq \tau'$, then if $\tau$ is $T_D$, then $\tau'$ is $T_D$

Example: the Sierpiński space
- $X = \{x, y\}$
- $\tau = \{\emptyset, \{x\}, X\}$
Topologies and derivative operators

Topologies

Given a topology $\tau$ on $X$

- let $L_\tau$ be the set of all topologies $\tau'$ on $X$ such that $\tau \leq \tau'$

Remark that

- the least element of $L_\tau$ is $\tau$
- the greatest element of $L_\tau$ is the topology $\mathcal{P}(X)$
- the least upper bound of a family $\{\tau'_i: i \in I\}$ in $L_\tau$ is the intersection of all $\tau' \in L_\tau$ such that $\bigcup\{\tau'_i: i \in I\} \subseteq \tau'$
- the greatest lower bound of a family $\{\tau'_i: i \in I\}$ in $L_\tau$ is $\bigcap\{\tau'_i: i \in I\}$
- $(L_\tau, \leq)$ is a complete lattice
A derivative operator on $X$ is a function $d: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ such that

- $d(\emptyset) = \emptyset$
- for all $A, B \subseteq X$, $d(A \cup B) = d(A) \cup d(B)$
- for all $A \subseteq X$, $d(d(A)) \subseteq d(A) \cup A$
- for all $x \in X$, $x \not\in d(\{x\})$

$A \subseteq X$ is said to be

- d-closed iff $d(A) \subseteq A$
Topologies and derivative operators

Derivative operators

We shall say that $d$ is $T_D$ iff

- for all $A \subseteq X$, $d(d(A)) \subseteq d(A)$

$d$ is said to be **Alexandroff** iff

- for all $x \in X$, there exists a greatest $A \subseteq X$ such that $A$ is $d$-closed and $x \notin A$
Topologies and derivative operators

Derivative operators

Let $\leq$ be the binary relation between derivative operators on $X$ such that

$\quad d \leq d'$ iff for all $A \subseteq X$, $d(A) \subseteq d'(A)$

Remark that for all derivative operators $d, d'$ on $X$

$\quad$ if $d \leq d'$, then if $d'$ is $T_D$, then $d$ is $T_D$

Example

$\quad X = \{x, y\}$

$\quad d(\emptyset) = \emptyset$

$\quad d(\{x\}) = \{y\}$

$\quad d(\{y\}) = \emptyset$

$\quad d(X) = \{y\}$
Given a derivative operator $d$ on $X$

- let $L_d$ be the set of all derivative operators $d'$ on $X$ such that $d' \leq d$

Remark that

- the least element of $L_d$ is the derivative operator $d_\emptyset : \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$, $d_\emptyset(A) = \emptyset$
- the greatest element of $L_d$ is $d$
- we do not know any representation of the least upper bound and the greatest lower bound of a family $\{d'_i : i \in I\}$ in $L_d$
- $(L_d, \leq)$ is a complete lattice
Topologies and derivative operators

Topologies v. derivative operators

Given a topology $\tau$ on $X$

- let $d_\tau$ be the function $d_\tau : \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$, $d_\tau(A) = \{x : x$ is a $\tau$-limit point of $A\}$

Remark that

- $d_\tau$ is a derivative operator on $X$
- for all $A \subseteq X$, $A$ is $d_\tau$-closed iff $A$ is $\tau$-closed
- $d_\tau$ is $T_D$ iff $\tau$ id $T_D$
- $d_\tau$ is Alexandroff iff $\tau$ is Alexandroff
- $d_\tau' \leq d_\tau$ iff $\tau \leq \tau'$
Given a derivative operator $d$ on $X$
- let $\tau_d$ be the set of $d$-open subsets of $X$

Remark that
- $\tau_d$ is a topology on $X$
- for all $A \subseteq X$, $A$ is $\tau_d$-closed iff $A$ is $d$-closed
- $\tau_d$ is $T_D$ iff $d$ id $T_D$
- $\tau_d$ is Alexandroff iff $d$ is Alexandroff
- $\tau_d' \leq \tau_d$ iff $d \leq d'$
Topologies and derivative operators
Topologies v. derivative operators

Let us further remark that

\[ \tau_{d\tau} = \tau \]
\[ d_{\tau d} = d \]

Given a topology \( \tau \) on \( X \)

\[ \text{the function } f: L_\tau \rightarrow L_{d\tau} \text{ such that } f(\tau') = d_{\tau'} \text{ is an anti-isomorphism between } (L_{d\tau}, \leq) \text{ and } (L_\tau, \leq) \]

Given a derivative operator \( d \) on \( X \)

\[ \text{the function } f: L_d \rightarrow L_{\tau d} \text{ such that } f(d') = \tau_{d'} \text{ is an anti-isomorphism between } (L_{\tau d}, \leq) \text{ and } (L_d, \leq) \]
Alexanderoff $T_D$ derivative operators and strict partial orders

Alexanderoff $T_D$ derivative operators

Given an Alexanderoff $T_D$ derivative operator $d$ on $X$

- let $L_d^A$ be the set of all Alexanderoff $T_D$ derivative operators $d'$ on $X$ such that $d' \leq d$

Remark that

- the least element of $L_d^A$ is the derivative operator $d_\emptyset$
- the greatest element of $L_d^A$ is $d$
- we do not know any representation of the least upper bound and the greatest lower bound of a family $\{d'_i: i \in I\}$ in $L_d^A$
- $(L_d^A, \leq)$ is a complete lattice
A strict partial order on $X$ is a binary relation $R$ on $X$ such that

- for all $x \in X$, $x \not\in R(x)$
- for all $x \in X$, $R(R(x)) \subseteq R(x)$

We shall say that $A \subseteq X$ is

- **R-closed** iff $R^{-1}(A) \subseteq A$
Alexandroff $T_D$ derivative operators and strict partial orders

Strict partial orders

Let $\leq$ be the binary relation between strict partial orders on $X$ such that

$R \leq R'$ iff $R \subseteq R'$
Alexandroff $T_D$ derivative operators and strict partial orders

Strict partial orders

Given a strict partial order $R$ on $X$

- let $L_R$ be the set of all strict partial orders $R'$ on $X$ such that $R' \leq R$

Remark that

- the least element of $L_R$ is the strict partial order $\emptyset$
- the greatest element of $L_R$ is $R$
- the least upper bound of a family $\{R'_i: i \in I\}$ in $L_R$ is the transitive closure of $\bigcup \{R'_i: i \in I\}$
- the greatest lower bound of a family $\{R'_i: i \in I\}$ in $L_R$ is $\bigcap \{R'_i: i \in I\}$
- $(L_R, \leq)$ is a complete lattice
Alexandroff $T_D$ derivative operators and strict partial orders

Given an Alexandroff $T_D$ derivative operator $d$ on $X$

- let $R_d$ be the binary relation on $X$ such that for all $x, y \in X$, $x R_d y$ iff $x \in d(\{y\})$

Remark that

- $R_d$ is a strict partial order on $X$
- for all $A \subseteq X$, $A$ is $R_d$-closed iff $A$ is $d$-closed
- $R_d \leq R_{d'}$ iff $d \leq d'$
Alexandroff $T_D$ derivative operators and strict partial orders

Given a strict partial order $R$ on $X$

- let $d_R$ be the function $d_R : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ such that for all $A \subseteq X$, $d_R(A) = R^{-1}(A)$

Remark that

- $d_R$ is an Alexandroff $T_D$ derivative operator on $X$
- for all $A \subseteq X$, $A$ is $d_R$-closed iff $A$ is $R$-closed
- $d_R \leq d_{R'}$ iff $R \leq R'$
Alexandroff $T_D$ derivative operators and strict partial orders

Let us further remark that

- $d_{R_d} = d$
- $R_{d_R} = R$

Given an Alexandroff $T_D$ derivative operator $d$ on $X$

- the function $f: L^A_d \rightarrow L_{R_d}$ such that $f(d') = R_{d'}$ is an isomorphism between $(L_{R_d}, \leq)$ and $(L^A_d, \leq)$

Given a strict partial order $R$ on $X$

- the function $f: L_R \rightarrow L^A_{d_R}$ such that $f(R') = d_{R'}$ is an isomorphism between $(L^A_{d_R}, \leq)$ and $(L_R, \leq)$
Given an Alexandroff $T_D$ derivative operator $d$ on $X$

- let $\theta_d$ be the function $\theta_d: L_d \rightarrow L_d$ such that for all $d' \in L_d$,
  $\theta_d(d') = d \circ d'$

Clearly

- $\theta_d$ is monotonic
- $\theta_d$ has a least fixpoint $\text{lfp}(\theta_d)$ and a greatest fixpoint $\text{gfp}(\theta_d)$
- $\text{lfp}(\theta_d) = d_\emptyset$
- $\text{gfp}(\theta_d)$ is the least upper bound of the family $\{d': d' \leq \theta_d(d')\}$ in $L_d$
For all ordinals $\alpha$, we inductively define $\theta_d \downarrow \alpha$ as follows

- $\theta_d \downarrow 0$ is $d$
- for all successor ordinals $\alpha$, $\theta_d \downarrow \alpha$ is $\theta_d(\theta_d \downarrow (\alpha - 1))$
- for all limit ordinals $\alpha$, $\theta_d \downarrow \alpha$ is the greatest lower bound of the family $\{\theta_d \downarrow \beta : \beta \in \alpha\}$ in $L_d$

There exists an ordinal $\alpha$ such that

- $\theta_d \downarrow \alpha = \text{gfp}(\theta_d)$
Cantor-Bendixson ranks
Cantor-Bendixson ranks of Alexandroff $T_D$ derivative operators

The least ordinal $\alpha$ such that
\[ \theta_d \downarrow \alpha = \text{gfp}(\theta_d) \]
is called the **Cantor-Bendixson rank** of $d$

**Example**
\[ X = \mathbb{Z} \]
\[ d_\mathbb{Z}(A) = \{ x : \text{there exists } y \in A \text{ such that } x <_\mathbb{Z} y \} \]
\[ \text{obviously} \]
\[ \theta_{d_\mathbb{Z}}(\theta_{d_\mathbb{Z}} \downarrow \omega) = \theta_{d_\mathbb{Z}} \downarrow \omega \]
\[ \text{the Cantor-Bendixson rank of } d_\mathbb{Z} \text{ is } \omega \]
Cantor-Bendixson ranks
Cantor-Bendixson ranks of strict partial orders

Given a strict partial order $R$ on $X$

- let $\theta_R$ be the function $\theta_R: L_R \to L_R$ such that for all $R' \in L_R$, $\theta_R(R') = R \circ R'$

Clearly

- $\theta_R$ is monotonic
- $\theta_R$ has a least fixpoint $\text{lfp}(\theta_R)$ and a greatest fixpoint $\text{gfp}(\theta_R)$
- $\text{lfp}(\theta_R) = \emptyset$
- $\text{gfp}(\theta_R)$ is the least upper bound of the family $\{ R': R' \leq \theta_R(R') \}$ in $L_R$
For all ordinals $\alpha$, we inductively define $\theta_R \downarrow \alpha$ as follows

- $\theta_R \downarrow 0$ is $R$
- For all successor ordinals $\alpha$, $\theta_R \downarrow \alpha$ is $\theta_R(\theta_R \downarrow (\alpha - 1))$
- For all limit ordinals $\alpha$, $\theta_R \downarrow \alpha$ is the greatest lower bound of the family $\{\theta_R \downarrow \beta : \beta \in \alpha\}$ in $L_R$

There exists an ordinal $\alpha$ such that

- $\theta_R \downarrow \alpha = \text{gfp}(\theta_R)$
Cantor-Bendixson ranks
Cantor-Bendixson ranks of strict partial orders

The least ordinal \( \alpha \) such that

\[ \theta_R \downarrow \alpha = \text{gfp}(\theta_R) \]

is called the **Cantor-Bendixson rank** of \( R \)

Example

\[ X = \mathbb{Q} \]

\[ x \; R_\mathbb{Q} \; y \text{ iff } x <_\mathbb{Q} y \]

obviously

\[ \theta_{R_\mathbb{Q}} (\theta_{R_\mathbb{Q}} \downarrow 0) = \theta_{R_\mathbb{Q}} \downarrow 0 \]

the Cantor-Bendixson rank of \( R_\mathbb{Q} \) is 0
Let $d$ be an Alexandroff $T_D$ derivative operator on $X$ and $R$ be a strict partial order on $X$ such that

- for all $x, y \in X$, $x R y$ iff $x \in d(\{y\})$
- for all $A \subseteq X$, $d(A) = R^{-1}(A)$

One can prove by induction on the ordinal $\alpha$ that

- for all $x, y \in X$, $x \theta_{R \downarrow \alpha} y$ iff $x \in \theta_{d \downarrow \alpha}(\{y\})$
- for all $A \subseteq X$, $\theta_{d \downarrow \alpha}(A) \supseteq \theta_{R \downarrow \alpha^{-1}}(A)$
Cantor-Bendixson ranks
Alexandroff $T_D$ derivative operators v. strict partial orders

Let
- $\alpha_d$ be the Cantor-Bendixson rank of $d$
- $\alpha_R$ be the Cantor-Bendixson rank of $R$

The above considerations prove that
- for all $x, y \in X$, $x \theta_R \downarrow \alpha y$ iff $x \in \theta_d \downarrow \alpha(\{y\})$
- for all $A \subseteq X$, $\theta_d \downarrow \alpha(A) \supseteq \theta_R \downarrow \alpha^{-1}(A)$
Cantor-Bendixson ranks
Alexandroff $T_D$ derivative operators v. strict partial orders

Example
A modal logic

Syntax

Formulas are defined as follows

- $\phi ::= p \mid \bot \mid \neg \phi \mid (\phi \lor \psi) \mid \Box \phi \mid \Box^* \phi$

Abbreviations

- Standard definitions for the remaining Boolean operations
- $\Diamond \phi ::= \neg \Box \neg \phi$
- $\Diamond^* \phi ::= \neg \Box^* \neg \phi$
A modal logic
Relational semantics

A **relational frame** is a structure of the form \( \mathcal{F} = (X, R, S) \) such that

- \( X \) is a nonempty set
- \( R \) is a strict partial order on \( X \)
- \( S \) is the greatest fixpoint of the function \( \theta_R \) in \( L_R \)

**Lemma:** If \( \mathcal{F} = (X, R, S) \) is a relational frame, then

1. \( R \circ R \leq R \)
2. \( S \circ S \leq S \)
3. \( S \leq R \)
4. \( R \circ S \leq S \)
5. \( S \circ R \leq S \)
6. \( S \leq R \circ S \)
A modal logic

Relational semantics

A **relational model** is a structure of the form $\mathcal{M} = (X, R, S, V)$ such that

- $(X, R, S)$ is a relational frame
- $V$ is a valuation on $X$

Satisfiability

- $\mathcal{M}, x \models p$ iff $x \in V(p)$
- $\mathcal{M}, x \not\models \bot$
- $\mathcal{M}, x \models \neg \phi$ iff $\mathcal{M}, x \not\models \phi$
- $\mathcal{M}, x \models \phi \lor \psi$ iff either $\mathcal{M}, x \models \phi$, or $\mathcal{M}, x \models \psi$
- $\mathcal{M}, x \models \Box \phi$ iff for all $y \in X$, if $x R y$, then $\mathcal{M}, y \models \phi$
- $\mathcal{M}, x \models \Box^* \phi$ iff for all $y \in X$, if $x S y$, then $\mathcal{M}, y \models \phi$
Lemma: if $\mathcal{F} = (X, R, S)$ is a relational frame, then

1. $\mathcal{F} \models □\phi \rightarrow □□\phi$
2. $\mathcal{F} \models □^*\phi \rightarrow □^*□^*\phi$
3. $\mathcal{F} \models □\phi \rightarrow □^*\phi$
4. $\mathcal{F} \models □^*\phi \rightarrow □□^*\phi$
5. $\mathcal{F} \models □^*\phi \rightarrow □^*□^*\phi$
6. $\mathcal{F} \models □□^*\phi \rightarrow □^*\phi$
A modal logic
Topological semantics

A topological frame is a structure of the form $\mathcal{F} = (X, d, e)$ such that

- $X$ is a nonempty set
- $d$ is an Alexandroff $T_D$ derivative operator on $X$
- $e$ is the greatest fixpoint of the function $\theta_d$ in $L_d$

Lemma: if $\mathcal{F} = (X, d, e)$ is a topological frame, then

1. $d \circ d \leq d$
2. $e \circ e \leq e$
3. $e \leq d$
4. $d \circ e \leq e$
5. $e \circ d \leq e$
6. $e \leq d \circ e$
A modal logic
Topological semantics

A **topological model** is a structure of the form \( M = (X, d, e, V) \) such that

- \((X, d, e)\) is a topological frame
- \( V \) is a valuation on \( X \)

**Interpretation**

- \( \| p \|_M = V(p) \)
- \( \| \bot \|_M = \emptyset \)
- \( \| \neg \phi \|_M = X \setminus \| \phi \|_M \)
- \( \| \phi \lor \psi \|_M = \| \phi \|_M \cup \| \psi \|_M \)
- \( \| \Box \phi \|_M = X \setminus d(X \setminus \| \phi \|_M) \)
- \( \| \Box^* \phi \|_M = X \setminus e(X \setminus \| \phi \|_M) \)
Lemma: if $\mathcal{F} = (X, d, e)$ is a topological frame, then

1. $\mathcal{F} \models \Box \phi \rightarrow \Box \Box \phi$
2. $\mathcal{F} \models \Box^* \phi \rightarrow \Box^* \Box^* \phi$
3. $\mathcal{F} \models \Box \phi \rightarrow \Box^* \phi$
4. $\mathcal{F} \models \Box^* \phi \rightarrow \Box \Box^* \phi$
5. $\mathcal{F} \models \Box^* \phi \rightarrow \Box^* \Box \phi$
6. $\mathcal{F} \models \Box \Box^* \phi \rightarrow \Box^* \phi$
Axiomatization and completeness

Axiomatization

Let $L$ be the least normal logic in our language containing

1. $\square \phi \rightarrow \square \square \phi$
2. $\square^* \phi \rightarrow \square^* \square^* \phi$
3. $\square \phi \rightarrow \square^* \phi$
4. $\square^* \phi \rightarrow \square \square^* \phi$
5. $\square^* \phi \rightarrow \square^* \square \phi$
6. $\square \square^* \phi \rightarrow \square^* \phi$

Proposition (Soundness)

- if $\phi \in L$, then $\phi$ is valid in all relational frames
- if $\phi \in L$, then $\phi$ is valid in all topological frames
Axiomatization and completeness

Axiomatization

Proposition (Completeness)
► if $\phi$ is valid in all relational frames, then $\phi \in L$

Example
► if $\phi = \Box(p \rightarrow \lozenge p) \rightarrow (\lozenge p \rightarrow \lozenge^* p)$, then
  ► $\phi$ is valid in all topological frames
  ► $\phi$ is not valid in all relational frames
Axiomatization and completeness

Axiomatization

A set $\Gamma$ of formulas is said to be an $L$-theory iff
- $\Gamma$ contains $L$
- $\Gamma$ is closed under the rule of modus ponens

We shall say that an $L$-theory $\Gamma$
- is consistent iff $\bot \notin \Gamma$
- is maximal iff for all formulas $\phi$, either $\phi \in \Gamma$, or $\neg \phi \in \Gamma$

Given an $L$-theory $\Gamma$ and a formula $\phi$
- $\Gamma + \phi = \{ \psi : \phi \rightarrow \psi \in \Gamma \}$
- $\Box \Gamma = \{ \phi : \Box \phi \in \Gamma \}$
- $\Box^* \Gamma = \{ \phi : \Box^* \phi \in \Gamma \}$
Axiomatization and completeness

Axiomatization

Lemma

1. $\Gamma + \phi$ is the least $L$-theory containing $\Gamma$ and $\phi$
2. $\Gamma + \phi$ is consistent iff $\neg\phi \not\in \Gamma$
3. $\square\Gamma$ is an $L$-theory
4. $\square^*\Gamma$ is an $L$-theory

Lemma (Lindenbaum’s Lemma)

- if $\Gamma$ is a consistent $L$-theory, then there exists a maximal consistent $L$-theory $\Delta$ such that $\Gamma \subseteq \Delta$
Axiomatization and completeness

Axiomatization

Lemma (Existence Lemma)

1. if $\Gamma$ is a maximal consistent $L$-theory such that $\Box \phi \not\in \Gamma$, then there exists a maximal consistent $L$-theory $\Delta$ such that $\Box \Gamma \subseteq \Delta$ and $\phi \not\in \Delta$

2. if $\Gamma$ is a maximal consistent $L$-theory such that $\Box^* \phi \not\in \Gamma$, then there exists a maximal consistent $L$-theory $\Delta$ such that $\Box^* \Gamma \subseteq \Delta$ and $\phi \not\in \Delta$

Lemma

- if $\Box^* \Gamma \subseteq \Delta$ then there exists a maximal consistent $L$-theory $\Lambda$ such that $\Box \Gamma \subseteq \Lambda$ and $\Box^* \Lambda \subseteq \Delta$
Axiomatization and completeness

Axiomatization

A subordination structure is a structure of the form $S = (X, R, S, \mu)$ such that

- $X$ is a finite nonempty set
- $R$ and $S$ are strict partial orders on $X$
- $S \subseteq R$
- $R \circ S \subseteq S$
- $S \circ R \subseteq S$
- $\mu$ is an interpretation on $X$, i.e. $\mu$ associates a maximal consistent $L$-theory $\mu(x)$ to any $x \in X$

Proposition

- if $\phi$ is true in the class of all subordination structures of cardinality 1 then $\phi \in L$
Axiomatization and completeness

Axiomatization

Given a subordination structure \( S = (X, R, S, \mu) \), it may contain imperfections

\( \Box \)-imperfections: triples of the form \((x, \Box, \phi)\) where \( x \in X \) is such that

\begin{itemize}
  \item \( \Box \phi \notin \mu(x) \)
  \item for all \( y \in X \), if \( x R y \), then \( \phi \in \mu(y) \)
\end{itemize}

\( \Box^* \)-imperfections: triples of the form \((x, \Box^*, \phi)\) where \( x \in X \) is such that

\begin{itemize}
  \item \( \Box^* \phi \notin \mu(x) \)
  \item for all \( y \in X \), if \( x S y \), then \( \phi \in \mu(y) \)
\end{itemize}

imperfections of density pairs of the form \((x, y)\) where \( x, y \in X \) are such that

\begin{itemize}
  \item \( x S y \)
  \item for all \( z \in X \), either not \( x R z \), or not \( z S y \)
\end{itemize}
Lemma: Given a □-imperfection \((x, □, \phi)\) in a subordination structure \(S\)

- there exists a subordination structure \(S'\) such that \(S'\) contains \(S\) and \((x, □, \phi)\) is not a □-imperfection in \(S'\)

Proof
Lemma: Given a □-imperfection \((x, □, φ)\) in a subordination structure \(S\)

- there exists a subordination structure \(S'\) such that \(S'\) contains \(S\) and \((x, □, φ)\) is not a □-imperfection in \(S'\)

Proof

\[ S = (X, R, S, μ) \]

\[ □φ \notin μ(x) \]
Lemma: Given a $\Box$-imperfection $(x, \Box, \phi)$ in a subordination structure $S$

there exists a subordination structure $S'$ such that $S'$ contains $S$ and $(x, \Box, \phi)$ is not a $\Box$-imperfection in $S'$

Proof

$S = (X, R, S, \mu)$

$x$\hspace{1cm} $\Box \phi \notin \mu(x)$

$y'$\hspace{1cm} $\Box \mu(x) \cup \{\neg \phi\} \subseteq \mu'(y')$
Axiomatization and completeness
Repairing imperfections

Lemma: Given a $\square^*$-imperfection $(x, \square^*, \phi)$ in a subordination structure $S$

- there exists a subordination structure $S'$ such that $S'$ contains $S$ and $(x, \square^*, \phi)$ is not a $\square^*$-imperfection in $S'$

Proof
Lemma: Given a □*-imperfection \((x, □^*, \phi)\) in a subordination structure \(S\)

- there exists a subordination structure \(S'\) such that \(S'\) contains \(S\) and \((x, □^*, \phi)\) is not a □*-imperfection in \(S'\)

Proof

\[
S = (X, R, S, \mu)
\]

\[
\square^* \phi \not\in \mu(x)
\]
Axiomatization and completeness
Repairing imperfections

Lemma: Given a \( \square^* \)-imperfection \((x, \square^*, \phi)\) in a subordination structure \(S\)

- there exists a subordination structure \(S'\) such that \(S'\) contains \(S\) and \((x, \square^*, \phi)\) is not a \(\square^*\)-imperfection in \(S'\)

Proof

\[
S = (X, R, S, \mu)
\]

\(\square^* \phi \notin \mu(x)\)

\(R', S'\)

\(y'\)

\(\square^* \mu(x) \cup \{\neg \phi\} \subseteq \mu'(y')\)
Axiomatization and completeness
Repairing imperfections

Lemma: Given an imperfection of density \((x, y)\) in a subordination structure \(S\)

- there exists a subordination structure \(S'\) such that \(S'\) contains \(S\) and \((x, y)\) is not an imperfection of density in \(S'\)

Proof
Lemma: Given an imperfection of density \((x, y)\) in a subordination structure \(S\)
- there exists a subordination structure \(S'\) such that \(S'\) contains \(S\) and \((x, y)\) is not an imperfection of density in \(S'\)

Proof
\[
S = (X, R, S, \mu)
\]
Lemma: Given an imperfection of density \((x, y)\) in a subordination structure \(S\)

- there exists a subordination structure \(S'\) such that \(S'\) contains \(S\) and \((x, y)\) is not an imperfection of density in \(S'\)

Proof

\[ S = (X, R, S, \mu) \]
Axiomatization and completeness

Completeness

Theorem: The following conditions are equivalent

1. $\phi \in L$
2. $\phi$ is valid in the class of all relational frames
3. $\phi$ is true in the class of all subordination structures of cardinality 1
Proposition

- □* is not definable in the ordinary language of modal logic with respect to $L$
Definability
Modal definability

Proof:

1. Assume there exists a formula $\phi$ in the ordinary language of modal logic defining $\Box^*$ with respect to $L$

2. Let $\mathcal{M} = (\mathbb{Z}, <, \emptyset, V)$ and $\mathcal{M}' = (\mathbb{Q}, <, V')$ with $V(q) = \emptyset$ and $V'(q) = \emptyset$ for all Boolean variables $q$

3. Remark that for all formulas $\psi$ in the ordinary language of modal logic, for all $x \in \mathbb{Z}$ and for all $x' \in \mathbb{Q}$, $\mathcal{M}, x \models \psi$ iff $\mathcal{M}', x' \models \psi$

4. Hence, $\mathcal{M}, 0 \models \phi$ iff $\mathcal{M}', 0 \models \phi$

5. Remark that $\mathcal{M}, 0 \models \Box^* \rho$ and $\mathcal{M}', 0 \not\models \Box^* \rho$

6. Since $\phi$ defines $\Box^*$ with respect to $L$, $\mathcal{M}, 0 \models \phi$ and $\mathcal{M}', 0 \not\models \phi$: a contradiction
Proposition
- the class of all relational frames is not first-order definable
Definability
First-order definability

Proof:

1. assume there exists a first-order sentence \( \phi \) defining the class of all relational frames

2. for all \( n \in \mathbb{N} \), let \( \mathcal{F}_n = (X_n, R_n, S_n) \) be the relational frame defined by \( X_n = \{0, \ldots, n\} \), \( R_n = X_n \) and \( S_n = \emptyset \)

3. obviously, for all \( n \in \mathbb{N} \)
   1. \( \mathcal{F}_n \models \phi \)
   2. \( \mathcal{F}_n \models \exists y \ \forall x \ (R(x, y) \lor x \equiv y) \)
   3. \( \mathcal{F}_n \models \forall x \ \forall y \ \neg S(x, y) \)
Definability
First-order definability

4. let \( U \) be an ultrafilter over \( \mathbb{N} \) and \( \mathcal{F}_U = (X_U, R_U, S_U) \) be the ultraproduct of the family \( \{ \mathcal{F}_n: n \in \mathbb{N} \} \) modulo \( U \)

5. by 3
   1. \( \mathcal{F}_U \models \phi \)
   2. \( \mathcal{F}_U \models \exists y \forall x (R(x, y) \lor x \equiv y) \)
   3. \( \mathcal{F}_U \models \forall x \forall y \neg S(x, y) \)

6. for all \( i \in \mathbb{N} \), let \([i]\) be the class of \((i, i, \ldots)\) modulo \( U \)

7. remark that for all \( i, j \in \mathbb{N} \), \([i]\) \( R_U \) \([j]\) iff \( i < j \)

8. by 5.2, there exists \( M_U \in X_U \) such that for all \( i \in \mathbb{N} \), either \([i]\) \( R_U \) \( M_U \), or \([i]\) = \( M_U \)

9. by 7, for all \( i \in \mathbb{N} \), \([i]\) \( R_U \) \( M_U \)
10. let $R'_U$ be the binary relation on $X_U$ such that for all $x, y \in X_U$, $x R'_U y$ iff there exists $i \in \mathbb{N}$ such that $x = [i]$ and $y = M_U$

11. remark that $R'_U$ is a strict partial order on $X_U$, $R'_U \subseteq R_U$ and $R'_U \neq \emptyset$

12. claim: $R'_U \leq \theta_{R_U}(R'_U)$

13. hence, $R'_U \leq \text{gfp}(\theta_{R_U})$

14. by 5.1 and 5.3, $\text{gfp}(\theta_{R_U}) = \emptyset$

15. by 13, $R'_U = \emptyset$: a contradiction
Notes

Open problems:

1. Philosophical interpretation of $\square^*$ in terms of beliefs?
2. What is the logic of $\square^*$ alone? $K4$?
3. Finite model property of $L$?
4. Decidability/complexity of the membership problem in $L$?
5. Modal definability of the class of all relational frames?
6. Generalization to other monotonic functions $\theta_R: L_R \rightarrow L_R$

